Calculus in coinductive form

D. Pavlović*　　M.H. Escardó†

Abstract

Coinduction is often seen as a way of implementing infinite objects [7, 4]. Since real numbers are typical infinite objects, it may not come as a surprise that calculus, when presented in a suitable way, is permeated by coinductive reasoning. What *is* surprising is that mathematical techniques, recently developed in the context of computer science, seem to be shedding a new light on some basic methods of calculus.

We introduce a coinductive formalization of elementary calculus that can be used as a tool for symbolic computation, and geared towards computer algebra and theorem proving. So far, we have covered ordinary differential and difference equations, Taylor series, Laplace transform and the basics of operator calculus.

Introduction

Our point of departure is the observation that the algebraic structure of streams, given by the equations

$$\mathrm{head}(a :: \beta) = a \quad (1)$$

$$\mathrm{tail}(a :: \beta) = \beta \quad (2)$$

$$\mathrm{head}(\alpha) :: \mathrm{tail}(\alpha) = \alpha \quad (3)$$

captures much of calculus. Given an analytic function f, define

$$\mathrm{head}(f) = f(0)$$

$$\mathrm{tail}(f) = f'$$

$$a :: f = \left(x \mapsto a + \int_0^x f\right)$$

Equation (3) now expresses the so-called *Fundamental Theorem of Calculus*, whereas equations (2) and (1) normalize the integral with respect to the subintegral function and the interval of integration.[1]

Reapplying equation (3) yields the Taylor (Maclaurin) expansion

$$\begin{aligned} f &= f(0) :: f' \\ &= f(0) :: f'(0) :: f'' \\ &\vdots \\ &= f(0) :: f'(0) :: \cdots :: f^{(n)}(0) :: \cdots \end{aligned}$$

Unfolding the above definition of ::, which amounts to iterated integration, one finally gets

$$f(x) = f(0) + f'(0)x + \cdots + f^{(n)}(0)x^n/n! + \cdots$$

The idea of infinitely applying (3) is formally captured by the notion of a stream (co)algebra. The set of infinite sequences forms a final stream coalgebra. Taylor expansions are then obtained using the unique homomorphism from the stream coalgebra of analytic functions.

From another point of view, $a :: f$ is the unique solution of the differential equation $g' = f$ with the initial value $g(0) = a$. The above derivation of Taylor series now leads to the usual power series method for solving differential equations [2, ch. 4]. For example, the equation $f^{(4)} = f$, with initial values $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$ and $f'''(0) = -1$, becomes $f = 0 :: 1 :: 0 :: -1 :: f$. Solving it amounts to running a corecursive program, which outputs the stream of Taylor coefficients (corresponding, in this case, to $f = \sin$).

*COGS, University of Sussex, Brighton, and Kestrel Institute, Palo Alto. E-mail: duskop@cogs.susx.ac.uk *(contact author)*

†Department of Computer Science, University of Edinburgh, Edinburgh. E-mail: mhe@dcs.ed.ac.uk

[1] This example may be suggested by Hoare's notation α_0 and α', respectively for the head and the tail of a trace [5].

Following these ideas, we introduce in sections 1 and 2 a formal setting for studying and implementing analytic structures by coalgebraic methods. Section 3 proceeds from our stream algebras of Taylor coefficients to derive an abstract characterisation of a different analytic method: Laplace transform. We show that it also arises, like Taylor series, as a coalgebra homomorphism induced by specific stream operations. We compute them and derive the corresponding integral expressions. As a byproduct of the coalgebraic treatment, we obtain a simple characterisation of the Laplace duals of analytic functions.

1 Stream algebras

Our main tool are the fixpoints of functors in the form $\Sigma \times (-) : \mathsf{Set} \to \mathsf{Set}$.

Definition 1.1 *Let Σ be a set. A* Σ-stream algebra *is a set A together with an isomorphism*

$$\Sigma \times A \underset{\langle h,t \rangle}{\overset{c}{\rightleftarrows}} A \qquad (\cong)$$

With $c(a,\beta)$ written in the infix form $a :: \beta$, this isomorphism exactly corresponds to equations (1–3).

The *stream homomorphisms* are required to preserve all three operations. (In fact, it suffices to require the preservation of c alone, or of h and t.)

1.1 Infinite lists

The basic example of a stream algebra is, of course, the set $A = \Sigma^\omega$ of infinite lists of elements from Σ. If $\alpha \in A$ is a list $[\alpha_0, \alpha_1, \alpha_2, \ldots]$, the operations will take it to

$$\begin{array}{rrccl} h = & \text{head}: & \Sigma^\omega & \longrightarrow & \Sigma \\ & & \alpha & \longmapsto & \alpha_0 \\ \\ t = & \text{tail}: & \Sigma^\omega & \longrightarrow & \Sigma^\omega \\ & & \alpha & \longmapsto & [\alpha_1, \alpha_2, \ldots] \\ \\ c = & \text{cons}: & \Sigma \times \Sigma^\omega & \longrightarrow & \Sigma^\omega \\ & & \langle a, \beta \rangle & \longmapsto & [a, \beta_0, \beta_1, \ldots] \end{array} \qquad (4)$$

With the pair $\langle \text{head}, \text{tail} \rangle$ as the structure map, Σ^ω is the final coalgebra for the functor $\Sigma \times (-) : \mathsf{Set} \to \mathsf{Set}$. The initial algebra is empty. The finality of Σ^ω means that every Σ-coalgebra $\langle k, s \rangle : A \longrightarrow \Sigma \times A$ induces a unique function

$$\begin{array}{rcl} \tau_{ks} : A & \longrightarrow & \Sigma^\omega \\ x & \longmapsto & [k(x),\ ks(x),\ ks^2(x), \ldots] \end{array}$$

making the diagram

$$\begin{array}{ccc} A & \xrightarrow{\langle k,s \rangle} & \Sigma \times A \\ {\scriptstyle \tau_{ks}} \downarrow & & \downarrow {\scriptstyle \Sigma \times \tau_{ks}} \\ \Sigma^\omega & \xrightarrow[\langle \text{head},\text{tail} \rangle]{} & \Sigma \times \Sigma^\omega \end{array}$$

commute. Most of the time, we shall be constructing $\Sigma\times$-colagebras and studying effects of the induced homomorphisms τ.

1.2 Sequences

Now suppose Σ is a group, say $\mathbb{Z}$, the integers. Besides the described list operations, the set $A = \mathbb{Z}^\omega$ supports various sequence operations, e.g.

$$\begin{array}{rccl} h: & \mathbb{Z}^\omega & \longrightarrow & \mathbb{Z} \\ & \alpha & \longmapsto & O\alpha \ = \ \alpha_0 \\ \\ t: & \mathbb{Z}^\omega & \longrightarrow & \mathbb{Z} \\ & \alpha & \longmapsto & \Delta\alpha \\ \\ c: & \mathbb{Z} \times \mathbb{Z}^\omega & \longrightarrow & \mathbb{Z}^\omega \\ & \langle a, \beta \rangle & \longmapsto & a + \Sigma\beta \end{array}$$

where

$$\begin{aligned} \Delta\alpha &= [\alpha_1 - \alpha_0,\ \alpha_2 - \alpha_1,\ \alpha_3 - \alpha_2, \ldots], \\ a + \Sigma\beta &= [a,\ a + \beta_0,\ a + \beta_0 + \beta_1, \ldots] \end{aligned}$$

Essentially employing the commutativity of $\mathbb{Z}$, one finds that this stream algebra structure on $\mathbb{Z}^\omega$ is actually isomorphic with (4), via

$$\begin{array}{ccc} \mathbb{Z}^\omega & \xrightarrow{\langle O,\Delta\rangle} & \mathbb{Z}\times\mathbb{Z}^\omega \\ \tau \big\downarrow \cong \big\uparrow \widetilde{\tau} & & \mathbb{Z}\times\tau \big\downarrow \cong \big\uparrow \mathbb{Z}\times\widetilde{\tau} \\ \mathbb{Z}^\omega & \xrightarrow[\langle \mathrm{head},\mathrm{tail}\rangle]{} & \mathbb{Z}\times\mathbb{Z}^\omega \end{array}$$

The n-th entries of the sequences $\tau(\alpha)$, resp. $\widetilde{\tau}(\alpha)$, are defined:

$$\begin{aligned} \tau(\alpha)_n &= \sum_{i=0}^{n} \binom{-n}{i} \alpha_i \\ \widetilde{\tau}(\alpha)_n &= \sum_{i=0}^{n} \binom{n}{i} \alpha_i \end{aligned}$$

Note that τ is actually the discrete Taylor transformation, because $\tau(\alpha)_n = (\Delta^n\alpha)_0$. The described isomorphism thus switches between a sequence of numbers and the sequence of its finite differences of finite orders, evaluated at 0.[2]

1.3 Analytic functions

Passing from integers $\mathbb{Z}$ to reals $\mathbb{R}$, and from the difference operator Δ to the derivative D, we get the stream algebra from the introduction, on the set $\mathbb{A}$ of functions analytic at 0.

$$\begin{array}{rlcl} h: & \mathbb{A} & \longrightarrow & \mathbb{R} \\ & f & \longmapsto & Of \;=\; f(0) \\[1ex] t: & \mathbb{A} & \longrightarrow & \mathbb{A} \\ & f & \longmapsto & Df \;=\; f' \\[1ex] c: & \mathbb{R}\times\mathbb{A} & \longrightarrow & \mathbb{A} \\ & \langle a, f\rangle & \longmapsto & a + \int_0^x f \end{array}$$

[2] On the other hand, extended to $\mathbb{R}^\omega$, τ and $\widetilde{\tau}$ can also be understood, along the lines of formula (9), as multiplying with the functions e^{-x} and e^x respectively.

As pointed out before, the essentials of calculus can be presented in terms of this algebra. Elementary functions arise as solutions of equations, e.g.

$$\begin{aligned} \exp &= 1 :: \exp \\ \sin &= 0 :: 1 :: 0 :: -1 :: \sin \\ \mathrm{ch} &= 1 :: 0 :: \mathrm{ch} \end{aligned}$$

In general, all initial value problems induce stream equations: e.g.

$$\begin{aligned} y'' &= y - 5\sin x + x^2 \\ y(0) &= 0,\ y'(0) = 3 \end{aligned} \tag{5}$$

corresponds to

$$y = 0 :: 3 :: (y - 5\sin x + x^2) \tag{6}$$

The point that we wish to make is that standard analytic methods of solving problems like (5) *conspicuously often* boil down to stream algebra manipulations with equations like (6). The upshot is that the procedures apparently based on the intuitions of continuum, and on resulting deply infinitistic concepts, hardly computational, can actually be formalized in terms of the familiar list operations, provided that circular and infinite lists — i.e. streams, are allowed.

So how do we deal with (6)?

2 Solving equations

2.1 Lifting the structure

In order to lift the real numbers into $\mathbb{A}$, we first define $\widehat{0} \in \mathbb{A}$ to be the unique solution of the equation

$$\widehat{0} = 0 :: \widehat{0}$$

Each real number $a \in \mathbb{R}$ can now be represented in $\mathbb{A}$ by the induced constant function $\widehat{a} = a :: \widehat{0}$. Since the mapping $\widehat{(-)} : \mathbb{R} \longrightarrow \mathbb{A}$ is injective, identifying $\widehat{a}$ and a should not cause confusion.

The variable x can now be defined as $0 :: 1$, x^2 is $0 :: 0 :: 2$, x^3 is $0 :: 0 :: 0 :: 6$ etc. Of course, x is just a way the identity function is usually denoted in calculus. Explaining x^n, however, requires a definiton of the multiplication of functions.

In general, the addition and the multiplication on $\mathbb{A}$ are determined by the systems

$$\begin{aligned} h(f+g) &= h(f)+h(g) \\ t(f+g) &= t(f)+t(g) \end{aligned}$$

and

$$\begin{aligned} h(f\cdot g) &= h(f)\cdot h(g) \\ t(f\cdot g) &= t(f)\cdot g + f\cdot t(g) \end{aligned}$$

Axioms of stream algebra (1–3) imply that these systems are respectively equivalent to

$$(a :: f) + (b :: g) = (a+b) :: (f+g) \quad (7)$$

and

$$(a :: f)\cdot(b :: g) = (a\cdot b) :: \Big(f\cdot(b :: g) + (a :: f)\cdot g\Big) \quad (8)$$

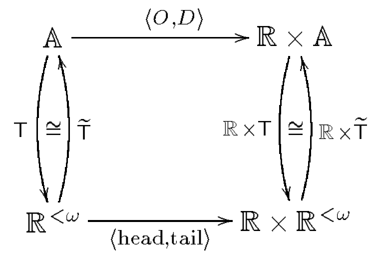
Note that these definitions are inductive only for the inductively defined f and g, i.e. those that are derived from the constants in a finite number of steps, not involving fixpoints. Such functions are, of course — the polynomials. For functions like exp, defined using fixpoints, the above definitions are circular — and resolved by fixpoints again:

$$\begin{aligned} \exp+\exp &= (1 :: \exp)+(1 :: \exp) = \\ &= (1+1) :: (\exp+\exp) = \\ &= 2 :: (\exp+\exp) \end{aligned}$$

But how do we know that there is a fixpoint? How do we know whether an equation like $y = 0 :: 3 :: (y - 5\sin x + x^2)$ has a solution, and when it is unique? A general answer to such questions is provided in [10].

2.2 Using Taylor series

To see things in a familiar setting, note that the Taylor representation again induces an isomorphism:



where $\mathbb{R}^{<\omega}$ is the set of sequences of Taylor coefficients, i.e. of $\alpha\in\mathbb{R}^\omega$ such that $\sum_{i=0}^{\infty}\frac{\alpha_i}{i!}x^i < \infty$ for some $x > 0$. The Taylor representation is, of course

$$\begin{aligned} \mathsf{T}\{f\} &= [f(0),\ f'(0),\ f''(0),\ldots] \\ \widetilde{\mathsf{T}}\{\alpha\} &= \sum_{i=0}^{\infty}\frac{\alpha_i}{i!}x^i \end{aligned}$$

Transferred along T, the stream algebra equations become the usual power series manipulations: the coefficients of the unknown function are determined recursively.

E.g., the T-image of the equation $y = 0 :: 3 :: (y - 5\sin x + x^2)$ will have $\mathsf{T}\{y\} = [y_0, y_1, y_2, \ldots]$ on the left-hand side; whereas the right-hand side will be the sum of the same $\mathsf{T}\{y\}$, with $\mathsf{T}\{-5\sin\} = [0,-5,0,5,\ldots]$ and $\mathsf{T}\{x^2\} = [0,0,2,0,\ldots]$, all of that prefixed with 0 and 3. The equation thus becomes:

$$\begin{array}{llllllll} [y_0, & y_1, & y_2, & y_3, & y_4, & y_5, & y_6, & \ldots] \; = \\ [0, & 3, & y_0, & y_1-5, & y_2+2, & y_3+5, & y_4, & \ldots] \end{array}$$

In this simple case, the coefficients can be extracted explicitly, and even eliminated by recognizing the elementary functions behind them

$$\begin{aligned} y &= [0,0,0,0,2,0,2,0,\ldots] + [0,3,0,-2,0,3,0,-2,\ldots] \\ &= 2\mathrm{ch}\ x - x^2 - 2 + \tfrac{1}{2}(\mathrm{sh}\ x + 5\sin x) \end{aligned}$$

In general, every first order initial value problem involving only analytic functions can be solved in this fashion [2, thms. 4.4–4.5], as well as many important higher order linear differential equations [3, ch. 10]. The recurrence relations on the coefficients tend to be tedious, though, and extracting the actual recursive formulas for them is not always feasible.

Other analytic methods are captured by different stream algebras and stream algebra homomorphisms between them.

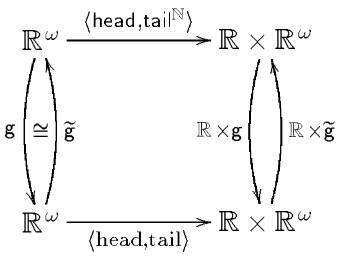
3 Laplace transform

3.1 Rings of streams

Now consider the algebra

$$\begin{array}{rlcl} \mathsf{head}: & \mathbb{R}^\omega & \longrightarrow & \mathbb{R} \\ & \alpha & \longmapsto & \alpha_0 \\[1ex] \mathsf{tail}^{\mathbb{N}}: & \mathbb{R}^\omega & \longrightarrow & \mathbb{R}^\omega \\ & \alpha & \longmapsto & [\alpha_1,\ 2\alpha_2,\ 3\alpha_3, \ldots] \\[1ex] \mathsf{cons}^{\mathbb{N}}: & \mathbb{R}\times\mathbb{R}^\omega & \longrightarrow & \mathbb{R}^\omega \\ & \langle a, \beta\rangle & \longmapsto & [a,\ \beta_0,\ \frac{\beta_1}{2},\ \frac{\beta_2}{3}, \ldots] \end{array}$$

This is yet another version of the stream algebra of infinite lists of numbers, isomorphic with the "original" via



where

$$\mathsf{g}(\alpha)_n = n!\alpha_n$$
$$\widetilde{\mathsf{g}}(\alpha)_n = \frac{\alpha_n}{n!}$$

Clearly, every sequence ν induces $\mathrm{tail}^\nu(\alpha) = [\nu_0\alpha_1,\ \nu_1\alpha_2,\ \nu_2\alpha_3, \ldots]$, and a stream algebra, provided that all $\nu_i \neq 0$. The importance of the algebra induced by $\nu = \mathbb{N}$ is that the composite $\mathsf{G} = \widetilde{\mathsf{T}} \circ \mathsf{g}$ assigns to a sequence α its *generating function*

$$\mathsf{G}\{\alpha\} = \sum_{i=0}^{\infty} \alpha_i x^i$$

Interpreted in terms of G, the operation $\mathrm{tail}^{\mathbb{N}}$ again corresponds to the derivation D, and $\mathrm{cons}^{\mathbb{N}}$ to the integration.

Transferred along T, the product of analytic functions f and g, with $\varphi = \mathsf{T}\{f\}$ and $\gamma = \mathsf{T}\{g\}$, induces the operation $\varphi \cdot \gamma = \mathsf{T}\{f \cdot g\}$ with the entries

$$(\varphi\cdot\gamma)_n = \sum_{i=0}^{n} \binom{n}{i} \varphi_i \gamma_{n-i} \tag{9}$$

Transferring, on the other hand, along $\widetilde{\mathsf{G}} = \widetilde{\mathsf{g}} \circ \mathsf{T}$,with $\alpha = \widetilde{\mathsf{G}}\{f\}$ and $\beta = \widetilde{\mathsf{G}}\{g\}$, we get $\alpha * \beta = \widetilde{\mathsf{G}}\{f \cdot g\}$, with

$$(\alpha * \beta)_n = \sum_{i=0}^{n} \alpha_i \beta_{n-i} \tag{10}$$

Both $\cdot$ and $*$ make $\mathbb{R}^\omega$ into a commutative monoid, even a ring, as they obviously distribute over the (componentwise) $+$. The isomorphism g switches between the two ring structures, and in particular satisfies

$$\mathsf{g}\{\alpha * \beta\} = \mathsf{g}\{\alpha\} \cdot \mathsf{g}\{\beta\}$$

On the other hand, for $x = [0, 1, 0, 0, \ldots]$ and any α holds

$$x * \alpha = [0,\ \alpha_0,\ \alpha_1,\ \alpha_2, \ldots] \tag{11}$$

But if α is a sequence of Taylor coefficients, prefixing by 0 corresponds to the integration! Integral can thus be presented as multiplication in a ring

$$\mathsf{T}\left\{\int_0^x f\right\} = x * \mathsf{T}\{f\}$$

It is not hard to see that this ring has no zero divisors, so that it can be extended into a field of fractions. The calculus of integrals and derivatives becomes algebra over this field.

This is, of course, the basic idea of operator calculus [8]. Indeed, $*$ as in (10) is a discrete convolution, and g reduces it to the multiplication, just like Laplace transform does with the continuous convolution.

But g does not correspond *precisely* to Laplace transform. An algebraic treatment of differential equations can be built upon it, but the induced formulas are different from those encountered in the Laplace tables.

We shall now proceed to modify g in order to capture the actual Laplace transform.

Lifting g from coefficients to functions in the same way as we shall do in a moment, the reader will find that g actually corresponds to what is sometimes called *Heavyside* transform: it maps f to the coefficients of $s\int_0^\infty e^{-st}f(t)dt$.

3.2 Laplace transform of Taylor coefficients

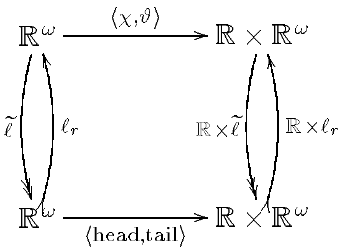
The crucial point about Laplace transform is that it is not an isomorphism, but a proper embedding of the convolution ring of real analytic functions into an *ideal* [6, 12] within the multiplicative ring of analytic (or better holomorphic) functions. We first spell this out on Taylor coefficients, and embed $\mathbb{R}^\omega$ into

$$\mathbb{R}_0^\omega \quad = \quad \{\alpha \in \mathbb{R}^\omega \mid \alpha_0 = 0\}$$

The embedding is realized using the following stream algebra:

$$\begin{array}{rlcl} \chi: & \mathbb{R}^\omega & \longrightarrow & \mathbb{R} \\ & \alpha & \longmapsto & \alpha_1 \\ \vartheta: & \mathbb{R}^\omega & \longrightarrow & \mathbb{R}^\omega \\ & \alpha & \longmapsto & [\alpha_0,\ \alpha_2,\ \frac{\alpha_3}{2},\ \frac{\alpha_4}{3}\ldots] \\ \varsigma: & \mathbb{R}\times\mathbb{R}^\omega & \longrightarrow & \mathbb{R}^\omega \\ & \langle a,\beta\rangle & \longmapsto & [\beta_0,\ a,\ \beta_1,\ 2\beta_2,\ 3\beta_3,\ldots] \end{array}$$

Since $\langle \mathsf{head}, \mathsf{tail}\rangle : \mathbb{R}^\omega \longrightarrow \mathbb{R}\times\mathbb{R}^\omega$ is the final $\mathbb{R}\times$-coalgebra, there is a unique homomorphism $\widetilde{\ell}$.



The sections ℓ_r of $\widetilde{\ell}$ are indexed by the real numbers r:

$$\widetilde{\ell}(\alpha)_n \quad = \quad \frac{\alpha_{n+1}}{n!} \tag{12}$$

$$\ell_r(\alpha)_n \quad = \quad \begin{cases} r & \text{if } n = 0 \\ n!\alpha_{n-1} & \text{if } n > 0 \end{cases} \tag{13}$$

Laplace transform corresponds to $\ell = \ell_0$. Its image $\mathbb{R}_0^\omega$, with χ, ϑ and ς restricted to it, is thus isomorphic to $\mathbb{R}^\omega$, as an $\mathbb{R}\times$-coalgebra. The isomorphism ℓ moreover preserves the ring operations

$$\ell\{\alpha+\beta\} \quad = \quad \ell\{\alpha\} + \ell\{\beta\} \tag{14}$$

$$\ell\{\alpha * \beta\} \quad = \quad \ell\{\alpha\}\cdot\ell\{\beta\} \tag{15}$$

and the zero $[0, 0, \ldots]$, but the unit $[1, 0, 0\ldots]$ lies outside $\mathbb{R}_0^\omega$, just as all constants $[a, 0, 0, \ldots]$, for $a \neq 0$. In the field of fractions over $\mathbb{R}_0^\omega$, each $a \in \mathbb{R}$ is represented by $\frac{\ell\{a\}}{\ell\{1\}}$, which ensures that the extension of ℓ to this field is $\mathbb{R}$-linear.

As $\ell\{1\} = [0, 1, 0, \ldots]$ is, on the other hand, the integrator x from (11), we have $\ell\{a\} = ax$ for all $a \in \mathbb{R}$. The ℓ-image of the equation $f = f(0) + \int f'$ is thus $\ell\{f\} = xf(0) + x\ell\{f'\}$. Multiplied with $s = \frac{1}{x}$, it yields the basic formula of operator calculus

$$\ell\{f'\} \quad = \quad s\ell\{f\} - f(0)$$

Together with (14–15), this formula determines the Laplace duals of a bulk of elementary functions. The algebra induced by ℓ thus looks exactly like the algebra induced by Laplace transform. Indeed, we shall now show that ℓ *exactly* mimicks Laplace transform on Taylor coefficients.

3.3 Laplace transform of analytic functions

Recall that Laplace transform $\mathcal{L}$ takes a real locally integrable function $f(x)$ and returns a function depending on a complex variable s

$$\mathcal{L}\{f\} \quad = \quad \int_0^\infty e^{-st} f(t)dt \qquad (16)$$

It is well-known that $\mathcal{L}\{f\}$ is analytic, for $\Re(s)$ sufficiently large, whenever the integral in (16) is absolutely (and therefore uniformly) convergent. The other way around, every function analytic for large $\Re(s)$ turns out to be the Laplace dual $\mathcal{L}\{f\}$ of some real function f, unique up to a set of measure zero [13].

A consequence of the present analysis is that restricting $\mathcal{L}$ to the real *analytic* functions leads to a simpler, perhaps even more instructive correspondence. It could be stated entirely within the framework of real analysis, but probably not really understood.

Definition 3.1 *Let f be a complex function analytic (holomorphic) at ∞: in other words, for sufficiently large z, there is a Laurent expansion $f(z) = \sum_{i=0}^\infty \frac{\alpha_i}{z^i}$.*

We say that a function f is coanalytic *if all of its coefficients $\alpha_0, \alpha_1, \dots$ are real. The set of coanalytic ($\mathbb{H}$olomorphic) functions will be denoted $\mathbb{H}$.*

Examples of coanalytic functions are $\exp\left(\frac{1}{z^2}\right)$, $\frac{1}{z^3+z}$, and similar.

Lemma 3.2 *$f(z)$ is a coanalytic function if and only if the real function $f\left(\frac{1}{x}\right)$ is analytic at 0. Coversely, every real function $g(x)$ gives rise to a coanalytic function $g\left(\frac{1}{z}\right)$.*

There is thus a one-to-one correspondence between $\mathbb{H}$ and $\mathbb{A}$. Extending it along the Taylor representation yields the bijection $\frac{1}{\mathsf{T}} : \mathbb{H} \to \mathbb{R}^{<\omega}$, assigning to each coanalytic function $f(z)$ the Taylor coefficients of $f\left(\frac{1}{x}\right)$. Its inverse is

$$\widetilde{\tfrac{1}{\mathsf{T}}}\{\alpha\} \quad = \quad \sum_{i=0}^\infty \frac{\alpha_i}{i! s^i}$$

Now we can finally prove that ℓ indeed captures $\mathcal{L}$.

Proposition 3.3 *Let $\alpha = \mathsf{T}\{f\}$ and $\beta = \frac{1}{\mathsf{T}}\{g\}$. Then*

$$\mathcal{L}\{f\} = g \quad \iff \quad \ell\{\alpha\} = \beta$$

Proof. Using the equation $\mathcal{L}\{x^i\} = \frac{i!}{s^{i+1}}$ and the fact that $\mathcal{L}$ is linear and continuous, i.e. $\mathcal{L}\left\{\sum_0^\infty \xi_i x^i\right\} = \sum_0^\infty \xi_i \mathcal{L}\{x^i\}$, we get

$$\begin{aligned}
\mathcal{L}\{f\} &= \mathcal{L}\left\{\sum_{i=0}^\infty \frac{\alpha_i}{i!} x^i\right\} = \\
&= \sum_{i=0}^\infty \frac{\alpha_i}{i!} \mathcal{L}\left\{x^i\right\} = \\
&= \sum_{i=0}^\infty \frac{\alpha_i}{i!} \cdot \frac{i!}{s^{i+1}} = \\
&= \sum_{n=1}^\infty \frac{\alpha_{n-1}}{s^n}
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathcal{L}\{f\} = \sum_{n=0}^\infty \frac{\beta_n}{n! s^n} \\
&\iff \quad \beta_n = \begin{cases} 0 & \text{if } n = 0 \\ n! \alpha_{n-1} & \text{if } n > 0 \end{cases}
\end{aligned}$$

□

The obvious consequence is that for every f analytic at 0 holds

$$\mathcal{L}\{f\} \quad = \quad \widetilde{\tfrac{1}{\mathsf{T}}} \circ \ell \circ \mathsf{T}\{f\} \qquad (17)$$

Laplace transform thus couples analytic and coanalytic functions. More precisely, it maps analytic functions $\mathbb{A}$ into coanalytic functions $\mathbb{H}_\infty$ *that vanish at* ∞, because ℓ maps their Taylor expansions $\mathbb{R}^{<\omega}$ into the Taylor expansions $\mathbb{R}_0^{<\omega}$ of functions that vanish at 0. We have thus proved

Corollary 3.4 *A real function f is analytic at 0 if and only if its Laplace dual $\mathcal{L}\{f\}$ is a coanalytic function vanishing at ∞. Laplace transform is a bijection $\mathcal{L} : \mathbb{A} \longrightarrow \mathbb{H}_\infty$.*

But it is not a mere bijection: as a morphism it is completely determined by its preservation properties:

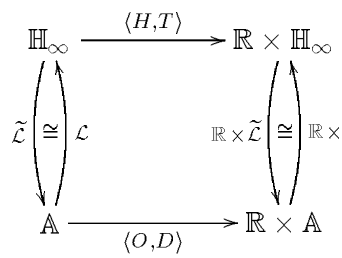
Corollary 3.5 *Laplace transform $\mathcal{L} : \mathbb{A} \longrightarrow \mathbb{H}_\infty$ is the only continuous linear operator satisfying*

$$\mathcal{L}\{x^n\} = \frac{n!}{s^{n+1}}$$

Proof. Since (17) implies that $\ell = \frac{1}{\mathsf{T}} \circ \mathcal{L} \circ \widetilde{\mathsf{T}}$, one shows, exactly as in the proof of proposition 3.3, that the assumptions about $\mathcal{L}$ pin down $\ell = \ell_0$ to formula (13). But the other way around, because of its linear continuity, $\mathcal{L}$ is also uniquely determined by ℓ. □

3.4 Laplace transform abstractly

The final point to be made is that the coanalitic ideal $\mathbb{H}_\infty$ *is a stream algebra* and that *both $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are induced by that structure, as coalgebra homomorphisms.*



This, of course, follows a priori from the established isomorphisms: $\mathcal{L}$ is just the lifting of ℓ along T and $\frac{1}{\mathsf{T}}$. The structure of $\mathbb{H}_\infty$ will thus be the lifting of the structure $\langle \chi, \vartheta, \varsigma\rangle$, which induces ℓ.

For simplicity, we first lift the χ, ϑ and ς along T to $\mathbb{A}_0$, the ideal of functions analytic and vanishing at 0.

$$\begin{array}{rlcl}
\Xi : & \mathbb{A}_0 & \longrightarrow & \mathbb{R} \\
& f & \longmapsto & f'(0) \\[1ex]
\Theta : & \mathbb{A}_0 & \longrightarrow & \mathbb{A}_0 \\
& f & \longmapsto & \int_0^x \frac{1}{t}\left(f'(t) - f'(0)\right) dt \\[1ex]
\Gamma : & \mathbb{R}\times\mathbb{A}_0 & \longrightarrow & \mathbb{A}_0 \\
& \langle a, g\rangle & \longmapsto & x(a+g(x)) + \int_0^x g(t)dt
\end{array}$$

The proof that this is a stream algebra is routine, but essentially depends on the properties of the functions from $\mathbb{A}_0$.

The desired structure of $\mathbb{H}_\infty$ is induced along the isomorphism $\mathbb{A}_0 \cong \mathbb{H}_\infty$ from lemma 3.2, substituting $\frac{1}{s}$ for x, and making use of the properties of the functions involved.

$$\begin{array}{rlcl}
H : & \mathbb{H}_\infty & \longrightarrow & \mathbb{R} \\
& F & \longmapsto & -\lim_{s\to\infty} s^2 F'(s) \\[1ex]
T : & \mathbb{H}_\infty & \longrightarrow & \mathbb{H}_\infty \\
& F & \longmapsto & \int_s^\infty \left(H(F) - \sigma F'(\sigma)\right) d\sigma \\[1ex]
C : & \mathbb{R}\times\mathbb{H}_\infty & \longrightarrow & \mathbb{H}_\infty \\
& \langle a, G\rangle & \longmapsto & \frac{a+G(s)}{s} - \int_s^\infty \frac{G(\sigma)d\sigma}{\sigma^2}
\end{array}$$

Checking that this structure corresponds, along $\frac{1}{\mathsf{T}}$ and T, to $\langle \chi, \vartheta, \varsigma\rangle$, and then using the fact that this structure determines ℓ, while ℓ determines $\mathcal{L}$, yields a "clean" proof of

Theorem 3.6 *$\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are completely determined by the commutativity of the last diagram. In other words, Laplace transform and its inverse are the unique stream homomorphisms between the analytic functions, and the coanalytic functions vanishing at ∞, with corresponding structures described above.*

4 Conclusion

In essence, calculus is coinductive programming. It consists mostly of using final fixpoints and constructing various transforms between them. When applying standard methods for solving differential equations, we are actually using coinduction even without knowing it!

Of course, real analysts probably do not need to know this. Our point is, however, that the *computational contents* of their infinitistic reasoning, in its standard forms, can usually be reduced to coinduction — and implemented as such.

In the present paper, we have provided some initial evidence for these claims. The examples described are, of course, very basic, but we believe that they are typical. Many other fundamental structures from differential and integral calculus are readily seen to give rise to similar final coalgebras. With some work, our stream algebras are extending in many directions: beyond analytic functions, and beyond functions, beyond Riemann integral and ordinary derivative, beyond real numbers, beyond deterministic analysis.

The upshot is, at least, a unified framework for presenting and implementating important analytic tools. But the conceptual value of an effective presentation is hard to estimate. So far, it is clear that Laplace transform implemented on streams is considerably easier to work with, and reason about, than in its original integral form[3].

Moreover, some frequently observed *analogies* — e.g. between differential and difference equations, and their operators [3], or between Taylor series and Laplace transform [13, ch. VII] — seem to be acquiring formal grounds in coinduction. We see that they are not precise correspondences: the coalgebras capturing Taylor and Laplace are quite different — but they are both coalgebras of the same kind. Hence the structural coincidence.

Attribution. The fact that analytic functions form a stream algebra, and that this can be used to explain Taylor series was discovered by Martín Escardó in 1992. He has also aware of the role of equation (7) in solving initial value problems.

[3]With no loss of generality, since the restriction to analytic functions can be avoided; and in fact, with a *gain* of generality, since ℓ can be implemented over any field instead of $\mathbb{R}$.

References

[1] J. Barwise and L.S. Moss, *Vicious Circles* (CSLI 1996)

[2] G. Birkhoff and G.-C. Rota, *Ordinary Differential Equations* (John Wiley and Sons 1959, 3^{rd} ed. 1978)

[3] L. Brand, *Differential and Difference Equations* (J. Wiley 1966)

[4] Th. Coquand, Infinite Objects in Type Theory, *Types for Proofs and Programs*, Lecture Notes in Computer Science 806 (Springer 1993)

[5] A. Hoare, *Communicating Sequential Processes* (Prentice Hall 1985)

[6] B. Malgrange, *Ideals of Differentiable Functions* (Oxford University Press 1966)

[7] N.P. Mendler, P. Panangaden and R.L. Constable, Infinite Objects in Type Theory, *Proceedings of the First LICS*, (IEEE 1986) 249–255

[8] J. Mikusiński, *Operational Calculus* (Pergamon Press, 1959)

[9] R. Milner and M. Tofte, Co-induction in Relational Semantics, *Theoret. Comput. Sci.* 87(1991) 209–220

[10] D. Pavlović, Semantics of guarded induction, *in preparation*

[11] E. Phragmén, Sur une extension d'un théorème classique de la theorie des fonctions, *Acta Math.*, 28(1904) 331–368

[12] J.-C. Tougeron, *Ideaux de Fonctions Differentiables*, Ergebnisse der Mathematik 71 (Springer 1971)

[13] D.V. Widder, *The Laplace Transform* (Princeton University Press 1946)