Self-test for prospective MSc students

Congratulations on your MSc offer and welcome to our program!
We would like you to arrive at Sussex well-prepared for your course and to that end we have prepared this self-test in order to help you check your readiness and identify any areas that might need revision or further study. These problems are not meant to be extensive and covering everything you should know, but they do cover many of the basics required.

We do not expect prior specialist area knowledge for entrants in our MSc programs (e.g. not prior Astronomy knowledge is needed for the Astronomy MSc). However, to be successful you should arrive with a solid background in both Mathematics and basic Physics at an university degree level.

If you are able to do the majority of the problems below by yourself (possibly after some revision, as required) you are reasonably well prepared.

Mathematics:

- **Vector Calculus**: Show that, for any vector A and scalar ϕ the following identities hold:

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \\ \nabla \cdot (\nabla \times \mathbf{A}) &= 0 \\ \nabla \times (\nabla \phi) &= 0 \end{aligned}$$

- **Calculus:** Calculate the integrals:

$$\int_0^\pi \sin^3 \alpha d\alpha \qquad \int e^x \cos x dx \qquad \int \frac{5x^2}{\sqrt{9-x^2}} dx$$

- **Differential equations:** Solve the following equations:

$$\frac{dy}{dx} = \frac{y(x^2-1)}{y+1}, \qquad y \neq 1$$

$$\frac{dy}{dx} - 2xy = x$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 30y = e^{6t}$$

- **Fitting data points:**

  Fit a straight line and a quadratic to the data in the Table using any numerical method you like (e.g. if you know Python you can use `curve_fit` function from `scipy.optimize`). Which fit is better? (Hint: Compare the standard deviations.)

x	1.0	2.5	3.5	4.0	1.1	1.8	2.2	3.7
y	6.008	15.722	27.130	33.772	5.257	9.549	11.098	28.828

Figure 1: Data for fitting.

- **Diffraction of light:** In optics, you have learned that light bends around objects, i.e. exhibits diffraction. One of the simplest cases to study is the bending of light around a straight edge. In this case, we find that the intensity of light varies as we move away from the edge according to:

  $$I = 0.5I_0\{[C(v) + 0.5]^2 + [S(v) + 0.5]^2\}$$

  where I_0 is the intensity of the incident light, v is proportional to the distance travelled, and $C(v)$ and $S(v)$ are the Fresnel integrals:

  $$C(v) = \int_0^v \cos(\pi w^2/2)dw$$

  and

  $$S(v) = \int_0^v \sin(\pi w^2/2)dw$$

  Using any method, numerically integrate the Fresnel integrals, and thus evaluate I/I_0 as a function of v for $0 \leq v \leq 10$. Plot your results for C, S and I/I_0. Do they agree with what you have learned about diffraction in Optics? As an extra twist, try to this *computationally efficiently*.

  Challenge: Derive the formula for I.

Physics:

- **Mechanics:** Two particles move about each other in circular orbits under the influence of their mutual gravitational force, with a period τ. At some time $t = 0$, they are suddenly stopped and then they are released and allowed to fall into each other. Find the time T after which they collide, in terms of τ.

- **Mechanics:** Small oscillations: A particle moves under the influence of the potential $V(x) = Cx^n e^{-ax}$. Find the frequency of small oscillations around the equilibrium point.

- **Mechanics:** Escape velocity:

  (a) Find the escape velocity (that is, the velocity above which a particle will escape to $r = \infty$) for a particle on a spherical planet of radius R and mass M. What is the numerical value for the Earth? The Moon? The Sun?

  (b) Approximately how small must a spherical planet be in order for a human to be able to jump off? (Assume a density roughly equal to the Earths.)

- **Mechanics:** Ball hitting stick: A ball of mass M hits a stick with moment of inertia $I = \eta m\ell^2$. The ball is initially travelling with velocity V_0, perpendicular to the stick. The ball strikes the stick at a distance d from its centre The collision is elastic. Find the resulting translational and rotational speeds of the stick, and also the resulting speed of the ball.

- **Relativity:** A photon moves at an angle θ with respect to the x'-axis in frame S'. Frame S' moves at speed v with respect to frame S (along the x' axis). Calculate the components of the photons velocity in S, and verify that the speed is still c.

- **Relativity:** Relativistic rocket: Assume that a rocket propels itself by continually converting mass into photons and firing them out the back. Let m be the instantaneous mass of the rocket, and let v be the instantaneous speed with respect to the ground. Show that

$$\frac{dm}{m} + \frac{dv}{1 - v^2} = 0$$

If the initial mass is M, and the initial v is zero, integrate the above equation to obtain

$$m = M\sqrt{\frac{1-v}{1+v}}.$$

- **Relativity:** Colliding photons: Two photons each have energy E. They collide at an angle θ and create a particle of mass M. What is M?

- **Electrodynamics:** Consider an electrostatic potential Φ given by

  $$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-ar}}{r}(1+br)$$

  where a and b are constants.

  (a) Find the distribution of charge (both continuous and discrete) that will give this potential.

  (b) What, if anything, is special when a = 2b ?

  (c) Interpret your results physically.

- **Electrodynamics/Waves:** A transverse plane wave impinges normally in vacuum on a perfectly absorbing flat screen.

  (a) From the law of conservation of linear momentum, show that the pressure radiation pressure exerted on the screen is equal to the field energy per unit volume in the wave.

  (b) Let the incident radiation have a flux of $1.4\,kW/m^2$. The absorbing screen has a mass of $1\,g/m^2$. What is the screens acceleration due to radiation pressure?

- **Electrodynamics/Waves:** A monochromatic plane wave has complex electric field $\mathbf{E}(\mathbf{r},t) = \mathbf{E_0}\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$. It travels in the $+z$ direction in a lossless isotropic medium with relative permittivity $\epsilon_r = 4$ and relative permeability $\mu_r \approx 1$. The field is linearly polarized in the x-direction, with frequency $\nu = 1\,\mathrm{GHz}$ and a peak field $+10^3\,V/m$ at $t = 5\,\mathrm{ns}$ and $z = 1\,\mathrm{m}$.

  (a) Define the medium refractive index and find the angular frequency, phase velocity, wavenumber, wave vector, and wavelength.

  (b) Obtain the real instantaneous expression for $\mathbf{E}(\mathbf{r},t)$ valid for any position and time.

(c) Obtain the real instanteneous expression for the magnetic field $\mathbf{B}(\mathbf{r}, t)$ valid for any position and time.

(d) Find the field energy density and Poynting vector. For both quantities derive the time-averaged value.

(e) Find the locations where the electric field is maximum when $t = 0$ s.

- **Quantum Mechanics:** Uncertain Dart: A dart of mass 1 kg is dropped from a height of 1 m, with the intention to hit a certain point on the ground. Estimate the limitation set by the uncertainty principle of the accuracy that can be achieved.

- **Quantum Mechanics:** Suppose that the wave function of a (spinless) particle of mass m is

$$\psi(r, \theta, \phi) = A\frac{e^{-\alpha r} - e^{-\beta r}}{r}$$

where A, α and β are constants such that $0 < \alpha < \beta$. Find the potential $V(r, \theta, \phi)$ and the energy E of the particle.