Faster Processing of Quantum Information with Trapped Ions

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- Unitaries
  - Householder reflection
  - discrete Fourier transform
- Highly entangled states
  - Dicke states
  - cluster states
- Quantum algorithms
  - Grover search
- Composite pulses
  - Local addressing by nonlocal pulses
  - Highly conditional gates

see posters 17 (S. Ivanov) and 47 (B. Torosov)

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Standard model of quantum computer

Single-qubit and two-qubit operations

- Hadamard gate
- phase gate
- two-qubit gate (C-NOT or C-phase)

A universal quantum computer can be built with these gates only.

Trapped ions: C-NOT gate fidelity $> 99\%$ demonstrated in Innsbruck

Problem: Too many gates needed to construct a single mathematical step.

Example 1: about 100 pulses used in NMR demonstration of Grover search with 3 qubits ($\mathcal{N} = 8$ states, $2 + 2$ logical steps).

Example 2: about 10^3 pulses needed for factoring the number 15 with ions.

Preskill (1996): $396N^3$ pulses and $5N + 1$ qubits needed for N-bit number

Alternative: use the symmetries of the ion system to construct the operations in fewer steps (single-purpose QC, quantum simulator)

ideally: 1 logical step = 1 physical step

Householder Reflection

Householder Reflection

$$\mathbf{M}(\chi;\varphi) = \mathbf{I} + (e^{i\varphi} - 1)|\chi\rangle\langle\chi|$$

arbitrary matrix $\longrightarrow$ triangular matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} \overset{\mathbf{M}(\chi_1;\varphi_1)}{\longrightarrow} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ 0 & b_{22} & b_{23} & b_{24} & b_{25} \\ 0 & b_{32} & b_{33} & b_{34} & b_{35} \\ 0 & b_{42} & b_{43} & b_{44} & b_{45} \\ 0 & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}$$

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$$\overset{\mathbf{M}(\chi_4;\varphi_4)}{\longrightarrow} \begin{bmatrix} b_{11} & e_{12} & e_{13} & e_{14} & e_{15} \\ 0 & c_{22} & e_{23} & e_{24} & e_{25} \\ 0 & 0 & d_{33} & e_{34} & e_{35} \\ 0 & 0 & 0 & e_{44} & e_{45} \\ 0 & 0 & 0 & 0 & e_{55} \end{bmatrix}$$

Householder Reflection

$$\mathbf{M}(\chi;\varphi) = \mathbf{I} + (e^{i\varphi} - 1)|\chi\rangle\langle\chi|$$

Hermitean matrix $\longrightarrow$ tridiagonal matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} \overset{\mathbf{M}(\chi;\varphi)}{\longrightarrow} \begin{bmatrix} b_{11} & b_{12} & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 \\ 0 & b_{32} & b_{33} & b_{34} & 0 \\ 0 & 0 & b_{43} & b_{44} & b_{45} \\ 0 & 0 & 0 & b_{54} & b_{55} \end{bmatrix}$$

Implication: Any Hamiltonian can be reduced to an effective one with nearest-neighbor interactions

Householder Reflection

$$\mathbf{M}(\chi;\varphi) = \mathbf{I} + (e^{i\varphi} - 1)|\chi\rangle\langle\chi|$$

unitary matrix $\longrightarrow$ diagonal matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} \overset{\mathbf{M}(\chi_1;\varphi_1)}{\longrightarrow} \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ 0 & b_{22} & b_{23} & b_{24} & b_{25} \\ 0 & b_{32} & b_{33} & b_{34} & b_{35} \\ 0 & b_{42} & b_{43} & b_{44} & b_{45} \\ 0 & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}$$

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$$\xrightarrow{\mathbf{M}(\chi_2;\varphi_2)} \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ 0 & c_{22} & 0 & 0 & 0 \\ 0 & 0 & c_{33} & c_{34} & c_{35} \\ 0 & 0 & c_{43} & c_{44} & c_{45} \\ 0 & 0 & c_{53} & c_{54} & c_{55} \end{bmatrix}$$

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Householder Reflection

generalized HR $\mathbf{M}(\chi;\varphi) = \mathbf{I} + (e^{i\varphi} - 1)|\chi\rangle\langle\chi|, \quad \mathbf{M}(\chi;\varphi)^\dagger = \mathbf{M}(\chi;\varphi)^{-1}$

ordinary HR $\mathbf{M}(\chi) \equiv \mathbf{M}(\chi;\varphi=\pi) = \mathbf{I} - 2|\chi\rangle\langle\chi| = \mathbf{M}(\chi)^\dagger = \mathbf{M}(\chi)^{-1}$

unitary matrix $\longrightarrow$ diagonal matrix

generalized HR $\mathbf{M}(\chi_{N-1};\varphi_{N-1})\cdots\mathbf{M}(\chi_2;\varphi_2)\mathbf{M}(\chi_1;\varphi_1)\mathbf{U} = \mathbf{I}$

ordinary HR $\mathbf{M}(\chi_{N-1})\cdots\mathbf{M}(\chi_2)\mathbf{M}(\chi_1)\mathbf{U} = \mathrm{diag}\,(e^{i\alpha_1}, e^{i\alpha_2}, \ldots, e^{i\alpha_N})$

Any N-dimensional unitary matrix $\mathbf{U}$ can be represented as a product of $N-1$ Householder reflections:

generalized HR $\mathbf{U} = \mathbf{M}(\chi_1;\varphi_1)\mathbf{M}(\chi_2;\varphi_2)\cdots\mathbf{M}(\chi_{N-1};\varphi_{N-1})$

ordinary HR $\mathbf{U} = \mathbf{M}(\chi_1)\mathbf{M}(\chi_2)\cdots\mathbf{M}(\chi_{N-1})\mathrm{diag}\,(e^{i\alpha_1}, e^{i\alpha_2}, \ldots, e^{i\alpha_N})$

Synthesis of unitaries: General case

Any N-dimensional unitary matrix can be expressed as a succession of

- $N-1$ generalized HRs $\mathbf{M}(\chi_n;\varphi_n)$ $(n=1,2,\ldots,N-1)$ and a one-dimensional phase gate:

$$\mathbf{U}(N)=\mathbf{M}(\chi_1;\varphi_1)\mathbf{M}(\chi_2;\varphi_2)\cdots\mathbf{M}(\chi_{N-1};\varphi_{N-1})\mathbf{F}(0,0,\ldots,0,\varphi_N)$$

$$|\chi_1\rangle=(|u_1\rangle-|e_1\rangle)/\text{norm};\quad \varphi_1=2\arg(1-u_{11})-\pi \qquad |\chi_2\rangle=\ldots$$

- $N-1$ standard HRs $\mathbf{M}(\chi_n)$ $(n=1,2,\ldots,N-1)$ and an N-dimensional phase gate $\mathbf{F}(\phi_1,\phi_2,\ldots,\phi_N)=\text{diag}\left\{e^{i\phi_1},e^{i\phi_2},\ldots,e^{i\phi_N}\right\}$:

$$\mathbf{U}(N)=\mathbf{M}(\chi_1)\mathbf{M}(\chi_2)\cdots\mathbf{M}(\chi_{N-1})\mathbf{F}(\phi_1,\phi_2,\ldots,\phi_N)$$

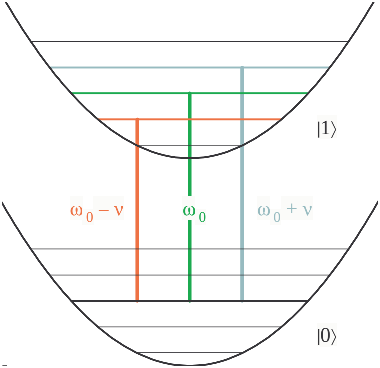
$$|\chi_1\rangle=(|u_1\rangle-e^{i\arg u_{11}}|e_1\rangle)/\text{norm};\qquad |\chi_2\rangle=\ldots$$

$\Longrightarrow$ any N-dimensional unitary transformation $\mathbf{U}(N)$ can be constructed by at most N steps

Standard methods (Givens SU(2) rotations) use $\mathcal{O}(N^2)$ steps!

M Reck, A Zeilinger, HJ Bernstein, P Bertani, PRL **73**, 58 (1994)

Linear ion chain: Energy levels



Vibrational energy levels
in the $|0\rangle$ and $|1\rangle$ manifolds, with
red-sideband $(\omega_L = \omega_0 - \nu)$,
carrier $(\omega_L = \omega_0)$, and
blue-sideband $(\omega_L = \omega_0 + \nu)$
transitions.

Linear ion chain: Hamiltonians

- Laser tuned near red-sideband resonance: $\omega_L(t) = \omega_0 - \nu - \delta(t)$

$$\mathbf{H}_I(t) = \hbar g(t) \sum_{n=1}^{N} \left[a\sigma_n^+ e^{i\int_{t_i}^{t} \delta(\tau)d\tau - i\phi_n} + a^\dagger \sigma_n^- e^{-i\int_{t_i}^{t} \delta(\tau)d\tau + i\phi_n} \right]$$

Jaynes-Cummings model

conserves the **SUM** of ionic excitations and phonons

$$|0\rangle_{\text{ion}}|n\rangle_{\text{phonon}} \xrightarrow{\text{red}} |1\rangle_{\text{ion}}|n-1\rangle_{\text{phonon}}$$

$\sigma_n^+ = |1_n\rangle\langle 0_n|$ and $\sigma_n^- = |0_n\rangle\langle 1_n|$: raising and lowering ionic operators
$a^\dagger$ and a: phonon creation and annihilation operators $\quad \nu \gg 2.6\Omega_n\eta/\sqrt{N}$

- Laser tuned near blue-sideband resonance: $\omega_L(t) = \omega_0 + \nu - \delta(t)$

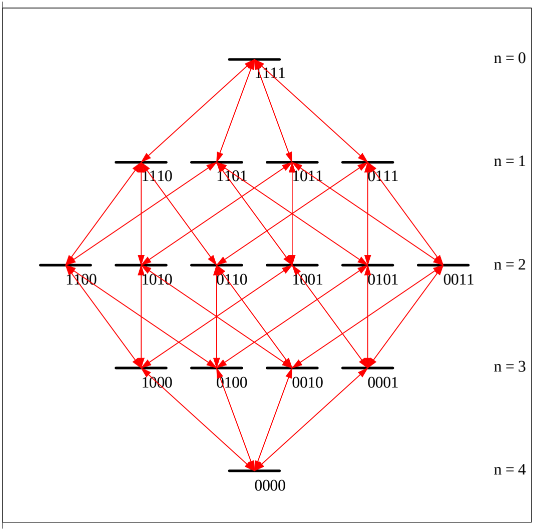
$$\mathbf{H}_I(t) = \hbar g(t) \sum_{n=1}^{N} \left[a^\dagger \sigma_n^+ e^{i\int_{t_i}^{t} \delta(\tau)d\tau - i\phi_n} + a\sigma_n^- e^{-i\int_{t_i}^{t} \delta(\tau)d\tau + i\phi_n} \right]$$

anti-Jaynes-Cummings model

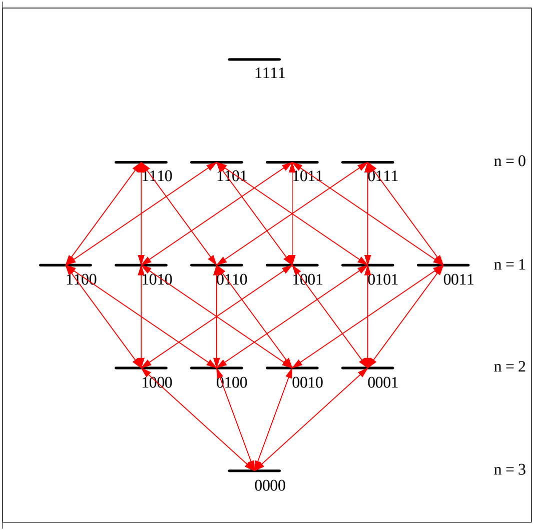
conserves the **DIFFERENCE** of ionic excitations and phonons

$$|0\rangle_{\text{ion}}|n\rangle_{\text{phonon}} \xrightarrow{\text{blue}} |1\rangle_{\text{ion}}|n+1\rangle_{\text{phonon}}$$

Linkage pattern: 4 ions, red-sideband



Linkage pattern: 4 ions, red-sideband



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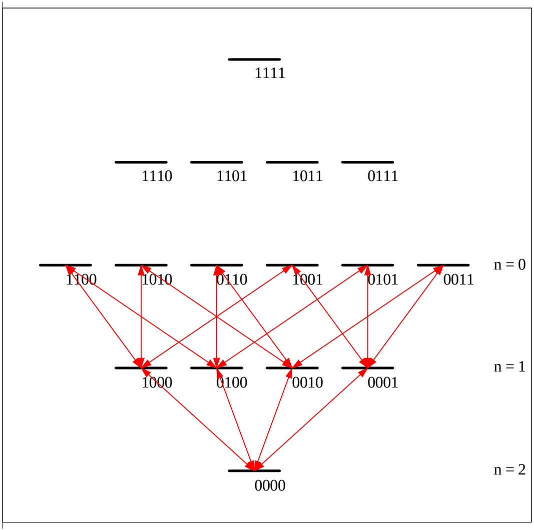
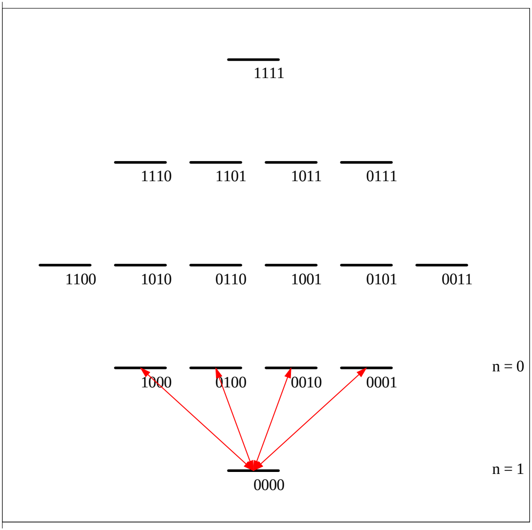
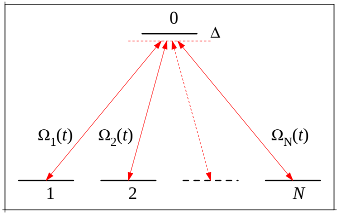


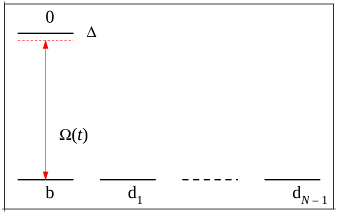
Linkage pattern: 4 ions, red-sideband



Morris-Shore transformation



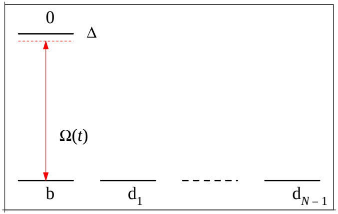
$$\mathbf{H}(t) = \frac{\hbar}{2}\begin{bmatrix} 0 & 0 & \cdots & 0 & \Omega_1(t) \\ 0 & 0 & \cdots & 0 & \Omega_2(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \Omega_N(t) \\ \Omega_1^*(t) & \Omega_2^*(t) & \cdots & \Omega_N^*(t) & 2\delta \end{bmatrix}$$



$$\mathbf{H}_{MS}(t) = \frac{\hbar}{2}\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \Omega(t) \\ 0 & 0 & \cdots & \Omega(t) & 2\delta \end{bmatrix}$$

$$\Omega(t) = \sqrt{\sum_{n=1}^{N} |\Omega_n(t)|^2}$$

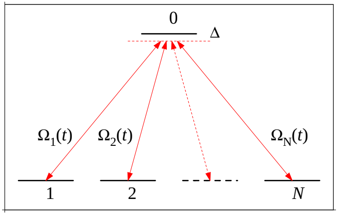
Propagator: Householder reflection



$$\mathbf{U}_{MS} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha & \beta \\ 0 & 0 & \cdots & -\beta^* & \alpha^* \end{bmatrix}$$

α, β are Cayley-Klein parameters

$$|\alpha|^2 + |\beta|^2 = 1$$



For $|\beta| = 0$ and $\alpha = e^{i\varphi}$

the propagator of the degenerate set is

$$\mathbf{U} = \mathbf{M}(\chi;\varphi) = \mathbf{I} + \left(e^{i\varphi} - 1\right) |\chi\rangle\langle\chi|$$

Householder reflection

$|\chi\rangle = [\Omega_1, \Omega_2, \ldots, \Omega_N]$ (complex vector)

ES Kyoseva, NVV, Phys. Rev. A **73**, 023420 (2006)

Householder reflections: Implementations

fulfill the conditions $|\beta| = 0$ and $\alpha = e^{i\varphi}$

- **Standard HR**: $\mathbf{M}(\chi) = \mathbf{I} - 2|\chi\rangle\langle\chi| \quad (\varphi = \pi)$

Exact resonance $(\Delta = 0)$: for any pulse shape $f(t)$ and rms pulse area
$A = \Omega \int_{-\infty}^{\infty} f(t)dt = 2(2k+1)\pi \quad (k = 0, 1, 2, \ldots) \qquad \Omega^2 = \sum_{n=1}^{N} |\Omega_n|^2$

- **Generalized HR**: $\mathbf{M}(\chi; \varphi) = \mathbf{I} + \left(e^{i\varphi} - 1\right) |\chi\rangle\langle\chi|$

Specific detunings off resonance

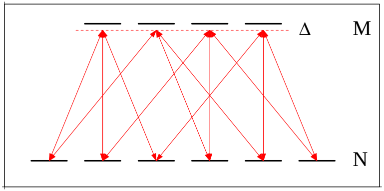
Example: for $f(t) = \operatorname{sech}(t/T)$, with rms area $A = \pi g T = 2\pi l$ $(l = 1, 2, \ldots)$, the desired phase φ is produced by a detuning δ obeying

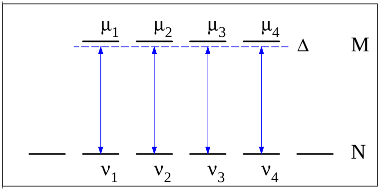
$$\varphi = 2 \arg \prod_{k=0}^{l-1} [\delta T + i(2k+1)]$$

Far-off-resonant fields: Generalized HR is realized automatically, with

$$\varphi \approx \frac{g^2}{\delta} \int_{-\infty}^{\infty} f^2(t)dt$$

Degenerate levels: Coupled reflections





$|\mu_m\rangle$ and $|\nu_m\rangle$ are eigenstates resp. of $\mathbf{V}^\dagger\mathbf{V}$ and $\mathbf{V}\mathbf{V}^\dagger$

For $|\beta_m| = 0$ and $\alpha_m = e^{i\varphi_m}$ $(m = 1, 2, \ldots, M;\ M \leq N)$
the propagators in the two degenerate sets are

$$\mathbf{U}_M = \mathbf{I} + \sum_{m=1}^{M} \left(e^{-i\varphi_m} - 1\right) |\mu_m\rangle\langle\mu_m| = \prod_{m=1}^{M} \mathbf{M}(\mu_m; -\varphi_m)$$
$$\mathbf{U}_N = \mathbf{I} + \sum_{m=1}^{M} \left(e^{i\varphi_m} - 1\right) |\nu_m\rangle\langle\nu_m| = \prod_{m=1}^{M} \mathbf{M}(\nu_m; \varphi_m)$$

products of Householder reflections

$\mathbf{U}_M$ and $\mathbf{U}_N$ can be reduced to single reflections by using $\mathbf{M}(\mu_m; 2k\pi) = \mathbf{I}$!

ES Kyoseva, NVV, BW Shore, J. Mod. Opt. **54**, S393 (2007)

Householder reflections: Applications

We used Householder reflections to:

- create highly entangled states
- navigate between entangled states in a single step
- create arbitrary preselected partially mixed states
- construct arbitrary N-dimensional unitaries in $< N$ steps [$\mathcal{O}(N^2)$ by standard methods]
- synthesize discrete (quantum) Fourier transforms in $\approx \frac{2}{3}N$ steps
- generate random matrices
- implement quantum algorithms (Grover search)

Two steps

- mathematical: by Householder reflections
- physical: uses the implementation with degenerate levels

Peter Ivanov, Elica Kyoseva, Boyan Torosov, Svetoslav Ivanov, Ian Linington
Phys. Rev. A 73, 023420 (2006); 74, 022323 (2006); 74, 053402 (2006); 75, 012323 (2007); 77, 012335 (2008); 77, 010302(R); 77, 062327 (2008); 77, 063837 (2008); 78, 012323 (2008); 78, 030301(R) (2008); 79, 012322 (2009); 80, 022329 (2009); 81, 042328 (2010); J. Mod. Opt. 54, S393 (2007)

Quantum Fourier transform (QFT)

QFT: unitary operator with the following action on a set $|n\rangle$ $(n = 1, 2 \ldots, N)$

$$\mathbf{U}_N^F |n\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{2\pi i (n-1)(k-1)/N} |k\rangle$$

$$\mathbf{U}_N^F = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{N-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & \cdots & w^{(N-1)(N-1)} \end{bmatrix} \qquad w = e^{2\pi i/N}$$

QFT can be represented as a product of HRs

N	2	3	4	5	6	7	8	9	10
steps	1	2	2	3	4	5	5	6	7

Standard methods use $\mathcal{O}(N^2)$ steps.

PA Ivanov, ES Kyoseva, NVV, Phys. Rev. A **74**, 022323 (2006)

QFT: Examples

- $\mathbf{U}_2^F = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \mathbf{M}(\chi)$, with $|\chi\rangle = \frac{1}{2}\left[-\sqrt{2-\sqrt{2}}, \sqrt{2+\sqrt{2}}\right]^T$

- $\mathbf{U}_3^F = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{bmatrix} = \mathbf{M}(\chi_1;\pi)\mathbf{M}(\chi_2;\pi/2)$

  with $|\chi_1\rangle = \frac{1}{2}\sqrt{1+\frac{1}{\sqrt{3}}}\left[1-\sqrt{3},1,1\right]^T$, $|\chi_2\rangle = \frac{1}{\sqrt{2}}\left[0,1,-1\right]^T$

- $\mathbf{U}_4^F = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} = \mathbf{M}(\chi_1;\pi)\mathbf{M}(\chi_2;\pi/2)$

  with $|\chi_1\rangle = \frac{1}{2}\left[-1,1,1,1\right]^T$ $|\chi_2\rangle = \frac{1}{\sqrt{2}}\left[0,1,0,-1\right]^T$

PA Ivanov, ES Kyoseva, NVV, Phys. Rev. A **74**, 022323 (2006)

Dicke States

Dicke states

Dicke-symmetric states of N particles and m excitations
robust against decoherence, particle loss and measurement
useful resource for quantum computing

$$|W_m^N\rangle \equiv \frac{1}{\sqrt{C_m^N}} \sum_k P_k | \underbrace{1,1,\ldots,1}_{m \text{ excitations}}, 0, \ldots, 0\rangle,$$

$\{P_k\}$ is the set of all distinct combinations of ions; $C_m^N \equiv \frac{N!}{m!(N-m)!} = \binom{N}{m}$

W-state: $|1_1 0_2 0_3 \ldots 0_N\rangle + |0_1 1_2 0_3 \ldots 0_N\rangle + \ldots + |0_1 0_2 0_3 \ldots 1_N\rangle$

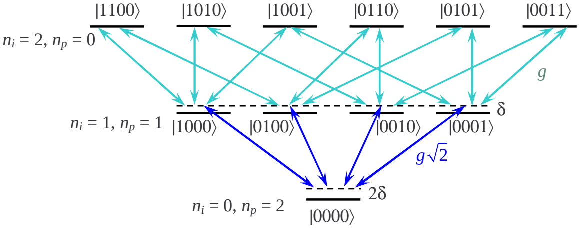
W_2-state: 2 excitations shared among N particles

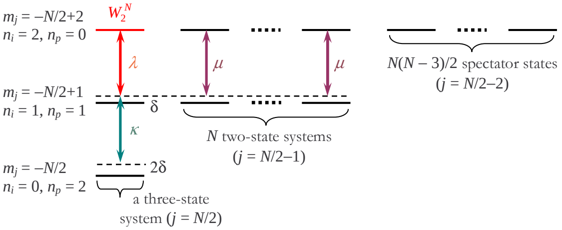
W_m-state: m excitations shared among N particles

addressing the ions one or two at a time requires a costly increase in the number of steps as the complexity of the state grows

FAST APPROACH [Linington, PRA 77, 010302 (2008); 77, 062327 (2008)]

- global addressing with only a single chirped adiabatic pulse
- applicable to any number of ions and excitations





couplings

$$\kappa(t) = g(t)\sqrt{2N}$$
$$\lambda(t) = g(t)\sqrt{2(N-1)}$$
$$\mu(t) = g(t)\sqrt{N-2}$$

Morris-Shore transformation for 4 ions and 2 phonons.

Morris-Shore Hamiltonian

If we start in state $|0_1 0_2 \dots 0_N\rangle$ then the evolution is confined to the longest $(m+1)$-state MS ladder:

- the lowest state is $|0_1 0_2 \dots 0_N\rangle |m\rangle$
- the highest is $|W_m^N\rangle$
- all intermediate states $(n = 1, \dots, m-1)$ are symmetric Dicke states

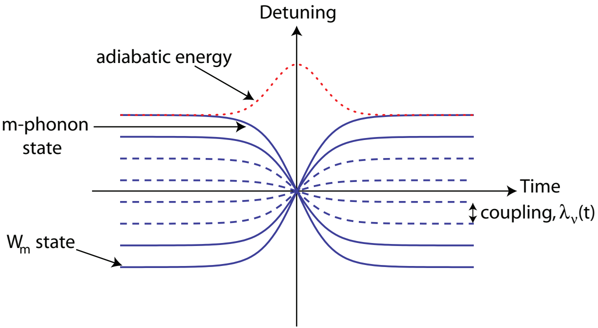
Morris-Shore Hamiltonian for the longest chain

$$\mathbf{H}_{N+1}(t) = \hbar \begin{bmatrix} 0 & \lambda_{0,1} & 0 & \dots & 0 & 0 \\ \lambda_{0,1} & \delta & \lambda_{1,2} & \dots & 0 & 0 \\ 0 & \lambda_{1,2} & 2\delta & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & (m-1)\,\delta & \lambda_{m-1,m} \\ 0 & 0 & 0 & \dots & \lambda_{m-1,m} & m\delta \end{bmatrix}$$

$$\lambda_{n,n-1}(t) = g(t)\sqrt{n(m-n+1)(N-m+n)}$$

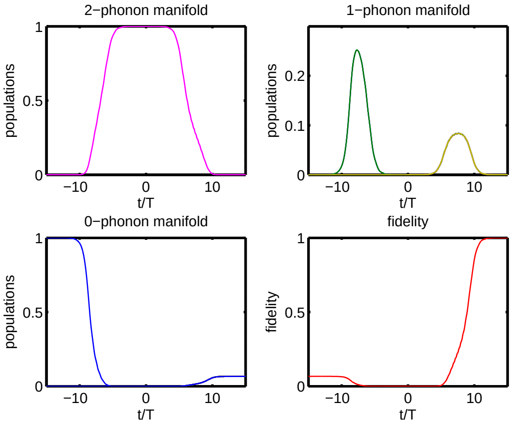
coupling between adjacent levels in the MS chain

Creation of Dicke states: Bowtie crossing



- Start in the m-phonon Fock state $|0_1 0_2 \cdots 0_N\rangle |m\rangle$.
- Apply an adiabatic chirped pulse addressing all N ions simultaneously
  The m-phonon state and the Dicke state $|W_m^N\rangle$ are connected adiabatically via a bowtie level-crossing.
  $\Longrightarrow$ the system is transferred adiabatically into the Dicke state $|W_m^N\rangle$:

$$|0_1 0_2 \cdots 0_N\rangle |m\rangle \xrightarrow{\text{red}} |W_m^N\rangle$$



Evolution of the populations of all 22 states for the creation of a $|W_2^6\rangle$ state (0-phonon: 1 state; 1-phonon: 6 states; 2-phonon: 15 states) for the sech-tanh model with $\Omega_0 T = 10; BT = 6$. The final fidelity is 99.996%. (Even when laser intensity is allowed to fluctuate by 10% across the chain, the overall fidelity is above 99.3%.)

Cluster States

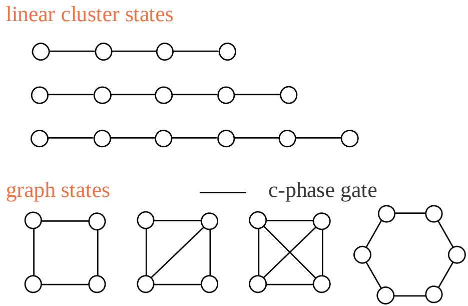
Cluster states

One-way quantum computer: Qubits are initialized in a highly entangled cluster state; the quantum computation proceeds by a sequence of single-qubit measurements with classical feedforward of their outcomes

A linear cluster state can be constructed as follows:

- each qubit is prepared in the superposition state $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$
- a control-phase gate is then applied between every nearest neighbor pair

$$|\Psi\rangle_{\mathfrak{C}} = \prod_{n=1}^{N-1} \mathbf{\Phi}_{n,n+1} |+\rangle^{\otimes N}$$



Demonstrated with photons.

R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. 86, 910 (2001); 86, 5188 (2001)

Cluster states: examples

Four-qubit cluster state

$$|\Psi_4\rangle = \frac{1}{2}\left[|0000\rangle + |1100\rangle + |0011\rangle - |1111\rangle\right]$$

Five-qubit cluster state

$$|\Psi\rangle_{\mathfrak{C}_5} = \frac{1}{\sqrt{8}}(|00000\rangle + |00011\rangle + |00101\rangle + |00110\rangle + |11000\rangle + |11011\rangle - |11101\rangle - |11110\rangle)$$

Six-qubit cluster state

$$|\Psi\rangle_{\mathfrak{C}_6} = \frac{1}{4}[|000000\rangle + |000011\rangle + |000101\rangle + |000110\rangle + |011000\rangle + |011011\rangle - |011101\rangle - |011110\rangle + |101000\rangle + |101011\rangle - |101101\rangle - |101110\rangle + |110000\rangle + |110011\rangle + |110101\rangle + |110110\rangle]$$

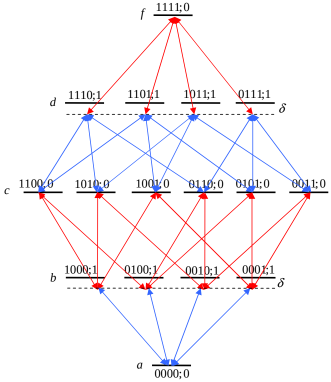
Cluster states: our technique

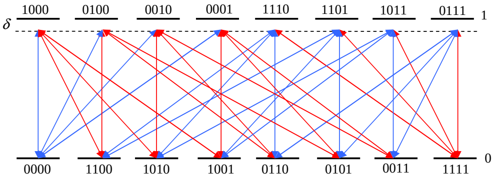
- N identical two-state ions, with a resonance frequency ω_0, in a linear Paul trap.
- Each ion interacts with two laser fields with frequencies tuned near the blue- and red-sideband resonance of a selected vibrational mode ν_p, with detunings $\pm\delta$:
  - $\omega_b = \omega_0 + \nu_p - \delta$
  - $\omega_r = \omega_0 - \nu_p + \delta$
- The Hamiltonian is $\mathbf{H}_I = \hbar \sum_{k=1}^{N} \sigma_k^{+} \left[a^{\dagger} g_k^b \mathrm{e}^{\mathrm{i}(\delta t + \phi_k^b)} + a g_k^r \mathrm{e}^{-\mathrm{i}(\delta t - \phi_k^r)} \right] + \text{h.c.}$

  $g_k^c(t) = s_k^p \eta_k^c \Omega_k^c(t)/(2\sqrt{N})$ $(c = r, b)$: laser coupling of the kth ion
- the Rabi frequencies $\Omega_k^c(t)$ have the same time dependence $f(t)$.
- the detuning δ from the sideband to be sufficiently large ($|\delta| \gg g_k^{b,r}$), so that all transitions with detunings $l\delta$ $(l = \pm 2, \pm 3, \ldots)$ can be neglected
- the blue and red couplings for each ion are equal, $g_k^b(t) = g_k^r(t) = g_k f(t)$
- the laser phases satisfy $\phi_k^b = l_k\pi - \phi$, and $\phi_k^r = l_k\pi + \phi$ $(l_k = 0, 1, \ldots)$

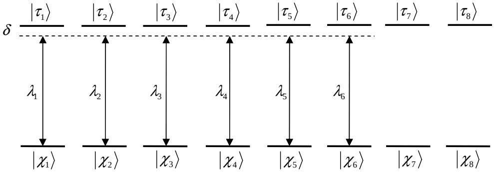
PA Ivanov, NVV, MB Plenio, Phys. Rev. A 78, 012323 (2008)

$N = 4$ CLUSTER STATE LINKAGE PATTERN



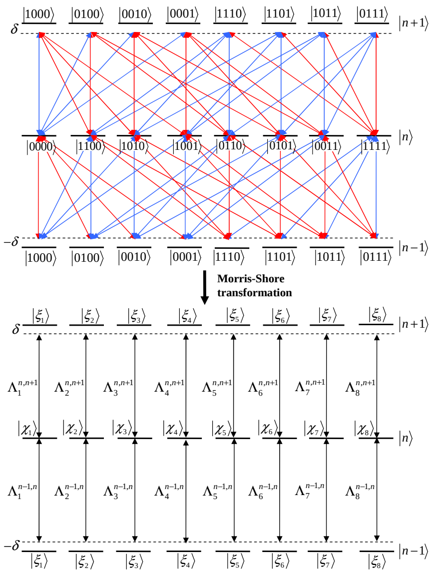


⇓ Morris-Shore transformation ⇓



PA Ivanov, NVV, MB Plenio, Phys. Rev. A 78, 012323 (2008)

$N = 4$ cluster state linkage pattern



Linear cluster state: Propagator for $N = 4$

$$\mathbf{U} = \mathbf{1} + \sum_{k=1}^{8} \left(e^{i\varphi_k} - 1\right) |\chi_k\rangle\langle\chi_k| = \prod_{k=1}^{8} \mathbf{M}(\chi_k; \varphi_k)$$

$$\varphi_k = \frac{(\Lambda_k^{n,n+1})^2 - \Lambda_k^{n-1,n})^2}{\delta} \int_{-\infty}^{\infty} f^2(t) dt$$

the dependence on the phonon number n is removed
(in 1st order of PT, as in Mølmer-Sörensen's gate)

The generalized Householder reflection (HR) $\mathbf{M}(\chi; \varphi) = \mathbf{1} + \left(\mathrm{e}^{\mathrm{i}\varphi} - 1\right) |\chi\rangle\langle\chi|$

- For $\varphi = 2l\pi$ (with l integer) we have $\mathbf{M}(\chi; 2l\pi) = \mathbf{1}$.
- For $\varphi = (2l+1)\pi$, the HR reduces to a standard HR: $\mathbf{M}(\chi; (2l+1)\pi) = \mathbf{M}(\chi) = \mathbf{1} - 2|\chi\rangle\langle\chi|$

We observe that $\mathbf{M}(\chi_8)\, \mathbf{M}(\chi_7)\, |0000\rangle = |\Psi\rangle_{\mathfrak{C}_4}$
$\Downarrow$ we must have

$$\varphi_k = 2m_k\pi \quad (k = 1, 2, \ldots, 6)$$
$$\varphi_k = (2m_k + 1)\pi \quad (k = 7, 8)$$

PA Ivanov, NVV, MB Plenio, Phys. Rev. A 78, 012323 (2008)

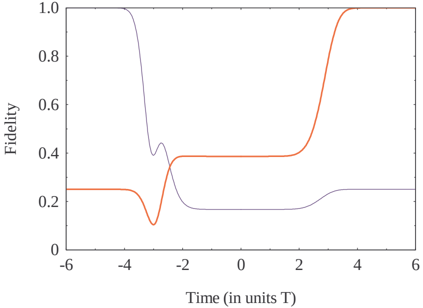
Linear cluster states: $N = 4$

Values for the scaled couplings $\widetilde{g}_k = g_k/\sqrt{\delta T\sqrt{2\pi}}$ $(k = 1, 2, 3, 4)$ for $\delta T = 1000$ and a Gaussian pulse shape $f(t) = \exp(-t^2/T^2)$.

step	$\widetilde{g}_1$	$\widetilde{g}_2$	$\widetilde{g}_3$	$\widetilde{g}_4$
1	$1/4$	$1/4$	$1/4$	$1/4$
2	$\sqrt{3}/4$	$\sqrt{3}/4$	$-\sqrt{3}/4$	$-\sqrt{3}/4$

Implementation

- apply a global pulse with amplitudes $\widetilde{g}_k = \frac{1}{4}$ $(k = 1, 2, 3, 4)$
- flip the signs of qubits 3 and 4
- apply a global pulse with amplitude $\widetilde{g}_k = \frac{\sqrt{3}}{4}$ $(k = 1, 2, 3, 4)$



PA Ivanov, NVV, MB Plenio, Phys. Rev. A 78, 012323 (2008)

Grover Search

Grover's quantum search

What it does?

- searches an arbitrary element in an unsorted database with $\mathcal{N}$ entries
- finds the marked item with only $\mathcal{O}(\sqrt{\mathcal{N}})$ calls to an oracle (returns "yes" or "no") (classical search requires $\mathcal{N}/2$ tries),

$$N_G = \left[\pi/(2\sin^{-1}(2\sqrt{\mathcal{N}-1}/\mathcal{N}))\right] \sim [(\pi/4)\sqrt{\mathcal{N}}] \quad \text{for large } \mathcal{N}$$

- as $\mathcal{N}$ increases, the fidelity approaches unity, with error $\mathcal{O}(1/\mathcal{N})$ (fully deterministic version also available)

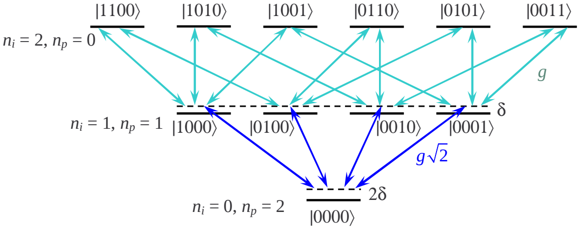
Implementation

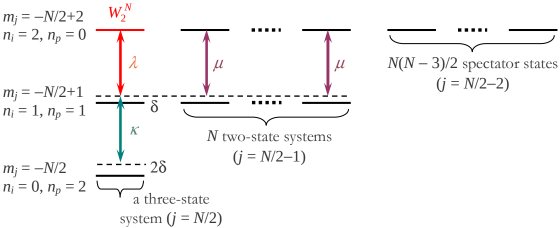
- initialize the database in an equal coherent superposition of states $|a\rangle = [1, 1, \ldots, 1]^T/\sqrt{\mathcal{N}}$
- an oracle flips the phase of the marked element $|m\rangle$: $\mathbf{M}(m) = \mathbf{1} - 2|m\rangle\langle m|$
- a reflection of the state vector about the mean: $\mathbf{M}(a) = \mathbf{1} - 2|a\rangle\langle a|$

Experimental demonstration

- two ($N = 4$) and three ($N = 8$) qubits in NMR
- two qubits ($N = 4$) in ion traps
- $N = 32$ items in classical optics

Grover search in a nonclassical database





couplings

$\kappa(t) = g(t)\sqrt{2N}$

$\lambda(t) = g(t)\sqrt{2(N-1)}$

$\mu(t) = g(t)\sqrt{N-2}$

Morris-Shore transformation for 4 ions and 2 phonons.

Reflection about the average

Propagator within the $n_i = 2$ manifold in the computational basis

$$\mathbf{U} = \mathbf{I} + (e^{i\varphi_a} - 1)|a\rangle\langle a| + (e^{i\varphi_b} - 1)\sum_{k=1}^{N} |\chi_k\rangle\langle\chi_k| = \mathbf{M}(a, \varphi_a) \prod_{k=1}^{N} \mathbf{M}(\chi_k, \varphi_b)$$

$\mathbf{M}(a, \varphi_a)$: exactly the reflection about the mean needed for Grover's search!

We wish that $\mathbf{U} \equiv \mathbf{M}(a, \varphi)$

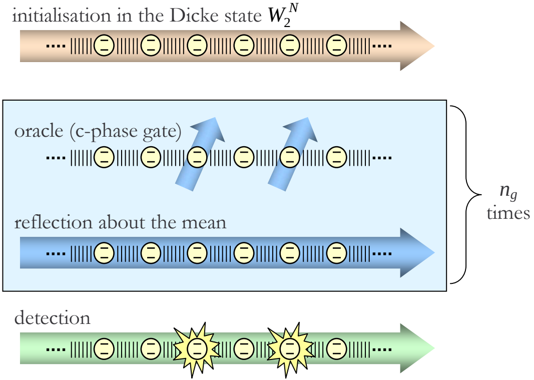
$$\mathbf{M}(\chi_k, 2l\pi) = \mathbf{I}$$

$$\Downarrow$$

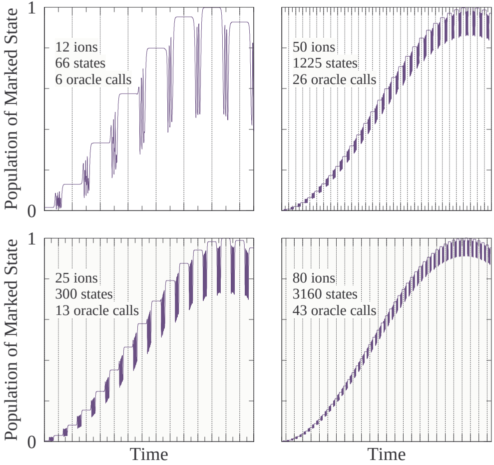
$$\varphi_a = \varphi + 2j\pi,$$
$$\varphi_b = 2l\pi.$$

IE Linington, PA Ivanov, NVV, Phys. Rev. A 79, 012322 (2009)

Grover search: implementation



Grover search in a nonclassical database



N ions and two excitations

register dim. $\mathcal{N} = N(N-1)/2$

The ions are initialized

in the Dicke state $|W_2^N\rangle$.

HR parameters

$\delta T = 10$ in all cases

$g(t) = g_0 \mathrm{e}^{-(t-t_n)^2/T^2}$

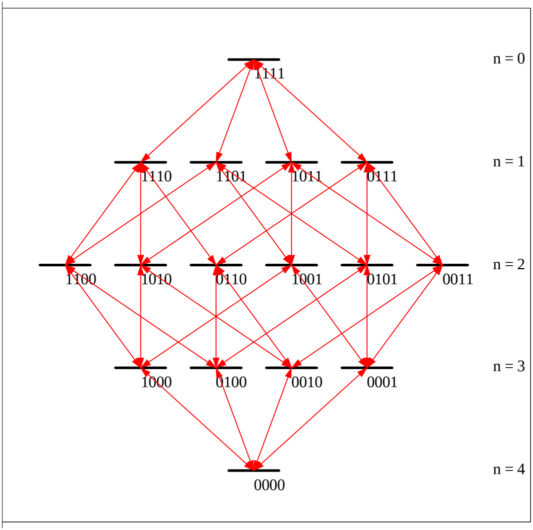
$g_0 T \approx 10.739, 13.587, 17.954, 21.547$

The oracle phase is

$\varphi \approx -0.94\pi, -0.98\pi, \pi, 0.95\pi$.

IE Linington, PA Ivanov, NVV, Phys. Rev. A 79, 012322 (2009)

Grover search in a Dicke database



Dicke database
N ions and $N/2$ excitations
largest set of states

$$\mathcal{N} = C^N_{N/2} \sim 2^N \sqrt{\frac{2}{\pi N}}$$

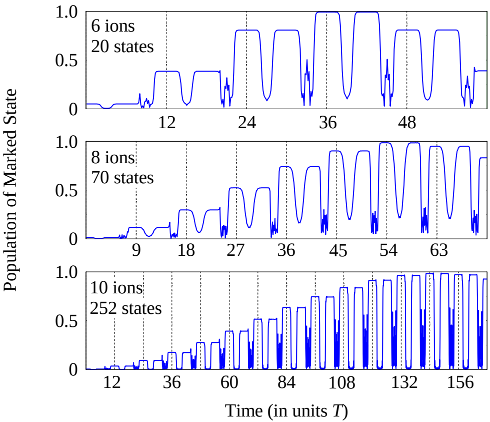
Initial state
the Dicke state $|W^N_{N/2}\rangle$
equal superposition

Reflection about the mean
reflection about $|W^N_{N/2}\rangle$

Oracle
$C^{N/2}$-phase gate

SS Ivanov, PA Ivanov, IE Linington, NV Vitanov, Phys. Rev. A 81, 042328 (2010)

Grover search in a Dicke database



N ions and $N/2$ excitations

$\mathcal{N} = C^N_{N/2} \sim 2^N \sqrt{\frac{2}{\pi N}}$

The ions are initialized in the Dicke state $|W^N_{N/2}\rangle$.

$$\Omega(t) = \Omega_n e^{-(t-t_n)^2/T^2}$$

#ions	#elements	#steps	oracle		reflection	
N	$\mathcal{N}$	n_G	δT	$\Omega_n T$	δT	$\Omega_n T$
6	20	3	19.470	28.610	10.320	25.830
8	70	6	21.400	10.800	21.050	24.400
10	252	12	15.687	70.322	88.565	87.142

SS Ivanov, PA Ivanov, IE Linington, NV Vitanov, Phys. Rev. A 81, 042328 (2010)

Conclusions

Universal quantum computer: Too many gates needed to construct a single mathematical step.

Example 1: about 100 pulses used in NMR demonstration of Grover search with 3 qubits ($\mathcal{N} = 8$ states, $3 + 3$ logical steps).

Example 2: about 10^3 pulses needed for factoring the number 15 with ions.

Preskill (1996): $396N^3$ pulses and $5N + 1$ qubits needed for N-bit number (later reduced)

Alternative: use the symmetries of the ion system to construct the operations in fewer steps

$\Rightarrow$ single-purpose quantum computer (like quantum simulator)
ideally: 1 logical step = 1 physical step

Linear ion chain: ideally suited for Householder reflection $\rightarrow$ Grover's search

Ring trap: ideal for quantum Fourier transform $\rightarrow$ Shor's factoring etc.

circulant Hamiltonian $\rightarrow$ discrete Fourier transform