A complete axiomatization of timed bisimulation for a class of timed regular behaviours

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ABSTRACT. One of the most satisfactory results in process theory is Milner's axiomatization of strong bisimulation for regular CCS. This result holds for open terms with finite-state recursion. Wang has shown that timed bisimulation can also be axiomatized, but only for closed terms without recursion. In this paper, we provide an axiomatization for timed bisimulation of open terms with finite-state recursion.

KEY WORDS AND PHRASES. Complete axiomatizations, ω-completeness, equational logic, timed bisimulation equivalence, regular processes, timed CCS.

1 Introduction

Much research in concurrency theory has recently been devoted to the development of extensions of standard process algebras like CCS [15], CSP [10] and ACP [3] with constructs allowing for the modelling of timing aspects in the behaviour of processes. By now, most process algebras have a timed counterpart (see, e.g., [1, 5, 17, 20]), and the development of results and techniques for these languages is becoming comparable with that for the standard process description languages. For example, complete axiomatizations of behavioural congruences for subsets of timed process algebras have been presented in, e.g., [9, 13, 17, 21]—showing that behavioural congruences which deal with timing considerations are as mathematically tractable as the standard untimed ones.

Two of the most beautiful results in the theory of process algebras are the complete axiomatizations of strong bisimulation equivalence and observational congruence for regular CCS processes provided by Milner in his classic papers [14] and [16], respectively. These results have put the notions of behaviour used in the theory of CCS on an equal footing with the one common in formal language theory, and have contributed to the realization that the notion of *process* is at least as elegant and mathematically tractable as that of *language*.

The main purpose of this paper is to show that the techniques developed by Milner in [14, 16] can be adapted to provide a complete axiomatization of the notion of *timed bisimulation equivalence*, due to Wang Yi [20], over a class of regular timed CCS processes [11, 21]. More precisely, we shall offer a complete axiomatization of timed bisimulation over the language of action guarded regular expressions studied in [11]. This complete axiomatization is obtained by combining an improved version of the laws which were shown in [21] to characterize timed bisimulation over finite trees with standard laws for recursively defined processes, *viz*, laws to unwind recursive definitions of expressions, and a version of unique fixed-point induction.

The paper is organized as follows. Section 2 is devoted to preliminaries, and background material on timed CCS and timed bisimulation. The axiomatization of timed bisimulation is presented and discussed in Section 3, where its soundness is also proved. The proof of completeness of the axiomatization is given in detail in Section 4, and relies on an adaptation of the techniques used by Milner in [14, 16].

As this is not an introductory paper on timed CCS, we have taken the liberty to refer the reader to the original papers by Wang Yi for motivations and examples. We hope, however, that the paper will still be sufficiently readable for the uninitiated reader.

2 Timed regular behaviours and timed bisimulation

The language for expressions that we shall consider in this paper is a generalization of the regular subcalculus of Wang Yi's timed CCS [20, 22]. This language has been investigated by Holmer, Larsen and Wang Yi in [11], and we shall mostly follow the notation and definitions given in that reference.

As usual, we shall assume a countably infinite set Δ of action names, ranged over by a and b, and a distinguished action $\tau \notin \Delta$. Let $\mathrm{Act} = \Delta \cup \{\tau\}$, be the set of actions, ranged over by μ and ν.

Following [12], we define a monoid $(X, +, 0)$ to be:

- *left-cancellative* iff $(x+y = x+z) \Rightarrow (y = z)$, and
- *anti-symmetric* iff $(x+y = 0) \Rightarrow (x = y = 0)$.

Examples of left-cancellative anti-symmetric monoids include:

- The singleton set $(\mathbf{1}, +, 0)$.
- The natural numbers $(\mathbf{N}, +, 0)$.
- The non-negative rationals $(\mathbf{Q}^+, +, 0)$.
- The non-negative reals $(\mathbf{R}^+, +, 0)$.

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Research was partly carried out during a stay at Aalborg University Centre, Denmark, and was partially supported by the Danish Natural Science Research Council project DART.
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Additional funding was received from EC BRA 7166 CONCUR2.

University of Sussex Computer Science Report 4/94

- The countable ordinals $(\omega_1, +, 0)$.

We can define a partial order on X as:

$$x \leq y \text{ iff } \exists z\,.\, x + z = y$$

It is simple to verify that $\leq$ is a partial order if $(X, +, 0)$ is a left-cancellative anti-symmetric monoid. A *time domain* is a left-cancellative anti-symmetric monoid $(\mathsf{Tim}, +, 0)$, ranged over by t, u and v, such that $\leq$ is a total order. Define:

$$\begin{aligned} t \wedge u = &\ \text{the minimum of } t \text{ and } u \\ t \vee u = &\ \text{the maximum of } t \text{ and } u \end{aligned}$$

and when $t \geq u$:

$$t - u = \text{ the unique } v \text{ such that } u + v = t$$

Let $\mathsf{Tim}_+ = \mathsf{Tim} \setminus \{0\}$ be the set of positive delays, ranged over by c, d and e.

Let $\mathsf{Lab} = \mathsf{Act} \cup \{\varepsilon(c) \mid c \in \mathsf{Tim}_+\}$ be the set of labels, ranged over by σ.

Let Var be a countably infinite set of process variables, ranged over by x, y and z.

The set of *regular process expressions* over Act, Tim and Var is given by the following grammar:

$$E ::= \mathbf{0} \mid x \mid \mu.E \mid \varepsilon(t).E \mid E + E \mid \mathbf{fix}(x = E)$$

The interested reader is referred to [20, 22] for intuition on the operators used in the above definition.

We shall assume the standard notions of free and bound variables in expressions, with $\mathbf{fix}(x = \_)$ as the binding construct. The set of free variables in an expression E is denoted by $\mathsf{fv}E$. Throughout this paper we shall restrict ourselves to considering regular process expressions in which recursions are action guarded, a notion that is defined below.

DEFINITION. *A variable x is* action guarded *in E iff $x \in \mathbf{AG}(E)$, defined:*

$$\begin{aligned} \mathbf{AG}(\mathbf{0}) &= \mathsf{Var} \\ \mathbf{AG}(x) &= \mathsf{Var} \setminus \{x\} \\ \mathbf{AG}(\mu.E) &= \mathsf{Var} \\ \mathbf{AG}(\varepsilon(t).E) &= \mathbf{AG}(E) \\ \mathbf{AG}(E + F) &= \mathbf{AG}(E) \cap \mathbf{AG}(F) \\ \mathbf{AG}(\mathbf{fix}(x = E)) &= \mathbf{AG}(E) \setminus \{x\} \end{aligned}$$

A regular process expression E is well-formed *iff for every subexpression of E of the form $\mathbf{fix}(x = F)$, x is action guarded in F.*

For example, the expression $(\mathbf{fix}(x = \tau.x)) + y$ is well-formed, while the expression $\mathbf{fix}(x = \varepsilon(c).x)$ is not. The above definition departs slightly from the one

$$\frac{}{\mathbf{0} \xrightarrow{\varepsilon(c)} \mathbf{0}} \qquad \frac{}{\mu.P \xrightarrow{\mu} P} \qquad \frac{}{a.P \xrightarrow{\varepsilon(c)} a.P}$$

$$\frac{}{\varepsilon(c+t).P \xrightarrow{\varepsilon(c)} \varepsilon(t).P} \qquad \frac{P \xrightarrow{\varepsilon(c)} P'}{\varepsilon(t).P \xrightarrow{\varepsilon(t+c)} P'} \qquad \frac{P \xrightarrow{\mu} P'}{\varepsilon(0).P \xrightarrow{\mu} P'}$$

$$\frac{P \xrightarrow{\mu} P'}{P + Q \xrightarrow{\mu} P'} \qquad \frac{Q \xrightarrow{\mu} Q'}{P + Q \xrightarrow{\mu} Q'} \qquad \frac{P \xrightarrow{\varepsilon(c)} P',\ Q \xrightarrow{\varepsilon(c)} Q'}{P + Q \xrightarrow{\varepsilon(c)} P' + Q'}$$

$$\frac{E\{\mathbf{fix}(x = E)/x\} \xrightarrow{\sigma} P}{\mathbf{fix}(x = E) \xrightarrow{\sigma} P}$$

FIGURE 1. The operational semantics for $\mathbf{TC}_0$

given in [11, Definition 2.1]. In particular, the expression $(\mathbf{fix}(x = \tau.x)) + y$ would not be well-formed according to the definition of [11] because the free variable y does not occur within a subexpression of the form $\mu.F$.

The set of all well-formed regular process expressions is $\mathbf{TC}$, ranged over by E, F and G. The set of all closed, well-formed regular process expressions is $\mathbf{TC}_0$, ranged over by P, Q and R. Elements of this set will often be referred to as *processes*.

Following Milner [16], we shall identify expressions which differ only by the renaming of bound variables. We shall also write $E\{F_1, \ldots, F_n/x_1, \ldots, x_n\}$ for the result of simultaneously substituting F_i for each free occurrence of x_i in E, renaming bound variables as necessary.

The operational semantics for $\mathbf{TC}_0$ is given by the labelled transition system $(\mathbf{TC}_0, \mathsf{Lab}, \rightarrow)$ in Figure 1. The interested reader is referred to [20, 22] for comments on the rules. Note that, following Wang Yi [20, 22], $\varepsilon(0)$ has been excluded from the semantics of processes.

To conclude this introductory section, we shall now define the notion of *timed bisimulation equivalence*.

DEFINITION (TIMED BISIMULATION EQUIVALENCE [20]). *A relation $\mathcal{R}$ over $\mathbf{TC}_0$ is a* timed bisimulation *iff $P \mathrel{\mathcal{R}} Q$ implies, for all σ:*

- *whenever $P \xrightarrow{\sigma} P'$ then, for some Q', $Q \xrightarrow{\sigma} Q'$ and $P' \sim Q'$.*
- *whenever $Q \xrightarrow{\sigma} Q'$ then, for some P', $P \xrightarrow{\sigma} P'$ and $P' \sim Q'$.*

The relation of timed bisimulation equivalence*, denoted by $\sim$, is the largest timed bisimulation.*

The interested reader is referred to the aforementioned papers by Wang Yi, and to [11] for intuition and examples of processes that are equivalent or inequivalent with respect to $\sim$. The definition of $\sim$ can be extended to expressions in the standard way as follows:

DEFINITION. *Let E and F be expressions with free variables in $\tilde{x} = x_1, \ldots, x_m$.*

$$
\begin{array}{lrcl}
\text{(S1)} & E+F & = & F+E \\
\text{(S2)} & E+(F+G) & = & (E+F)+G \\
\text{(S3)} & E+E & = & E \\
\text{(S4)} & E+\mathbf{0} & = & E \\
\\
\text{(TD)} & \varepsilon(t).(E+F) & = & \varepsilon(t).E+\varepsilon(t).F \\
\text{(TA)} & \varepsilon(t+u).E & = & \varepsilon(t).\varepsilon(u).E \\
\text{(T0)} & \varepsilon(0).E & = & E \\
\\
\text{(R1)} & \mathbf{fix}(x=E) & = & E\{\mathbf{fix}(x=E)/x\}
\end{array}
$$

(R2) If $F = E\{F/x\}$, then $F = \mathbf{fix}(x = E)$, provided x is action guarded in E

FIGURE 2. The axiom system $\mathcal{G}$

$$
\begin{array}{lrcl}
\text{(MP)} & \tau.E+\varepsilon(c).F & = & \tau.E \\
\text{(AP)} & a.E+\varepsilon(t).a.E & = & a.E \\
\text{(NP)} & \varepsilon(t).\mathbf{0} & = & \mathbf{0}
\end{array}
$$

FIGURE 3. The axiom system $\mathcal{F}$ is $\mathcal{G}$ plus (MP), (AP) and (NP)

$$
\begin{array}{lrcl}
\text{(MP)} & \tau.E+\varepsilon(c).F & = & \tau.E \\
\text{(P)} & E+\varepsilon(t).E & = & E
\end{array}
$$

FIGURE 4. The axiom system $\mathcal{E}$ is $\mathcal{G}$ plus (MP) and (P)

Then $E \sim F$ *iff for all vectors* $\tilde{P} = P_1,\ldots,P_m$ $E\{\tilde{P}/\tilde{x}\} \sim F\{\tilde{P}/\tilde{x}\}$.

PROPOSITION 1 ([20, THEOREM 5.1]). *Timed bisimulation equivalence forms a congruence over* **TC**.

In the remainder of this paper, we shall present a complete axiomatization of $\sim$ over **TC**.

3 Axiomatization and soundness

In [20] various equational laws were proved to hold for Wang Yi's timed CCS modulo timed bisimulation equivalence, and in [21] a set of such axioms was shown to be complete over the language of recursion-free $\mathbf{TC}_0$ processes with delays from the time domain of the positive reals. We shall now present an axiomatization which will be proven complete for $\sim$ over the whole of **TC**, i.e., complete for regular process expressions with action guarded recursion. The detailed proof of completeness occupies Section 4 of this paper.

Wang's axiomatization for recursion-free $\mathbf{TC}_0$ processes is given by the axiom system $\mathcal{F}$ in Figures 2 and 3. Our axiomatization for regular **TC** process expressions is given by the axiom system $\mathcal{E}$ in Figures 2 and 4.

The axioms (S1)–(S4) are the standard laws for a complete axiomatization of strong bisimulation equivalence over finite trees [8]. Together with axioms (R1) and (R2), these form a complete axiomatization of strong bisimulation equivalence for guarded regular CCS terms [14]. (In fact, the axiomatization in [14] can be obtained as a special case of that in Figure 2 by taking the time domain to be the singleton set $(\mathbf{1},+,0)$.)

The axioms (TD), (TA) and (T0) correspond to the operational properties of *time determinacy*, *time additivity* and *zero delay*. These axioms are present in Wang's [20, 21] axiomatization. As we shall see in Section 4, the axiom system $\mathcal{G}$ given in Figure 2 is powerful enough to prove Milner's [14] *Equational Characterization Theorem* for timed regular expressions. However, $\mathcal{G}$ is not powerful enough to give a complete axiomatization for recursion-free timed expressions.

Wang [20, 21] added the axioms (MP), (AP) and (NP) to $\mathcal{G}$ to provide a complete axiomatization for recursion-free $\mathbf{TC}_0$ processes. These axioms correspond to the operational properties of *maximal progress* and *persistency*, and are discussed in detail by Wang. However, the resulting axiom system $\mathcal{F}$, given in Figure 3, is not powerful enough to give a complete axiomatization for recursion-free **TC** process expressions.

Our axiomatization replaces (AP) and (NP) with one new persistency axiom (P). In Section 4 we show that $\mathcal{E}$ is complete for timed bisimulation equivalence over **TC** process expressions. In Section 5 we show that $\mathcal{E}$ is strictly stronger than $\mathcal{F}$, and thus that Wang's axiomatization is not complete for open **TC** process expressions.

We shall write $\mathcal{E} \vdash E = F$ when $E = F$ may be proved from $\mathcal{E}$ together with the structural rules for $=$ to be a congruence, and similarly for $\mathcal{F} \vdash E = F$ and $\mathcal{G} \vdash E = F$.

To conclude this section, we shall show that $\mathcal{E}$ is indeed sound with respect to timed bisimulation equivalence over **TC**.

PROPOSITION 2 (SOUNDNESS). *For all* **TC** *expressions* E,F, $\mathcal{E} \vdash E = F$ *implies* $E \sim F$.

PROOF. All the laws in $\mathcal{E}$ have been shown sound by Wang Yi in [21]. The only exception is the persistency axiom (P), the soundness of which is established by the timed bisimulation:

$$\{(Q+P,Q) \mid \exists c\,.\,P \xrightarrow{\varepsilon(c)} Q\} \cup \{(P,P) \mid P \in \mathbf{TC}_0\}$$

The verification of the fact that the above relation satisfies the defining clause of $\sim$ uses the properties of *time determinacy*, *time additivity* and *persistency* of the operational semantics for $\mathbf{TC}_0$. (The interested reader is referred to [20] and [19] for details on these properties). □

4 Completeness

In this section, we shall present the proof of completeness of the set of laws $\mathcal{E}$ over **TC**. The structure of the proof of this result will follow closely the most beautiful arguments used by Milner in [14, 16] to prove the completeness of the axiomatizations for strong bisimulation and observational congruence over regular CCS processes.

The structure of the completeness proof will be as follows: first of all, we shall show that every **TC** expression E provably satisfies a certain kind of equation set. This is what Milner calls the *Equational Characterization Theorem.* Next, we shall show that if $E \sim F$ and E provably satisfies an equation set, while F provably satisfies another equation set, then both E and F provably satisfy a common equation set. Finally, we show that whenever two **TC** expressions provably satisfy the same equation set, then $\mathcal{E}$ proves that they are equal.

DEFINITION. *An* equation set $\tilde{x} = \tilde{E}$ *is a finite non-empty sequence of declarations* $x_1 = E_1, \ldots, x_n = E_n$*, where the* x_i*s are pairwise distinct variables, and the* E_i*s are* **TC** *expressions.*

A vector $\tilde{F} = F_1 \ldots F_n$ satisfies $\tilde{x} = \tilde{E}$ *iff* $\forall i \,.\, F_i \sim E_i\{\tilde{F}/\tilde{x}\}$.

For an equational theory $\mathcal{T}$*, a vector* $\tilde{F} = F_1 \ldots F_n$ $\mathcal{T}$-provably satisfies $\tilde{x} = \tilde{E}$ *iff* $\forall i \,.\, \mathcal{T} \vdash F_i = E_i\{\tilde{F}/\tilde{x}\}$.

An expression E *(*$\mathcal{T}$*-provably) satisfies* $\tilde{x} = \tilde{F}$ *iff we can find a vector* $\tilde{E}$ *which (*$\mathcal{T}$*-provably) satisfies* $\tilde{x} = \tilde{F}$ *and* $E \sim E_1$ *(*$\mathcal{T} \vdash E = E_1$*).*

We refer to x_1 *as the leading variable of the equation set* $\tilde{x} = \tilde{F}$.

For example, the equation set:

$$x_1 = \varepsilon(1).a.x_2 + \varepsilon(3).y \qquad x_2 = \varepsilon(2).b.x_1 \tag{1}$$

is satisfied by $\mathbf{fix}(z = \varepsilon(1).a.\varepsilon(2).b.z + \varepsilon(3).y)$.

DEFINITION. *An equation set* $\tilde{x} = \tilde{E}$ *is* standard *iff each* E_i *is of the form:*

$$\sum_{j \in J_i} \varepsilon(t_j).\mu_j.x_j + \sum_{k \in K_i} \varepsilon(u_k).w_k$$

where the vectors $\tilde{x}$ *and* $\tilde{w}$ *are disjoint. We call* $\tilde{x}$ *the* formal *variables of* $\tilde{x} = \tilde{E}$*, and* $\tilde{w}$ *the* free *variables of* $\tilde{x} = \tilde{E}$.

For example, the above equation set (1) is standard, but the following is not:

$$x_1 = \varepsilon(1).x_2 + \varepsilon(3).y, \qquad x_2 = a.\varepsilon(2).b.x_1$$

PROPOSITION 3. *If* $\tilde{x} = \tilde{E}$ *is standard and* w *is not a formal variable of* S*, then we can find a standard* $\tilde{x} = \tilde{F}$ *such that* $\forall i \,.\, \mathcal{G} \vdash F_i = E_i\{E_1/w\}$.

PROOF. Define $\tilde{F}$ as:

$$F_i \equiv \sum_{j \in J_i} \varepsilon(t_j).\mu_j.x_j + \sum_{\substack{k \in K_i \\ w_k \neq w}} \varepsilon(u_k).w_k + \sum_{\substack{k \in K_i \\ w_k = w \\ j' \in J_1}} \varepsilon(u_k + t_{j'}).\mu_{j'}.x_{j'} + \sum_{\substack{k \in K_i \\ w_k = w \\ k' \in K_1}} \varepsilon(u_k + u_{k'}).w_{k'}$$

It is simple to show that this is standard, and that $\forall i \,.\, \mathcal{G} \vdash F_i = E_i\{E_1/w\}$. □

PROPOSITION 4. *If* x *is action guarded in* E *and* $\mathcal{G} \vdash E = F$ *then* x *is action guarded in* F.

PROOF. Show that $\mathbf{AG}(\_)$ is a model for the equational theory $\mathcal{G}$. □

PROPOSITION 5. *We shall use the following standard results about substitution:*

1. $G\{\tilde{F}/\tilde{x}\}\{E/w\} \equiv G\{E/w\}\{\tilde{F}\{E/w\}/\tilde{x}\}$*, if* w *does not occur in* $\tilde{x}$*, and* $\tilde{x}$ *are not free in* E.
2. $F\{G/w\}\{\tilde{E}/\tilde{x}\} \equiv F\{G\{\tilde{E}/\tilde{x}\}/w\}\{\tilde{E}/\tilde{x}\}$*, if* $\tilde{x}$ *are not free in* $\tilde{E}$.

PROOF. Routine structural induction. □

THEOREM 6 (EQUATIONAL CHARACTERIZATION). *For any* E *we can find a standard equation set* $\tilde{x} = \tilde{G}$ *which* E $\mathcal{G}$*-provably satisfies.*

PROOF. An induction on E. The only difficult case is when $E \equiv \mathbf{fix}(w = F)$. In this case, by induction we find a $\tilde{x} = \tilde{H}$ which F $\mathcal{G}$-provably satisfies, and wlog we can assume that w is not a formal variable of $\tilde{x} = \tilde{H}$, and that $\tilde{x}$ are not free in E. Thus we have a $\tilde{F}$ such that:

$$\mathcal{G} \vdash F_1 = F \tag{2}$$

$$\forall i \,.\, \mathcal{G} \vdash F_i = H_i\{\tilde{F}/\tilde{x}\} \tag{3}$$

Define:

$$E_i \equiv F_i\{E/w\} \tag{4}$$

Let $\tilde{G}$ be the standard equation set given by Proposition 3 such that:

$$\mathcal{G} \vdash G_i = H_i\{H_1/w\} \tag{5}$$

Since w is action guarded in F, by Proposition 4 it must be action guarded in $H_1\{\tilde{F}/\tilde{x}\}$, so, as $w \notin \tilde{x}$, must be action guarded in H_1, so cannot be free in H_1. Then:

$$\begin{aligned} \mathcal{G} \vdash\ & E \\ &= F\{E/w\} && \text{(R1)} \\ &= F_1\{E/w\} && \text{(2)} \end{aligned}$$

$$= E_1 \qquad (4)$$

and:

$$\begin{aligned}
&\mathcal{G} \vdash E_1 \\
&\quad = F_1\{E/w\} && (4)\\
&\quad = H_1\{\tilde{F}/\tilde{x}\}\{E/w\} && (3)\\
&\quad = H_1\{E/w\}\{\tilde{F}\{E/w\}/\tilde{x}\} && \text{(Propn 5.1)}\\
&\quad = H_1\{E/w\}\{\tilde{E}/\tilde{x}\} && (4)\\
&\quad = H_1\{\tilde{E}/\tilde{x}\} && (w \notin \mathsf{fv} H_1)
\end{aligned}$$

and so:

$$\begin{aligned}
&\mathcal{G} \vdash E_i \\
&\quad = F_i\{E/w\} && (4)\\
&\quad = H_i\{\tilde{F}/\tilde{x}\}\{E/w\} && (3)\\
&\quad = H_i\{E/w\}\{\tilde{F}\{E/w\}/\tilde{x}\} && \text{(Propn 5.1)}\\
&\quad = H_i\{E/w\}\{\tilde{E}/\tilde{x}\} && (4)\\
&\quad = H_i\{H_1\{\tilde{E}/\tilde{x}\}/w\}\{\tilde{E}/\tilde{x}\} && \text{(above)}\\
&\quad = H_i\{H_1/w\}\{\tilde{E}/\tilde{x}\} && \text{(Propn 5.2)}\\
&\quad = G_i\{\tilde{E}/\tilde{x}\} && (5)
\end{aligned}$$

Thus we have found a standard $\tilde{x} = \tilde{G}$ which E $\mathcal{G}$-provably satisfies. □

Theorem 6 shows that every expression E in **TC** $\mathcal{G}$-provably satisfies a standard equation set $\tilde{x} = \tilde{G}$. The second stepping stone towards the promised completeness theorem is a result showing that if $E \sim F$, where F $\mathcal{G}$-provably satisfies a standard equation set $\tilde{y} = \tilde{H}$, then there exists a third standard equation set $\mathcal{E}$-provably satisfied by both E and F. Note that this part of the completeness proof requires the axioms (MP) and (P).

THEOREM 7. *Let E and E' be expressions in* **TC** *such that $E \sim E'$. Assume that E $\mathcal{E}$-provably satisfies a standard equation set $\tilde{x} = \tilde{F}$, and E' $\mathcal{E}$-provably satisfies a standard equation set $\tilde{x}' = \tilde{F}'$. Then there exists a standard equation set $\mathcal{E}$-provably satisfied by both E and E'.*

PROOF (FOLLOWING MILNER). Assume that:

$$F_i \equiv \sum_{j \in J_i} \varepsilon(t_j).a_j.x_j + \sum_{k \in K_i} \varepsilon(u_k).\tau.x_k + \sum_{l \in L_i} \varepsilon(v_l).w_l \qquad (6)$$

$$F'_{i'} \equiv \sum_{j' \in J'_{i'}} \varepsilon(t'_{j'}).a_{j'}.x'_{j'} + \sum_{k' \in K'_{i'}} \varepsilon(u'_{k'}).\tau.x'_{k'} + \sum_{l' \in L'_{i'}} \varepsilon(v'_{l'}).w'_{l'} \qquad (7)$$

As E $\mathcal{E}$-provably satisfies $\tilde{x} = \tilde{F}$ and E' $\mathcal{E}$-provably satisfies $\tilde{x}' = \tilde{F}'$, we can find $\tilde{E}$ and $\tilde{E}'$ such that:

$$\begin{aligned}
\mathcal{E} \vdash E &= E_1 && (8)\\
\forall i\,.\,\mathcal{E} \vdash E_i &= F_i\{\tilde{E}/\tilde{x}\} && (9)\\
\mathcal{E} \vdash E' &= E'_1 && (10)\\
\forall i\,.\,\mathcal{E} \vdash E'_i &= F'_i\{\tilde{E}'/\tilde{x}'\} && (11)
\end{aligned}$$

Let $\mathcal{R}$ be the relation $\{(i,i') \mid E_i \sim E'_{i'}\}$, let $\tilde{z}$ be the vector of fresh variables $\{z_{ii'} \mid i \,\mathcal{R}\, i'\}$ (with z_{11} as leading variable), and define the vectors $\tilde{G}$, $\tilde{H}$ and $\tilde{H}'$ as:

$$G_{ii'} \equiv \sum_{\substack{j \in J_i,\, j' \in J'_{i'} \\ j \,\mathcal{R}\, j',\, a_j = a'_{j'}}} \varepsilon(t_j \vee t'_{j'}).a_j.z_{jj'} + \sum_{\substack{k \in K_i,\, k \in K'_{i'} \\ k \,\mathcal{R}\, k'}} \varepsilon(u_k \vee u'_{k'}).\tau.z_{kk'} + \sum_{\substack{l \in L_i,\, l' \in L'_{i'} \\ w_l = w'_{l'}}} \varepsilon(v_l \vee v'_{l'}).w_l \qquad (12)$$

$$H_{ii'} \equiv E_i \qquad (13)$$

$$H'_{ii'} \equiv E'_i \qquad (14)$$

Note that the equation set $\tilde{z} = \tilde{G}$ is standard by construction. We now show that the vector $\tilde{H}$ $\mathcal{E}$-provably satisfies $\tilde{z} = \tilde{G}$. To this end, we prove, first of all, that, for each $i \,\mathcal{R}\, i'$, every summand of $G_{ii'}\{\tilde{H}/\tilde{z}\}$ can be absorbed into $H_{ii'}$. We consider three cases, depending on the form taken by the summand of $G_{ii'}\{\tilde{H}/\tilde{z}\}$.

For any $i \,\mathcal{R}\, i'$, $j \in J_i$ and $j' \in J'_{i'}$ such that $j \,\mathcal{R}\, j'$ and $a_j = a'_{j'}$:

$$\begin{aligned}
&\mathcal{E} \vdash H_{ii'} \\
&\quad = E_i && (13)\\
&\quad = F_i\{\tilde{E}/\tilde{x}\} && (9)\\
&\quad = F_i\{\tilde{E}/\tilde{x}\} + \varepsilon(t_j).a_j.E_j && \text{(S1–S3,6)}\\
&\quad = F_i\{\tilde{E}/\tilde{x}\} + \varepsilon(t_j).(a_j.E_j + \varepsilon((t_j \vee t'_{j'}) - t_j).a_j.E_j) && \text{(P)}\\
&\quad = F_i\{\tilde{E}/\tilde{x}\} + \varepsilon(t_j).a_j.E_j + \varepsilon(t_j).\varepsilon((t_j \vee t'_{j'}) - t_j).a_j.E_j && \text{(TD)}\\
&\quad = F_i\{\tilde{E}/\tilde{x}\} + \varepsilon(t_j).a_j.E_j + \varepsilon(t_j + ((t_j \vee t'_{j'}) - t_j)).a_j.E_j && \text{(TA)}\\
&\quad = F_i\{\tilde{E}/\tilde{x}\} + \varepsilon(t_j).a_j.E_j + \varepsilon(t_j \vee t'_{j'}).a_j.E_j && (t + (u - t) = u)\\
&\quad = F_i\{\tilde{E}/\tilde{x}\} + \varepsilon(t_j \vee t'_{j'}).a_j.E_j && \text{(S1–S3,6)}\\
&\quad = E_i + \varepsilon(t_j \vee t'_{j'}).a_j.E_j && (9)\\
&\quad = H_{ii'} + \varepsilon(t_j \vee t'_{j'}).a_j.H_{jj'} && (13)\\
&\quad = H_{ii'} + \varepsilon(t_j \vee t'_{j'}).a_j.z_{jj'}\{\tilde{H}/\tilde{z}\} && \text{(substitution)}
\end{aligned}$$

Similarly, for any $i \,\mathcal{R}\, i'$, $k \in K_i$ and $k' \in K'_{i'}$ such that $k \,\mathcal{R}\, k'$:

$$\mathcal{E} \vdash H_{ii'} = H_{ii'} + \varepsilon(u_k \vee u'_{k'}).\tau.z_{kk'}\{\tilde{H}/\tilde{z}\}$$

and for any $i \,\mathcal{R}\, i'$, $l \in L_i$ and $l' \in L'_{i'}$ such that $w_l = w'_{l'}$:

$$\mathcal{E} \vdash H_{ii'} = H_{ii'} + \varepsilon(v_l \vee v'_{l'}).w_l\{\tilde{H}/\tilde{z}\}$$

We remark here that the proof of the above equality makes an essential use of axiom (P), and could not have been carried out using Wang's persistency axioms (AP) and (NP).

Thus each summand of $G_{ii'}\{\tilde{H}/\tilde{z}\}$ can be absorbed into $H_{ii'}$, and by (S1)–(S4):

$$\mathcal{E} \vdash H_{ii'} = H_{ii'} + G_{ii'}\{\tilde{H}/\tilde{z}\} \tag{15}$$

We now show that the converse also holds, namely that $H_{ii'}$ can be absorbed into $G_{ii'}\{\tilde{H}/\tilde{z}\}$. To this end, by (9) and (13), it is sufficient to prove that each summand of $F_i\{\tilde{E}/\tilde{x}\}$ can be absorbed into $G_{ii'}\{\tilde{H}/\tilde{z}\}$. Again, we distinguish three cases depending on the form the summand takes.

For any $i \mathcal{R} i'$ and $j \in J_i$, either:

- $t_j \leq u_k$, for every $k \in K_i$, or:
- there exists $k \in K_i$ such that $t_j > u_k$.

We proceed to show that in either case:

$$\mathcal{E} \vdash G_{ii'}\{\tilde{H}/\tilde{z}\} = G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(t_j).a_j.x_j\{\tilde{E}/\tilde{x}\}$$

- *Case* $\forall k \in K_i \,.\, t_j \leq u_k$. In this case, by the operational semantics for $\mathbf{TC}_0$, it follows that $F_i\{\tilde{E}/\tilde{x}\} \xrightarrow{\varepsilon(t_j)}\xrightarrow{a_j} E_j$. As $E_i \sim E'_{i'}$ and $\mathcal{G}$ is sound for $\sim$, we have that $F_i\{\tilde{E}/\tilde{x}\} \sim F'_{i'}\{\tilde{E}/\tilde{x}'\}$. So $F'_{i'}\{\tilde{E}'/\tilde{x}'\} \xrightarrow{\varepsilon(t_j)}\xrightarrow{a_j} E'_{j'}$ for some j' with $t_j \geq t'_{j'}$, $a_j = a'_{j'}$ and $j \mathcal{R} j'$. Thus:

$$\begin{aligned}
\mathcal{E} \vdash\ & G_{ii'}\{\tilde{H}/\tilde{z}\} \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(t_j \vee t'_{j'}).a_j.H_{jj'} && \text{(S1–S3,12)} \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(t_j).a_j.H_{jj'} && (t_j \geq t'_{j'}) \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(t_j).a_j.E_j && \text{(13)} \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(t_j).a_j.x_j\{\tilde{E}/\tilde{x}\} && \text{(substitution)}
\end{aligned}$$

- *Case* $\exists k \in K_i \,.\, t_j > u_k$. Choose k such that u_k is minimal in the set $\{u_h \mid h \in K_i\}$. Then, by the operational semantics for $\mathbf{TC}_0$, $F_i\{\tilde{E}/\tilde{x}\} \xrightarrow{\varepsilon(u_k)}\xrightarrow{\tau} E_k$. Therefore, as in the previous case, we have $F'_{i'}\{\tilde{E}'/\tilde{x}'\} \xrightarrow{\varepsilon(u_k)}\xrightarrow{\tau} E'_{k'}$ for some $k' \in K'_{i'}$ with $u_k \geq u'_{k'}$ and $k \mathcal{R} k'$. In fact, by symmetry and the fact that u_k is minimal in the set $\{u_h \mid h \in K_i\}$, it is easy to see that $u_k = u'_{k'}$. Thus:

$$\begin{aligned}
\mathcal{E} \vdash\ & G_{ii'}\{\tilde{H}/\tilde{z}\} \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(u_k \vee u'_{k'}).\tau.H_{kk'} && \text{(S1–S3,12)} \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(u_k).\tau.H_{kk'} && (u_k = u'_{k'}) \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(u_k).(\tau.H_{kk'} + \varepsilon(t_j - u_k).a_j.H_{jj'}) && (\text{MP}, t_j > u_k) \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(u_k).\tau.H_{kk'} + \varepsilon(u_k).\varepsilon(t_j - u_k).a_j.H_{jj'} && \text{(TD)} \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(u_k).\tau.H_{kk'} + \varepsilon(u_k + (t_j - u_k)).a_j.H_{jj'} && \text{(TA)} \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(u_k).\tau.H_{kk'} + \varepsilon(t_j).a_j.H_{jj'} && (t + (u - t) = u) \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(u_k \vee u'_{k'}).\tau.H_{kk'} + \varepsilon(t_j).a_j.H_{jj'} && (u_k = u'_{k'}) \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(t_j).a_j.H_{jj'} && \text{(S1–S3,12)} \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(t_j).a_j.E_j && \text{(13)} \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(t_j).a_j.x_j\{\tilde{E}/\tilde{x}\} && \text{(substitution)}
\end{aligned}$$

Note that the above reasoning uses the equation (MP).

Thus:

$$\mathcal{E} \vdash G_{ii'}\{\tilde{H}/\tilde{z}\} = G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(t_j).a_j.x_j\{\tilde{E}/\tilde{x}\}$$

Similarly, for any $i \mathcal{R} i'$ and $k \in K_i$, it is not too difficult to prove that:

$$\mathcal{E} \vdash G_{ii'}\{\tilde{H}/\tilde{z}\} = G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(u_k).\tau.x_k\{\tilde{E}/\tilde{x}\}$$

We are now left to show that for any $i \mathcal{R} i'$ and $l \in L_i$:

$$\mathcal{E} \vdash G_{ii'}\{\tilde{H}/\tilde{z}\} = G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(v_l).w_l\{\tilde{E}/\tilde{x}\} \tag{16}$$

As before, we prove this statement by considering the following two sub-cases:

- $v_l \leq u_k$, for every $k \in K_i$, or:
- there exists $k \in K_i$ such that $v_l > u_k$.

The proof of (16) when there exists $k \in K_i$ such that $v_l > u_k$ follows the lines spelled out in detail above. We shall thus concentrate on presenting a detailed proof of (16) in the case $v_l \leq u_k$, for every $k \in K_i$.

Assume that $l \in L_i$ and that $v_l \leq u_k$, for every $k \in K_i$. We claim that there exists $l' \in L'_{i'}$ such that $v'_{l'} \leq v_l$ and $w_l = w'_{l'}$. To see that this is indeed the case, note that $F_i\{\tilde{E}/\tilde{x}\} \sim F'_{i'}\{\tilde{E}'/\tilde{x}'\}$, by (9), (11) and the soundness of $\mathcal{E}$. Let $\tilde{w}$ denote the set of free variables occurring in either $F_i\{\tilde{E}/\tilde{x}\}$ or $F'_{i'}\{\tilde{E}'/\tilde{x}'\}$. Choose a vector $\tilde{a}$ of distinct actions, one action a_w for each $w \in \tilde{w}$, that do not occur in $F_i\{\tilde{E}/\tilde{x}\}$ and $F'_{i'}\{\tilde{E}'/\tilde{x}'\}$. (This is always possible as the set of action names Δ is countably infinite.) Take the vector $\tilde{P}$ of processes given by $P_w \equiv a_w.\mathbf{0}$. We then have that:

$$F_i\{\tilde{E}/\tilde{x}\}\{\tilde{P}/\tilde{w}\} \sim F'_{i'}\{\tilde{E}'/\tilde{x}'\}\{\tilde{P}/\tilde{w}\} \tag{17}$$

As $l \in L_i$ and $v_l \leq u_k$, for every $k \in K_i$, it follows that $F_i\{\tilde{E}/\tilde{x}\}\{\tilde{P}/\tilde{w}\} \xrightarrow{\varepsilon(v_l)}\xrightarrow{a_{w_l}}$. By (17) and the fact that a_{w_l} does not occur in $F'_{i'}\{\tilde{E}'/\tilde{x}'\}$, we then have that $F'_{i'}\{\tilde{E}'/\tilde{x}'\}\{\tilde{P}/\tilde{w}\} \xrightarrow{\varepsilon(v_l)}\xrightarrow{a_{w_l}}$ because, for some $l' \in L'_{i'}$, $v'_{l'} \leq v_l$ and $w'_{l'} = w_l$ as claimed.

Now we can easily prove (16) as follows:

$$\begin{aligned}
\mathcal{E} \vdash\ & G_{ii'}\{\tilde{H}/\tilde{z}\} \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(v_l \vee v'_{l'}).w_l && \text{(S1–S3,12)} \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(v_l).w_l && (v'_{l'} \leq v_l) \\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} + \varepsilon(v_l).w_l\{\tilde{E}/\tilde{x}\} && \text{(substitution)}
\end{aligned}$$

Thus each summand of $F_i\{\tilde{E}/\tilde{x}\}$ can be absorbed into $G_{ii'}\{\tilde{H}/\tilde{z}\}$, and by (S1)–(S4):

$$\mathcal{E} \vdash G_{ii'}\{\tilde{H}/\tilde{z}\} = G_{ii'}\{\tilde{H}/\tilde{z}\} + F_i\{\tilde{E}/\tilde{x}\} \tag{18}$$

Hence:

$$\mathcal{E} \vdash H_{ii'}$$

$$
\begin{aligned}
&= H_{ii'} + G_{ii'}\{\tilde{H}/\tilde{z}\} && (15)\\
&= E_i + G_{ii'}\{\tilde{H}/\tilde{z}\} && (13)\\
&= F_i\{\tilde{E}/\tilde{x}\} + G_{ii'}\{\tilde{H}/\tilde{z}\} && (9)\\
&= G_{ii'}\{\tilde{H}/\tilde{z}\} && (18)
\end{aligned}
$$

Thus $\tilde{H}$ $\mathcal{E}$-provably satisfies $\tilde{z} = \tilde{G}$, and $\mathcal{E} \vdash E = E_1 = H_{11}$ so E $\mathcal{E}$-provably satisfies $\tilde{z} = \tilde{G}$. Similarly, E' $\mathcal{E}$-provably satisfies $\tilde{z} = \tilde{G}$. □

The final ingredient of the proof of completeness is a result showing that every standard equation set has a unique solution up to provable equality.

THEOREM 8 (UNIQUE SOLUTION). *If $\tilde{x} = \tilde{H}$ is a standard equation set, then there is a* **TC** *expression E which $\mathcal{E}$-provably satisfies it. Moreover, if another* **TC** *expression F also $\mathcal{E}$-provably satisfies $\tilde{x} = \tilde{H}$, then $\mathcal{E} \vdash E = F$.*

PROOF. The claim follows from the following, slightly stronger statement:

> Let $\tilde{x} = x_1, \ldots, x_m$ and $\tilde{w} = w_1, \ldots, w_n$ be disjoint vectors of pairwise distinct variables, and $\tilde{H} = \{H_1, \ldots, H_m\}$ be well-formed expressions with free variables in $\tilde{x} \cup \tilde{w}$ in which each variable x_i is action guarded. Consider the equation set $\tilde{x} = \tilde{H}$. Then there exists an expression $E \in$ **TC** which $\mathcal{E}$-provably satisfies it. Moreover, if F also $\mathcal{E}$-provably satisfies $\tilde{x} = \tilde{H}$, then $\mathcal{E} \vdash E = F$.

This is proven by induction on m by a simple reworking of the proof of Theorem 5.7 in [14]. The interested reader will have no difficulty in filling in the details following Milner's proof. □

We are now in a position to prove the completeness of $\mathcal{E}$.

THEOREM 9 (COMPLETENESS). *For all* **TC** *expressions E, F, $E \sim F$ implies $\mathcal{E} \vdash E = F$.*

PROOF. By Theorem 6, E may be proved to satisfy a standard equation set; likewise F. By Theorem 7, E and F may be proved to satisfy a single standard equation set. Finally, Theorem 8 ensures that $\mathcal{E} \vdash E = F$. □

5 Comparison with Wang's axiomatization

In this section we show that the theory $\mathcal{E}$ is strictly stronger than Wang's $\mathcal{F}$ over **TC**. More precisely, we shall prove that if $\mathcal{F}$ proves an equality $E = F$, then so does $\mathcal{E}$. On the other hand, $\mathcal{F}$ is not strong enough to prove the new persistency axiom (P).

PROPOSITION 10. *For all $E, F \in$ **TC**, $\mathcal{F} \vdash E = F$ implies $\mathcal{E} \vdash E = F$.*

PROOF. A straightforward induction on the length of the proof of the equation $E = F$ from the theory $\mathcal{F}$. Note that axiom (AP) is an instance of axiom (P), and that an application of axiom (NP) can be mimicked using (P) and (S1)–(S4). □

PROPOSITION 11. $\mathcal{F} \nvdash E = E + \varepsilon(t).E$

PROOF. Define a denotational semantics for **TC** in the domain $\{0, 1, 2\}$ with the semantics:

$$
\begin{aligned}
[\![x]\!]\rho &= \rho(x)\\
[\![\mathbf{0}]\!]\rho &= 0\\
[\![\mu.E]\!]\rho &= 2\\
[\![\varepsilon(0).E]\!]\rho &= [\![E]\!]\rho\\
[\![\varepsilon(c).E]\!]\rho &= \begin{cases} 0 & \text{if } [\![E]\!]\rho = 0 \\ 2 & \text{otherwise} \end{cases}\\
[\![E + F]\!]\rho &= \max([\![E]\!]\rho, [\![F]\!]\rho)\\
[\![\mathbf{fix}(x = E)]\!]\rho &= \text{the least fixed point of the function } \lambda d.\ [\![E]\!]\rho[x \mapsto d]
\end{aligned}
$$

where $\rho : \mathsf{Var} \to \{0, 1, 2\}$, and $\rho[x \mapsto d]$ stands for the function that maps x to d and agrees with ρ on all the other variables.

Note that, because of our requirement that expressions be well-formed, the function $\lambda d.\ [\![E]\!]\rho[x \mapsto d]$ used in the definition of the semantics of recursive expressions has always a *unique* fixed point. It is now simple to check that this is a model for $\mathcal{F}$, but

$$[\![x + \varepsilon(c).x]\!](\lambda x.1) = 2 \neq 1 = [\![x]\!](\lambda x.1)$$

and so it is not a model for $\mathcal{E}$. □

However, all the closed instantiations of (P) can be derived from $\mathcal{F}$, as the following proposition shows.

PROPOSITION 12. *For every $P \in \mathbf{TC}_0$, $\mathcal{F} \vdash P = P + \varepsilon(t).P$.*

PROOF. By Theorem 6, for some finite index set I, actions $\mu_i \in \mathsf{Act}$, delays $t_i \in \mathsf{Tim}$ and processes $P_i \in \mathbf{TC}_0$:

$$\mathcal{G} \vdash P = \sum_{i \in I} \varepsilon(t_i).\mu_i.P_i \qquad (19)$$

Now:

$$
\begin{aligned}
&\mathcal{F} \vdash P\\
&\quad = \sum_{i \in I} \varepsilon(t_i).\mu_i.P_i && (19)\\
&\quad = \sum_{i \in I} \varepsilon(t_i).(\mu_i.P_i + \varepsilon((t + t_i) - t_i).\mu_i.P_i) && (\text{AP, or MP if } \mu_i = \tau)\\
&\quad = \sum_{i \in I} \varepsilon(t_i).\mu_i.P_i + \varepsilon(t_i).\varepsilon((t + t_i) - t_i).\mu_i.P_i && (\text{TD})\\
&\quad = \sum_{i \in I} \varepsilon(t_i).\mu_i.P_i + \varepsilon(t_i + ((t + t_i) - t_i)).\mu_i.P_i && (\text{TA})
\end{aligned}
$$

$$= \sum_{i\in I} \varepsilon(t_i).\mu_i.P_i + \varepsilon(t+t_i).\mu_i.P_i \quad (t+(u-t)=u)$$
$$= \sum_{i\in I} \varepsilon(t_i).\mu_i.P_i + \varepsilon(t).\varepsilon(t_i).\mu_i.P_i \quad \text{(TA)}$$
$$= \sum_{i\in I} \varepsilon(t_i).\mu_i.P_i + \sum_{i\in I} \varepsilon(t).\varepsilon(t_i).\mu_i.P_i \quad \text{(S1,S2)}$$
$$= \sum_{i\in I} \varepsilon(t_i).\mu_i.P_i + \varepsilon(t).\sum_{i\in I} \varepsilon(t_i).\mu_i.P_i \quad \text{(TD,NP)}$$
$$= P + \varepsilon(t).P \quad (19)$$

Thus $\mathcal{F}$ can show any closed instantiation of axiom (P). □

Note that throughout the above proof we have been careful not to assume that the monoidal operation $+$ on the time domain is commutative. Although this is true for most of the examples of time domain one encounters in the literature, it does not hold for, e.g., the time domain of the countable ordinals $(\omega_1, +, 0)$.

6 Concluding remarks

In this paper, we have presented a complete axiomatization of timed bisimulation equivalence over open terms with finite-state recursion in a generalization of the regular subcalculus of Wang's timed CCS. Our inference system for timed bisimulation equivalence is obtained by combining an improved version of Wang's complete axiomatization for finite trees [21] with standard laws for recursively defined processes. The proof of completeness of the proposed axiomatization uses an adaptation of Milner's classic arguments presented in [14, 16].

The axiomatization we have presented is parametric with respect to the chosen time domain, and will hold for many of the models of time that have been considered in the literature on timed process algebras, e.g., the natural numbers, the non-negative rationals and the non-negative reals. The definition of time domain that we have chosen in this paper is due to Jeffrey, Schneider and Vaandrager [12] and suits the purpose of this paper well. However, it is certainly not the only one possible, and several ones have been proposed in the literature (see [4] for a series of examples).

Complete axiomatizations of behavioural equivalences for several timed process algebras have been presented in the literature; see, e.g., [6, 9, 13, 17, 18, 21] for examples of such results. With the notable exception of the one presented in [9], all the aforementioned axiomatizations are restricted to recursion-free processes. Hennessy and Regan's axiomatization of their behavioural precongruence over the language TPL includes an infinitary conditional equation, the so-called ω-*induction rule*, whose validity is justified by the algebraicity [7] of their testing-based semantics. To the best of our knowledge, the work reported in this paper is the first to offer a finitary complete axiomatization for a class of timed behaviours with finite-state recursion.

The axiomatization of strong bisimulation equivalence presented by Milner in [14] is complete for arbitrary regular CCS expressions. Milner's inference system deals with unguarded recursive expressions by means of the law:

$$\mathbf{fix}(x = E + x) = \mathbf{fix}(x = E) \quad (20)$$

Such a law, however, is not sound with respect to timed bisimulation. For example, $\mathbf{fix}(x = a.\mathbf{0} + x)$ is not timed bisimulation equivalent to $\mathbf{fix}(x = a.\mathbf{0})$, as the latter can delay whereas the former cannot. We conjecture that our complete axiomatization of timed bisimulation can be extended to arbitrary timed regular expressions by extending the language **TC** with a new constant $\mho$ denoting the *time stop*, i.e., a process that cannot perform any action, and, unlike **0**, is not allowed to delay. Using $\mho$, we could then write a version of law (20) as follows:

$$\mathbf{fix}(x = E + x) = \mathbf{fix}(x = E + \mho)$$

The time stop process could then be axiomatized by means of the laws:

$$\mho + \varepsilon(c).E = \mho$$
$$\mho + \tau.E = \tau.E$$

It is interesting to note that axiom (MP) is derivable from these two laws for $\mho$.

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