*First order flow only available*

Lower limit on the distance-scaled transverse velocity.
Constraint on the immediacy of the surface patch.
Constraint on the immediacy of the plane.
Constraint on the spin.

*First order flow and tilt available*

The direction of the transverse velocity.
The immediacy of the surface path.
The immediacy of the plane.
The spin.

*First order flow and the immediacy of the plane available*

The direction of the transverse velocity (4 solutions, coupled to tilt).
The tilt (4 solutions, coupled to transverse velocity).
The immediacy of the surface patch.
The spin (two solutions).

*First order flow and spin available*

The direction of the transverse velocity (4 solutions, coupled to tilt).
The tilt (4 solutions, coupled to transverse velocity).
The immediacy of the surface patch (2 solutions).
The immediacy of the plane (2 solutions).

It is highly likely that a visual system will have some of this additional information available to it. The ambiguous solutions can be resolved with only coarse extra information.

It is clear that first-order flow, for even a single surface patch, can contribute much of the information needed for the control of actions.

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$$\frac{\partial v_D}{\partial Q_D} = \frac{V_D}{h}\cos G + \frac{V_A}{h}\sin G \tag{43}$$

$$\frac{\partial v_S}{\partial Q_S} = \frac{V_A}{h}\sin G \tag{44}$$

$$\frac{\partial v_D}{\partial Q_S} = -\omega_A \tag{45}$$

$$\frac{\partial v_S}{\partial Q_D} = \frac{V_S}{h}\cos G + \omega_A \tag{46}$$

These are the components of the deformation rate tensor in the *D–S* coordinate system. For the time being, we choose an image coordinate system which has its *x*-axis aligned with *D* and its *y*-axis aligned with *S*. Since we are working in a small patch round the fixed point, where the sphere is close to the tangent plane (Fig. 7), we then have $Q_D = x$ and $Q_S = y$ — that is, $\partial V_D / \partial Q_S = v_{x,y}$ etc. We can substitute Eqs. 43 to 46 into Eqs. 31 to 34 to get

$$D = \frac{V_A}{h}\sin G + \frac{V_D}{2h}\cos G \tag{47}$$

$$S = \frac{1}{2}\sqrt{\left(\frac{V_D}{h}\cos G\right)^2 + \left(\frac{V_S}{h}\cos G\right)^2} \tag{48}$$

$$R = \frac{V_S}{2h}\cos G + \omega_A \tag{49}$$

$$\tan 2\theta = V_S / V_D \qquad (|\theta| \le \pi/2,\ \mathrm{sign}\,\theta = \mathrm{sign}\,V_S) \tag{50}$$

The first three of these are identical to Eqs. 10 to 12. Fig. 4b shows that $V_S / V_D = \tan(\angle \boldsymbol{V_T} - \angle \boldsymbol{n_D})$, so that Eq. 13 is equivalent to Eq. 50.

Although we have chosen the *x-y* coordinates in our image along the *D* and *S* axes in order to link the equations, the relationships given by Eqs. 47 to 50 are independent of any particular choice of coordinate system: having established them, we can measure the differential flow using any axes we choose, and then use these equations to make the inferences discussed in Section 5.

# 7 Summary

The nature of first-order differential optic flow has been described in some detail, and a representation for motion relative to a planar surface patch set up. The central relationships between motion and first-order flow were stated in Eqs. 10 to 13 and derived in Section 6.

The information that can be obtained from the first-order flow of a single surface patch, with certain other kinds of information also available, was analysed in Section 5. The main results were that the following inferences can be made:

region about the line of sight the differences will be negligible. All the vectors are defined in a frame of reference which moves with the observer.

In this frame of reference, the surface is translating with a velocity $-\boldsymbol{V}$, and is rotating with an angular velocity $-\omega$. The velocity of the surface point is then given by

$$\frac{\partial}{\partial t}(\rho \boldsymbol{Q}) = -\boldsymbol{V} - \omega \times \rho \boldsymbol{Q} \tag{36}$$

from which it follows (using $\boldsymbol{Q} \cdot \boldsymbol{Q} = 1$) that

$$\boldsymbol{v} = -(\boldsymbol{V} - (\boldsymbol{V} \cdot \boldsymbol{Q})\boldsymbol{Q})/\rho - \omega \times \boldsymbol{Q} \tag{37}$$

This is the fundamental optic flow equation for rigid body motion.

*The equation of the plane*

The equation of the plane is simply

$$\boldsymbol{Q} \cdot \boldsymbol{p} = h/\rho \tag{38}$$

where $\boldsymbol{p}$ is the unit vector along the surface normal shown in Fig. 7.

*The partial derivatives of flow for a planar surface patch*

The next step is to differentiate Eq. 37 with respect to position in the image. To do this, we write it in component form:

$$v_i = (V_j Q_j Q_i - V_i)/\rho - \varepsilon_{ikl}\omega_k Q_l \tag{39}$$

where $\varepsilon$ is the alternating tensor and repeated suffices are summed. At this point, the suffices refer to components along arbitrary axes. Differentiation with respect to a component of $\boldsymbol{Q}$ then gives

$$\frac{\partial v_i}{\partial Q_m} = (V_j Q_j Q_i - V_i)\frac{\partial}{\partial Q_m}(1/\rho) + (V_m Q_i + V_j Q_j \delta_{im})/\rho - \varepsilon_{ikm}\omega_k \tag{40}$$

and differentiating both sides of Eq. 38 gives

$$\frac{\partial}{\partial Q_m}(1/\rho) = p_m/h \tag{41}$$

Substituting Eq. 41 into Eq. 40 gives an expression for the first-order flow in arbitrary axes:

$$\frac{\partial v_i}{\partial Q_m} = (V_j Q_j Q_i - V_i)p_m/h + (V_m Q_i + V_j Q_j \delta_{im})/\rho - \varepsilon_{ikm}\omega_k \tag{42}$$

We now simplify this expression by choosing a coordinate frame aligned with the *A, D* and *S* axes defined in Fig. 3. We want to evaluate the derivatives at the image of *F*, where $Q_D = Q_S = 0$ and $Q_A = 1$. We also have $p_S = 0$. From Fig 3b we have $1/\rho = (\sin G)/h$ and from Fig. 3b and Fig. 7 $p_D = -\cos G$. Eq. 42 then becomes

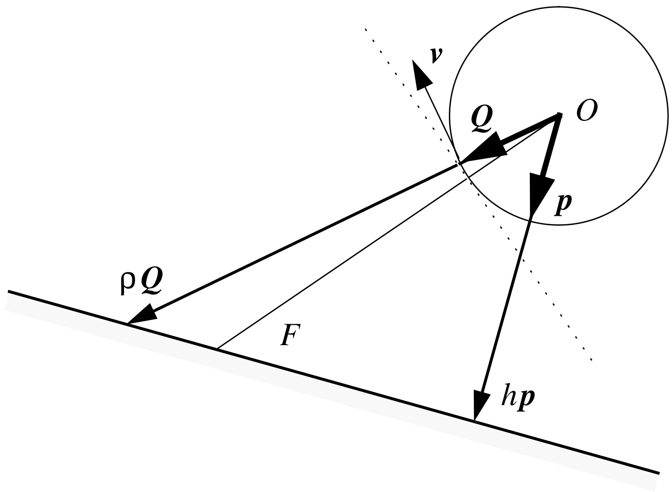


**Figure 7.** Spherical geometry for optic flow calculation. The vector to a point on the surface close to $F$ is $\rho\boldsymbol{Q}$. Its projection onto the unit sphere round $O$ is $\boldsymbol{Q}$. The sphere moves with the observer, and the motion of the tip of $\boldsymbol{Q}$ over its surface is described by the optic flow vector $\boldsymbol{v}$. The dotted line represents the position of the planar image surface for a camera looking along $OF$. For points close to the line of sight, $\boldsymbol{v}$ will be a good approximation to the flow of an image projected onto this plane. The unit vector along the normal to the plane is $\boldsymbol{p}$ and the perpendicular distance from $O$ to the plane is $h$.

$$R = \frac{1}{2}(v_{y,x} - v_{x,y}) \tag{33}$$

$$\tan 2\theta = \frac{v_{x,y} + v_{y,x}}{v_{x,x} - v_{y,y}} \tag{34}$$

There are two distinct solutions for $\theta$, at 90° to each other, corresponding to the expansion and contraction axes. The expansion axis is given by the solution with $|\theta| \leq 90°$ and $\text{sign}(\theta) = \text{sign}(v_{x,y} + v_{y,x})$ .

*The optic flow equation*

We now need the equation for the optic flow vector associated with a point on the surface. We denote by $\boldsymbol{Q}$ a unit vector from $O$, the point of observation, towards some point on the surface. We will write the distance from $O$ to the point as $\rho$, so that the position of the point relative to $O$ is given by $\rho\boldsymbol{Q}$. We define the optic flow vector associated with the point to be

$$\boldsymbol{v} = \frac{\partial \boldsymbol{Q}}{\partial t} \tag{35}$$

These vectors are shown in Fig. 7. The vector $\boldsymbol{Q}$ is the position of the image of the surface point onto a spherical image surface of unit radius round $O$, so $\boldsymbol{v}$ is the velocity vector for that image point. Although most practical image surfaces are planar, in a small

# 6 Derivation of the first-order flow equations

The equations for first-order optic flow have been derived in a variety of ways by different authors (e.g. Waxman & Wohn, 1988). It is possible to use a variety of formalisms. The following derivation aims to be succinct but reasonably complete and clear.

*The Taylor series for optic flow*

First, we need to make the connection between the first-order flow variables, as presented in Section 2, and the derivatives of optic flow. It will be convenient to write Eqs. 4 to 6 as a matrix equation:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} D+S & -R \\ R & D-S \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{28}$$

or

$$\boldsymbol{v} = \mathbf{T}\boldsymbol{r} \tag{29}$$

where $\boldsymbol{v}$ is the differential optic flow vector, $\boldsymbol{r}$ is the position vector in the image, and $\mathbf{T}$, the product of the three $2 \times 2$ matrices shown in Eq. 28, is the first-order flow deformation tensor.

We can express any optic flow field $\boldsymbol{v}(\boldsymbol{r})$ as a Taylor expansion about the origin. This looks like:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_{x0} \\ v_{y0} \end{bmatrix} + \begin{vmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} \end{vmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + O(r^2) \tag{30}$$

where the first vector on the right is the flow at the origin. We have taken this to be nulled by tracking. We ignore the nonlinear last term on the assumption (that can be justified) that it will be small for low-curvature surfaces and small surface patches. Comparing Eq. 28 with Eq. 30, we can now identify the elements of $\mathbf{T}$ with the partial derivatives of $\boldsymbol{v}$.

In order to leave the calculus out of the early development, we initially presented the first-order flow in terms of linear functions of the coordinates, and have now identified it with the first derivatives of the flow field. In fact, the usual approach is to *define* the first-order flow in terms of Eq. 30. This is more general, in that the first-order flow is then defined for all points in a flow-field, and not just around the fixed point.

Writing out the elements of $\mathbf{T}$ explicitly allows us to express the first-order flow parameters in terms of the derivatives of flow. Writing $v_{x,y}$ for $\partial v_x / \partial y$, etc. it is fairly straightforward to obtain

$$D = \frac{1}{2}(v_{x,x} + v_{y,y}) \tag{31}$$

$$S = \frac{1}{2}\sqrt{(v_{x,x} - v_{y,y})^2 + (v_{x,y} + v_{y,x})^2} \tag{32}$$

If we can find a way of choosing one of the solutions, then we have available the same information as in the case of known tilt, and the same remarks about its possible adequacy for control of action apply.

*Given spin*

The spin may also be known or tightly constrained. Cyclotorsional movements of the eyes are generally small in both animals and robots. Rotation of the head about the line of sight can occur — for example, as mentioned earlier, if you fixate the ground just to the right of your feet as you walk forward, your head rotates clockwise about the line of sight — but the rate will be small for most fixation points, and there may be mechanical information for it.

Consider the case where $\omega_A$ is zero. Eqs. 12 and 11, together with Fig. 4b give us

$$\sin\phi = V_S/V_T = R/S \tag{26}$$

Again, there is a twofold ambiguity in $\phi$, and in particular we do not know the sign of $\cos\phi$. The immediacy of $F$ therefore has two possible values given by $D \pm S\cos\phi$, and the immediacy of the plane has two values given by $D \pm 3S\cos\phi$. Combined with the twofold ambiguity in $\theta$, we get four values for the transverse velocity, this time given by

$$\angle \boldsymbol{V_T} = \theta + \begin{matrix} 0 \\ \pi \end{matrix} + \begin{matrix} \phi/2 \\ \pi/2 - \phi/2 \end{matrix} \tag{27}$$

and a corresponding four values for the tilt $\angle \boldsymbol{n_D}$. The combinations are shown in Fig. 6b. Again, we may be able to disambiguate the solution with only very approximate information from another source, and end up with all the same information as in the previous two cases.

*General solution for a single surface patch*

The preceding sections outline how specific kinds of additional information can be used to solve the first-order flow equations. In practice, it is likely that various kinds of extra information will be available, but with varying degrees of accuracy. The flow equations Eqs. 10 to 13 will then be best regarded as *constraints* in a more general scheme for finding structure and motion, or for direct control of action, which takes account of the accuracies of all the different kinds of information.

*Consistency in time and space*

Finally, a real vision system will pick up information relating to many surfaces in its environment, and will be able to integrate that information over some extended period of time. The velocity and orientation information obtained for any surface patch will have to be consistent both with other parts of the image and with earlier and later times. This consistency requirement probably provides not only the most powerful way of disambiguating the solutions for known perpendicular velocity or spin, but also a way to improve the accuracy and stability of the estimates of the structure and motion parameters. This introduces a trade-off between the time and amount of computation and the accuracy of the information obtained. This extension is beyond the scope of this paper.

which gives us two possible values for $\phi$, one equal to minus the other. The immediacy of $F$ is specified uniquely, from Eq. 22, but there are now two possible values for spin, given by $R \pm S \sin\phi$. There are four possible directions for the transverse velocity: from Eqs. 13 and 20 we find $\angle \boldsymbol{V_T} = \theta - \phi/2$, but as either of the two directions along the expansion axis is possible, and we have uncertainty in the sign of $\phi$, we can only state that

$$\angle \boldsymbol{V_T} = \theta + \begin{matrix} 0 \\ \pi \end{matrix} + \begin{matrix} \phi/2 \\ -\phi/2 \end{matrix} \quad (25)$$

where $\phi$ in the equation represents one of the possible values, the quantities stacked vertically are alternatives, and all four combinations are allowed. (The angle $\pi$ is 180°.) There are four corresponding directions for the tilt, with $+\phi/2$ replaced by $-\phi/2$ and *vice versa*. The possible directions for tilt and transverse velocity given the perpendicular velocity are shown in Fig. 6a.

The ambiguity may well be resolved by other information, which need only be approximate. The spin might be well enough controlled that only one of the two values for $\phi$ is acceptable, given $R$, or there might be enough information about the tilt or the direction of the transverse velocity to rule out some of the possible combinations in Eq. 25.

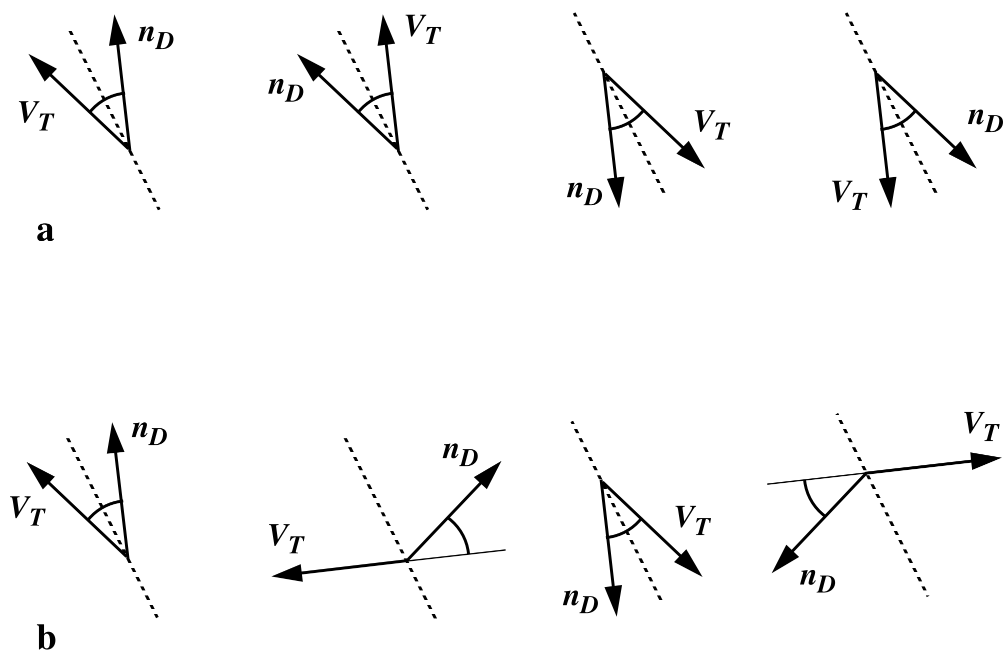


**Figure 6.** The fourfold ambiguities in tilt and transverse velocity direction. The dotted line is the shear expansion axis and the marked angle shows the value found for $\phi$ assuming $|\phi| < \pi/2$. **(a)** Perpendicular velocity $V_P$ is known. **(b)** Spin $\omega_A$ is known.

*Given tilt*

It may be the case that we can estimate the direction of the projection of the surface normal, $\angle \boldsymbol{n_D}$. This is called the *tilt* of the surface patch. For example, it might be known from the texture gradient, from the direction of the image of a linear object perpendicular to the surface, or from mechanical information. When the surface is a horizontal ground surface, the last two possibilities are very plausible: the observer may be able to see a vertical object or detect the direction of gravity. Using the axis of expansion, we can then deduce the direction of the transverse velocity, and hence the immediacy of $F$ and the spin.

That is, from Eq. 13

$$\angle \boldsymbol{V_T} = 2\theta - \angle \boldsymbol{n_D} \tag{19}$$

and from this and Fig. 4b

$$\phi = \angle \boldsymbol{V_T} - \angle \boldsymbol{n_D} = 2(\theta - \angle \boldsymbol{n_D}) \tag{20}$$

We know $\theta$ from the flow and $\angle \boldsymbol{n_D}$ from other information, so we know $\angle \boldsymbol{V_T}$ and $\phi$. (The equations refer only to $2\theta$, so it does not matter that we arbitrarily chose one end of the expansion axis to define $\theta$ in Section 2.) Note that $\phi$ can be negative. From Fig. 4b

$$V_D = V_T \cos\phi \qquad V_S = V_T \sin\phi \tag{21}$$

so from Eqs. 10 to 12

$$1/\tau_F = D - S\cos\phi \tag{22}$$

$$\omega_A = R - S\sin\phi \tag{23}$$

Using Eq. 17 we can also find the immediacy of the plane:

$$1/\tau_P = D - 3S\cos\phi \tag{24}$$

We now have available a good deal of information relevant to the control of locomotion. Even though we do not know the slant, we have the times to nearest approach both to the fixated point and to the plane of the surface, and we know the direction of the projection of our velocity onto the image, so we can predict our future course relative to visible features. In fact, we know everything about our motion and the surface apart from a scale factor in depth.

If we happen to know the surface slant, perhaps from the texture gradient, or because the surface is the ground plane and we know where the horizon is, then we can go further and determine the height-scaled dip, strike and approach velocities. It is by no means certain, however, that these are necessary for control of many actions: the immediacies and the direction of $V_T$ provide a great deal of predictive information as they stand.

*Given plane immediacy*

If the fixated point is on the ground surface, then plane immediacy will often be mechanically specified. Indeed, $V_P$ will normally be zero. From Eq. 24 we can then obtain $\cos\phi$,

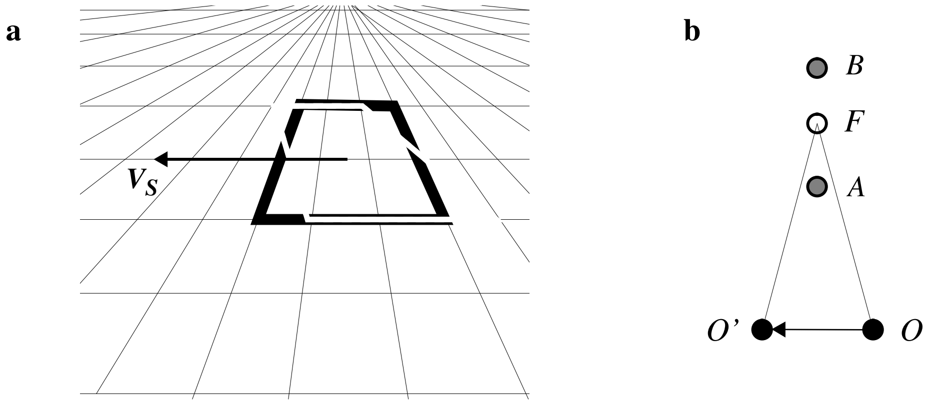


**Figure 5.** The effect of strike motion. **(a)** The image for an observer moving from right to left and fixating the centre of the outlined rectangle. The shape of the image patch will deform approximately to the shape shown by the dotted outline, so that the flow resembles that of Fig. 1e. There is stretching along the top-left to bottom-right direction and contraction along the bottom-left to top-right direction, and an overall anticlockwise rotation. **(b)** The way this flow arises can be partly understood by looking at a plan view of the scene. As the observer moves from *O* to *O'*, point *A* moves from the left of the line of sight *OF* to the right of it, whilst point *B* moves from the right of *OF* to the left.

From Eq. 9, $|V_D| \le V_T$. It follows from Eqs. 10 and 11 that

$$D - S \le 1/\tau_F \le D + S \tag{15}$$

That is, we have a constraint on the immediacy of *F*, and if *S* happens to be small compared to *D*, we will have a good estimate of it. Shear is small compared to dilations whenever the slant is small or the transverse velocity is small compared to the approach velocity.

Similarly:

$$R - S \le \omega_A \le R + S \tag{16}$$

so there is a limit on the spin.

We can also find a constraint on the immediacy of the plane. This is the inverse of the estimated time to pass through the plane tangent to the surface patch — the ground plane in the examples (and is quite different from the immediacy of *F*). It is equal to the *perpendicular* component of velocity $V_P$, divided by *h*: we have $1/\tau_P = V_P/h$. From Fig. 3b:

$$V_P = V_A \sin G - V_D \cos G \tag{17}$$

and it follows from Eqs. 10 and Eqs. 11 that

$$D - 3S \le 1/\tau_P \le D + 3S \tag{18}$$

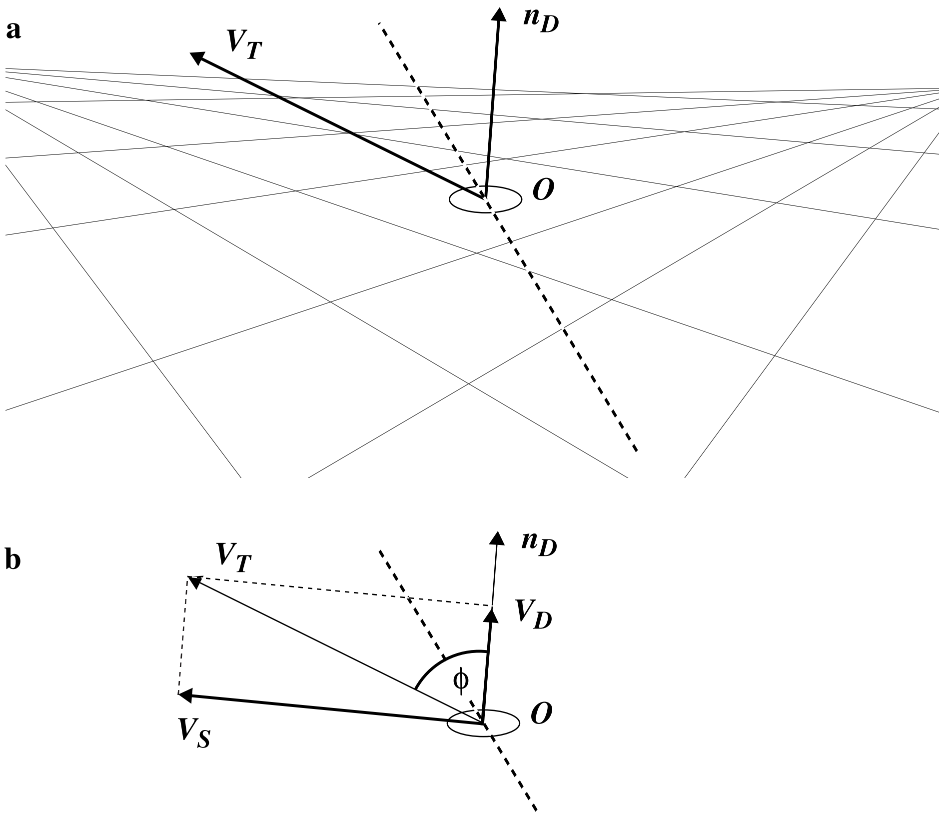


**Figure 4.** Relationships involving the axis of expansion. **(a)** The figure represents part of the image plane, seen from behind.The axis of expansion is shown by the dotted line, which bisects the angle between $\boldsymbol{n_D}$ and $\boldsymbol{V_T}$. Both of these vectors lie in the plane of the image. Note that $\boldsymbol{n_D}$ is in the direction of the image of a vertical rod with its foot at *F*, and $\boldsymbol{V_T}$ is in the direction of the image of the future path of the observer. **(b)** The same diagram with dip and strike components of velocity shown, and the bisected angle $\phi$ marked.

to obtain expressions for $G$ and the velocity components in terms of the flow components. In particular, note that $V_S$ and $V_D$ always appear in combination with cos $G$, so a change in the slant can always be exactly balanced by a change in the transverse velocity, as far as first-order flow is concerned.

Nonetheless, first-order flow clearly carries a great deal of information. Even on its own, the flow for a single patch places strong constraints on the surface slant and tilt and the observer's motion. If some additional information is available, constraining one or more of the quantities involved, a great deal can be inferred. We now explore these possibilities.

*No extra information*

Since cos $G$ lies between 0 and +1, we have a lower limit on the transverse velocity from Eq. 11:

$$V_T / h \geq 2S \tag{14}$$

# 4 First-order flow from structure and motion

The equations linking first-order flow to the orientation of the surface and the observer's motion are quite simple. We assume that the image surface is perpendicular to the line of sight, that it is roughly planar close to the fixed point, and that the image is formed by an ordinary optical system approximating to a pinhole, but without inversion. We then have:

$$D = \frac{V_A}{h}\sin G + \frac{V_D}{2h}\cos G \tag{10}$$

$$S = \frac{V_T}{2h}\cos G \qquad \text{or} \qquad S^2 = (V_S^2 + V_D^2)\,(\cos G/2h)^2 \tag{11}$$

$$R = \omega_A + \frac{V_S}{2h}\cos G \tag{12}$$

$$\theta = \frac{1}{2}(\angle \boldsymbol{V_T} + \angle \boldsymbol{n_D}) \tag{13}$$

In Eq. 13, $\angle \boldsymbol{n}$ means the angle between the $x$-axis and the vector $\boldsymbol{n}$, measured anticlockwise from the $x$-axis. A derivation of the equations is deferred until Section 6.

Let us examine the form of these equations. First, note that all the velocities are divided by $h$, so that $h$ provides, in effect, a distance scale for measuring speed. The dimension of every term in Eqs. 10 to 12 is inverse time.

We can see from Fig. 3b that $(V_A/h)\sin G$ is equal to $V_A/\rho$, or $1/\tau_F$, where $\tau_F$ is the "time-to-contact" — in fact, the time to nearest approach to $F$ if the observer keeps moving with its current velocity. We will call $1/\tau_F$ the *immediacy* of $F$. We can thus say:

Dilation = Immediacy + Dip Term

Shear squared = Strike Term + Dip Term

Rotation = Spin + Strike Term

where the strike and dip terms include the effect of cos $G$, the slant of the surface at $F$. The relationship for $\theta$ is elaborated in Fig. 4.

The immediacy contribution to the dilation is simply an expression of the fact that the image of an approaching object grows. The dip contribution expresses the stretching that occurs as a result of changing foreshortening: as the observer moves along the $D$ axis towards the surface normal through $F$, its view becomes more "straight down", and so the area of a surface patch round $F$ grows. This effect increases with the slant.

The foreshortening (or forelengthening) effect of dip velocity produces an extension of the image along the $D$ axis, but there is no corresponding contraction along the $S$ axis, so the flow due to dip is like that shown in Fig. 1c. Thus the dip velocity contributes to the shear as well.

Strike motion affects both shear and rotation. Figure 5 explains how this arises.

# 5 Structure and motion from first-order flow

How can an observer use first-order flow? Clearly, the flow does not, on its own, specify the surface orientation and the observer's motion: it is not possible to solve the equations

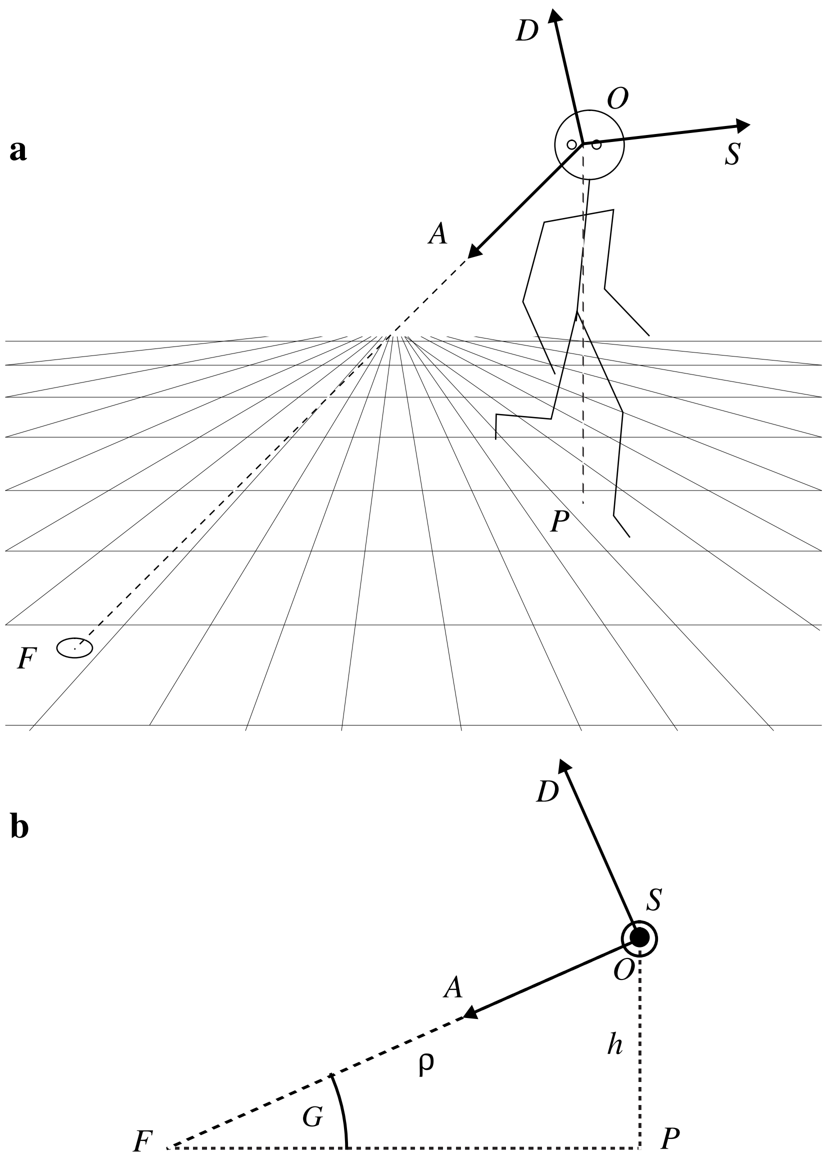


**Figure 3.** Coordinate frame for representing observer velocity. **(a)** The point of observation is at *O* and the point *F* on the ground is fixated. *P* is in the ground plane and the line *OP* is perpendicular to this plane. The three orthogonal axes are indicated: *A* is along the line of sight; *D* is perpendicular to this and in the plane of the triangle *OPF*; *S* is perpendicular to the other two. **(b)** Drawing the plane of the triangle *OPF* shows the directions of the axes more clearly; *S* points perpendicularly out from the page. The grazing angle of the line of sight to the plane, *G*, the height *h* and the depth ρ are also shown. Fig. 4b shows the axes in the *D–S* plane.

example, if you fixate a point on the ground just to the right of your feet as you walk along, your head and eyes will rotate clockwise about your line of sight. We denote the rate of rotation about the line of sight $\omega_A$, and refer to it as the *spin*. It is positive for a clockwise rotation of the eye, looking from behind.

We are now in a position to state the central relationships.

## 3 Approach, dip and strike

We now return to an observer moving relative to a planar surface, and fixating a point on it. To make the example more concrete, we will take the surface to be the ground surface. The optic flow depends on the instantaneous velocity of the observer's eye, $\boldsymbol{V}$. There are various ways to represent this vector, but it will be convenient to use its components along the *A, D* and *S* axes shown in Fig. 3. We make no use at this stage of the fact that motion is normally parallel to the ground plane, as we want the theory to apply to any surface.

The components of velocity along these axes are designated $V_A$, $V_D$ and $V_s$, and will be referred to as the *approach* velocity, the *dip* velocity and the *strike* velocity respectively. Approach is the component along the line of sight. Approach and dip lie in a plane perpendicular to the surface, whilst strike is perpendicular to that plane and parallel to the surface. Dip is positive away from the surface, and strike is positive for motion to the left of the observer's line of sight, as seen by the observer. The terms dip and strike are borrowed from geology, since there do not seem to be convenient words already in use for these components of motion. The decomposition into $V_D$ and $V_S$ is not well-defined if the observer is looking at its feet so that *P* and *F* coincide. If we write $\boldsymbol{n_A}$, $\boldsymbol{n_D}$ and $\boldsymbol{n_S}$ for the unit vectors along the *A, D* and *S* axes, the observer's velocity can be expressed as

$$\boldsymbol{V} = V_A \boldsymbol{n_A} + V_D \boldsymbol{n_D} + V_S \boldsymbol{n_S} \tag{7}$$

We will also refer to the *transverse* velocity $\boldsymbol{V_T}$ given by

$$\boldsymbol{V_T} = V_D \boldsymbol{n_D} + V_S \boldsymbol{n_S} \tag{8}$$

This is just the component of velocity at right-angles to the line of sight. Since dip and strike are orthogonal, the magnitude of the transverse velocity is

$$V_T = \sqrt{V_D^2 + V_S^2} \tag{9}$$

To get an idea of what the components mean, consider fixating different points on the ground plane. If you fixate a point on your path in front of you, $V_A$ and $V_D$ will be positive, but $V_S$ will be zero. If this point is far away (compared to your height), then $V_A$ will be large compared to $V_D$, and *vice versa* if it is close to you. If you fixate a point directly to your right, $V_S$ will be positive and $V_A$ and $V_D$ will be zero.

Although the ground plane has been used in order to give a definite example, the *ADS* coordinate system can be set up with reference to any planar surface in the environment, even if it is only a small patch. The point *P* might not then lie on the real surface, but it will still be the foot of the normal from *O* to the plane. For curved surfaces, we use the tangent plane of the surface at the fixated point. We will measure the *slant* of the surface using the grazing angle *G* between the line of sight *OF* and its perpendicular projection onto the surface *PF*. *G* is in the range $0 < G \leq 90°$. We will use *h* to stand for the distance $\overline{OP}$ and ρ to stand for the distance $\overline{OF}$.

It is also necessary to say something about the eye's rotational movement. Although our definition of differential optic flow requires the eye to turn so as to keep the image of the fixated point static, the eye remains free to rotate about the line of sight. If the head is not rotating, then this motion is cyclotorsion, which is normally small in people; but it is possible for the head itself to have a component of rotation about the line of sight. For where $R$ is the rate of rotation. The formulae are illustrated in Fig. 2c. As with dilation, these equations remain true however the axes are oriented. Positive and negative values of $R$ correspond to anticlockwise and clockwise rotation respectively.

A patch dragged along by a pure rotation clearly suffers no change in shape or in area. A combination of equal parts of rotation and shear produces the kind of parallel or lamellar flow pattern shown in Fig. 1e.

*General first-order flow*

General first-order flow is formed by simply adding the velocities from the three components. This gives

$$\begin{aligned} v_p &= (D+S)\,p - Rq \\ v_q &= (D-S)\,q + Rp \end{aligned} \tag{4}$$

However, we will not normally be able to work in a coordinate system conveniently aligned with the shear axes, as the *(p, q)* system is. Suppose that we wish to work in an *(x, y)* coordinate system with its origin at the fixed point, and that the axis of expansion is at an angle θ to the *x*-axis. We restrict θ to the range $|\theta| \le 90°$, with θ positive anticlockwise from the *x*-axis. (In Fig. 1b, if *x* runs from left to right and *y* runs from bottom to top in the conventional way, θ is +75˚.) We can calculate the flow for a point at *(x, y)* by converting to *(p, q)* with the standard equations for coordinate rotation:

$$\begin{aligned} p &= \phantom{-}x\cos\theta + y\sin\theta \\ q &= -x\sin\theta + y\cos\theta \end{aligned} \tag{5}$$

We then apply Eq. 4, and convert the results back to *(x, y)* velocity components with

$$\begin{aligned} v_x &= v_p\cos\theta - v_q\sin\theta \\ v_y &= v_p\sin\theta + v_q\cos\theta \end{aligned} \tag{6}$$

The general first-order flow is thus specified by four parameters, *D, S, R* and θ, relative to some fixed coordinate system in the image. An example of general flow, involving all the components, is given in Fig. 1f.

The decomposition of a first-order flow field into dilation, shear and rotation is unique: each set of values for *D, S, R* and θ gives a different flow field. In addition, any first-order flow (i.e. with velocity components that are linear functions of the image coordinates) can be expressed as a sum of dilation, shear and rotation.

The theory outlined here is intended to show how the different components of first-order flow can be defined, and what they mean in terms of image change. We have not discussed how they can be measured — that is a major research problem. In principle, if flow vectors are known at three non-collinear points, one of which may be the fixed point, then *D, S, R* and θ can be determined. In practice, such measurements are affected by noise, so it is helpful to measure more than three velocities. It may be better to measure the components directly, for example dilation from rate of change of area and rotation from change in orientation of linear features.

The geometrical interpretation of these equations is given in Fig. 2a. They hold good regardless of how the axes are oriented. The dilation rate $D$ can be positive or negative, corresponding to expansion or contraction respectively.

A patch of the image dragged along with a pure dilational flow suffers no change in shape, but its area changes at a rate of $2DA$, where $A$ is its current area.

*Shear*

Pure shear, or deformation, is the simplest kind of local change of shape. It involves an expansion along some axis and an equal contraction along the axis at right-angles. An example is shown in Fig. 1b.

To write down the equations for shear, assume that the $p$- and $q$-axes of our coordinate system lie along the expansion and contraction axes of the shear respectively. The equations for the velocity are

$$v_p = Sp \qquad v_q = -Sq \tag{2}$$

where $S$ is the rate of shear, and is always positive. The equations are illustrated in Fig. 2b.

A patch of image dragged along with a pure shear suffers no change in area, but is squeezed along one axis and stretched along another. A combination of equal rates of dilation and shear produces an expansion or contraction in one direction, and no change in the orthogonal direction, as in Fig. 1c.

*Rotation*

Rotation is just what one would expect. An example is shown in Fig. 1d. Each image point moves at right angles to the line joining it to the fixed point. The equations are

$$v_p = -Rq \qquad v_q = Rp \tag{3}$$

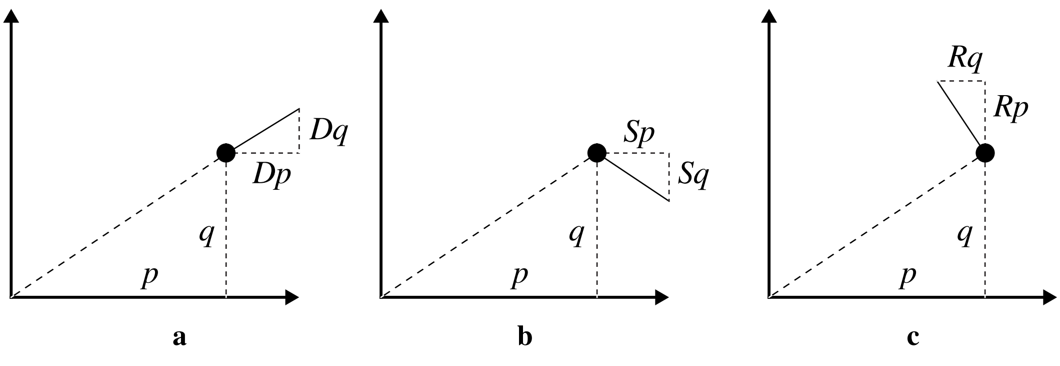


**Figure 2.** The geometry of first-order flow, illustrating Eqs. 1 – 3. The components of the differential optic flow vector for an image point at *(p, q)* are shown for **(a)** dilation, **(b)** shear, **(c)** rotation. Note how the signs of the components in the formulae translate into directions in the diagram. Similar triangles show how the dilation formula gives radial flow, whilst the rotation formula gives tangential flow. In b, the axis of expansion is horizontal; the diagram has to be rotated anticlockwise by 75° to make it correspond to Fig. 1b.

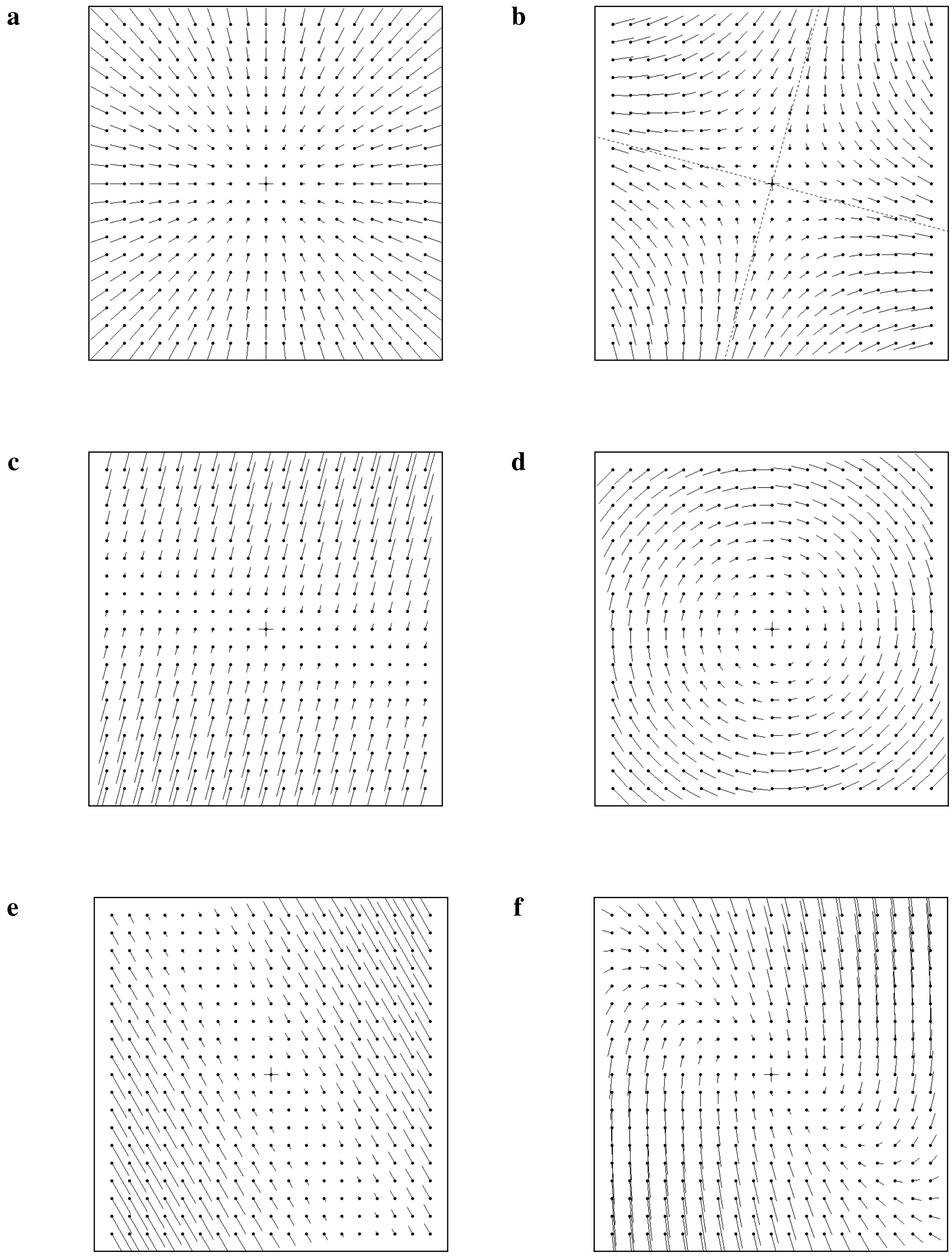


**Figure 1.** Examples of first-order flow fields. The image velocity for each point marked by a dot is shown by the length and direction of the attached line. The fixed point is shown by a cross. **(a)** Pure dilation. **(b)** Pure shear. The axes of expansion and contraction are shown by the dotted lines. **(c)** A combination of equal rates of dilation and shear. **(d)** Pure rotation. **(e)** A combination of equal rates of rotation and shear. **(f)** A general flow: the combination of a, b and d.

# 1 Introduction

Walking or driving down a street, you fixate a point on some nearby surface — the pavement or a wall, say. The image of the fixated point is kept static on your retina, but the image of the patch of surface round it changes as you move along, expanding or contracting, deforming, and rotating. This change is the differential optic flow, and it carries information about your motion and about the slant, tilt and shape of the patch of wall or pavement that you are looking at. For example, uniform expansion of the patch is related to approach to the surface.

Differential optic flow is potentially useful to a visual system even without knowledge of how the eye or camera is rotating. Ordinary (zero-order) optic flow is strongly affected by eye rotations, and it can be difficult to allow for these in order to obtain information about structure and motion. For example, the optic flow for the fixated point is zero, so carries no information unless the head and eye rotations producing fixation are known.

The relationships between differential optic flow, motion and surface shape have been derived by Koenderink & van Doorn (1975, 1992) and others (e.g. Waxman & Wohn, 1988), but relatively little use has been made of them. (For an exception, see Cipolla & Blake, 1992.) The present paper has three purposes. First, a review of first-order flow and its relationship to planar structure and observer motion is given, with a minimum of mathematical formalism, in Sections 2 to 4. Second, a summary of the information that can be obtained from the first-order flow for a single planar surface patch is given in Section 5. Third, for those interested in the derivation of the equations for first-order flow, the details are given in Section 6.

Differential optic flow can be defined for any point in the image, but is most easily envisaged by considering image motion relative to the image of the fixated point. (This will be referred to as the *fixed point*.) This paper is only concerned with first-order flow, which is the part of the differential flow which can be described as a linear function of image coordinates. It is possible to relate second-order flow, which is quadratic in the image coordinates, to the curvature of the surface, but this is not done here. If the surface patch is smooth and its curvature small, the differential flow in some small region round the fixed point can be closely approximated by first-order flow.

# 2 Dilation, Shear and Rotation

This section presents the components of first-order flow: dilation, shear and rotation, shown graphically in Fig. 1. The next section sets up a particular representation of the velocity of a moving point of observation relative to a planar surface patch, and the relationship between first-order flow, structure and motion is presented thereafter.

*Dilation*

Dilation describes uniform expansion or contraction. An example is shown in Fig. 1a.

In pure dilation, each image point moves along the line joining it to the fixed point, with a speed proportional to its distance from the fixed point. This can be written very simply in terms of image coordinates. Suppose Cartesian coordinates $(p, q)$ are set up in the image plane with the origin at the fixed point. Then if the rate of dilation is $D$, a point at $(p, q)$ will have velocity components along the axes given by

$$v_p = Dp \qquad v_q = Dq \tag{1}$$

# First-Order Optic Flow

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CSRP no. 313
February 1994