A Fully Abstract May Testing Semantics for Concurrent Objects

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Abstract

This paper provides a fully abstract semantics for a variant of the concurrent object calculus. We define may testing for concurrent object components and then characterise it using a trace semantics inspired by UML interaction diagrams. The main result of this paper is to show that the trace semantics is fully abstract for may testing. This is the first such result for a concurrent object language.

1 Introduction

Abadi and Cardelli's [1] object calculus is a minimal language for investigating features of object languages such as encapsulated state, subtyping, and self variables. Gordon and Hankin [7] added concurrent features to the object calculus, to produce the concurrent object calculus.

Prior work on the object calculus has concentrated on the operational behaviour of object systems, and type systems which provide type safety guarantees. The closest paper to ours is Gordon and Rees's [8] fully abstract semantics for the immutable single-threaded object calculus. There has been no work on providing fully abstract semantics for concurrent mutable objects.

In this paper, we present the first fully abstract testing semantics for a variant of Gordon and Hankin's concurrent object calculus without subtyping. The lack of subtyping here affords a simpler presentation of the labelled transitions and traces but we anticipate that the proof techniques used here are robust enough to cater for subtyping also. This semantics was inspired by UML interaction diagrams [4], which are a common tool for visualising interactions with object systems.

1.1 Interaction diagrams

Interaction diagrams (in particular sequence diagrams) were developed by Jacobson, and are now part of the Unified Modeling Language standard [4]. Interaction diagrams record the messages sent between objects of a component in an object system. These messages include method calls

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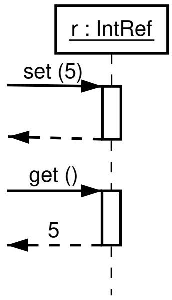
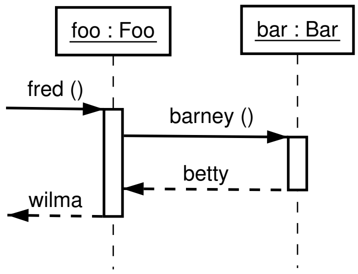
and returns (interaction diagrams include other forms of message, but we will not use these in this paper).

A simple interaction with an integer reference object `r` of type `IntRef` has it receive two incoming method calls `set(5)` and `get()`, for which it produces appropriate return values:



A more complex interaction allows a method call on one object to call methods on other objects:



Here, the object `foo` has one incoming call to `fred()`, makes one outgoing call to `barney()`, receives the result `betty` back, then returns `wilma` itself. This illustrates the four messages which may be sent during an interaction: incoming and outgoing method calls, and matching outgoing and incoming returns.

In this paper, we use a textual representation of an interaction, as a trace, which is just a sequence of messages. In the above example, `foo` has the trace:

$$\begin{array}{l}\langle \mathsf{call\,foo.fred()} \rangle ? \\ \langle \mathsf{call\,bar.barney()} \rangle ! \\ \langle \mathsf{return\,betty} \rangle ? \\ \langle \mathsf{return\,wilma} \rangle !\end{array}$$

where we mark incoming messages with ? and outgoing messages with !. The object `bar` has the matching trace:

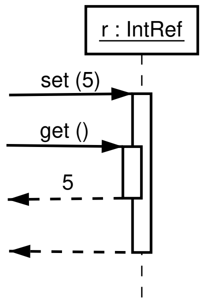
$$\begin{array}{l}\langle \mathsf{call\,bar.barney()} \rangle ? \\ \langle \mathsf{return\,betty} \rangle !\end{array}$$

and so composing these two traces together, we get that the whole system has the trace:

$$\begin{array}{l}\langle \mathsf{call\,foo.fred()} \rangle ? \\ \langle \mathsf{return\,wilma} \rangle !\end{array}$$

There are two additions we will make to the UML message notation: adding thread identifiers, and making name scope more explicit.

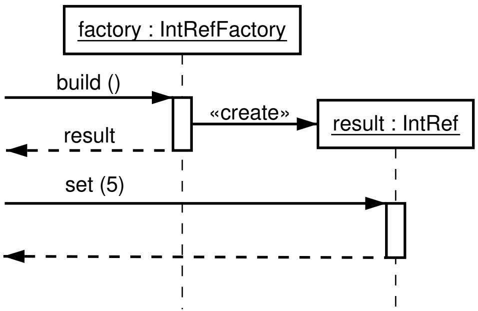
Sequence diagrams can be used for multithreaded applications, for example:



Here, two threads independently call methods of the object `r`, creating a race condition. In our textual representation, we give the threads names, and we decorate each message with the thread responsible for the message:

$$\begin{array}{l}\mathsf{thread1}\langle\mathsf{call\,r.set}(5)\rangle?\\ \mathsf{thread2}\langle\mathsf{call\,r.get}()\rangle?\\ \mathsf{thread2}\langle\mathsf{return\,5}\rangle!\\ \mathsf{thread1}\langle\mathsf{return}\rangle!\end{array}$$

The other addition we make to the notation is to make the scope of names more explicit. For example, consider the following interaction with a factory object, which builds new integer reference objects:



In the textual representation of this trace, we need to make clear that the `result` object has not been seen before by the environment (it is a genuinely new object, not a recycled object). We do this by decorating the label with ν to indicate that the `result` object is new:

$$\begin{array}{l}\mathsf{thread1}\langle\mathsf{call\,factory.build}()\rangle?\\ \nu(\mathsf{result}:\mathsf{IntRef})\,.\,\mathsf{thread1}\langle\mathsf{return\,result}\rangle!\\ \mathsf{thread1}\langle\mathsf{call\,result.set}(5)\rangle?\\ \mathsf{thread1}\langle\mathsf{return}\rangle!\end{array}$$

As well as allowing the system to generate new names on outgoing messages, we allow the environment to generate new names on incoming messages. This style of dealing with fresh names comes originally from the π-calculus [19, 18], and has since been used in other languages, notably the ν-calculus [23].

We have now presented informally all of the machinery required by our semantics for objects:

- The semantics of a system is given by a set of traces, where a trace is a sequence of messages corresponding to one interaction.

- Messages are incoming or outgoing message calls, or matching outgoing or incoming returns.
- Messages are decorated with thread identifiers.
- Messages may include fresh names.

We have only used a very small subset of sequence diagrams, which in turn is a very small subset of UML, but in this paper we will show that this small subset is very expressive, and in particular provides a fully abstract semantics.

1.2 The object calculus

The object calculus is a minimal language for modelling object-based programming. Abadi and Cardelli [1] provided a type system and operational semantics for a variety of object calculi, and proved type safety for them. Gordon and Hankin [7] have since extended this language to include concurrent features.

In this paper, we shall investigate a variant of Gordon and Hankin's concurrent object calculus, which includes:

- A heap of named objects and threads.
- Threads can call or update object methods, can compare object or thread names for equality, can create new objects and threads and can discover their own thread name.
- An operational semantics based on the π-calculus [19, 18], and a simple type system.
- A trace semantics as discussed in Section 1.1.

We are not considering many of the more advanced features of the object calculus or the concurrent object calculus, such as recursive types, object cloning and object locking. This is just for simplicity, and we do not see any technical problems with incorporating these features into our language.

In another strand of research Di Blasio and Fisher [3] also designed a calculus for modelling imperative, concurrent object-based systems. As with Abadi and Cardelli's object calculus and its various extensions, the emphasis in Di Blasio and Fisher's work is again on type systems and safety properties for them.

1.3 Full abstraction

The problem of full abstraction was first introduced by Milner [17], and investigated in depth by Plotkin [24]. Full abstraction was first proposed for variants of the λ-calculus, but has since been investigated for process algebras [9], the π-calculus [6, 10], the ν-calculus [23, 14], Concurrent ML [5, 15], and the immutable object calculus [8].

One way to define a semantics for a programming language is to define:

- A language of *typed components* C which can be *composed* $C_1 \parallel C_2$. (In this paper, components are programs in the concurrent object calculus.)
- A notion of when a component *is successful*. (In this paper, we use a special `succ` method call to indicate a successful component although the theory is robust enough that any other suitable observable would suffice).

We can then define the *may testing preorder* [21, 9] as $C_1 \sqsubseteq_{\text{may}} C_2$ whenever:

$$\begin{array}{l}\text{for any appropriately typed } C \\ \text{if } C_1 \parallel C \text{ is successful then } C_2 \parallel C \text{ is successful}\end{array}$$

Unfortunately, although it is very simple to define, and is quite intuitive, may testing is often very difficult to reason about directly, because of the quantification over 'any appropriately typed C'. In practice, we require a proof technique which we can use to show results about may testing.

One approach is to use a *trace* semantics, given by defining possible executions of components $C \stackrel{s}{\Longrightarrow} C'$ where s is a sequence of messages. We then write $\mathsf{Traces}(C)$ for the set of all traces of C. We say that:

- Traces are *sound* for may testing when $\mathsf{Traces}(C_1) \subseteq \mathsf{Traces}(C_2)$ implies $C_1 \sqsubseteq_{\text{may}} C_2$.
- Traces are *complete* for may testing when $C_1 \sqsubseteq_{\text{may}} C_2$ implies $\mathsf{Traces}(C_1) \subseteq \mathsf{Traces}(C_2)$.
- Traces are *fully abstract* for may testing when they are both sound and complete.

A fully abstract trace model can be a useful tool in understanding a behavioural equivalence in the sense that, in order to be sound, the traces used to build the model must, at minimum, account for all of the possible interactions a system of objects may have with its environment and, in order to be complete, the interactions described by the traces must be genuine. This is taken to mean that for each interaction described by a trace there is an actual system of objects which can play the role of the environment in that interaction. Therefore, to obtain a fully abstract trace model it is necessary to describe all possible interactions accurately.

Establishing full abstraction for a language which includes features such as higher-order programming, new name generation, and heap-based objects is often non-trivial. For example, Pitts and Stark introduced the ν-calculus [23], as a minimal higher-order language with name generation, by extending the simply typed λ-calculus with an abstract type of names, together with a name generator and an equality test. Even this minimal language is remarkably difficult to reason about, and there is no known fully abstract semantics for it [15].

1.4 Contribution of this paper

In this paper, we present a variant of Gordon and Hankin's concurrent object calculus, which is in turn an extension of Abadi and Cardelli's object calculus. The only significant departures from Gordon and Hankin's concurrent object calculus is that we use named threads, where they use anonymous threads and we restrict the calculus to disallow subtyping and recursive types. Whilst this latter restriction does move us away from the essence of object-oriented programming it is imposed so as to keep the technical presentation as simple as possible at this stage. The re-introduction of these features into the type system would affect the behavioural theory in what we expect to be a predictable way and anticipate that techniques employed in [11] and those presented here can be combined to give a similar treatment for a concurrent object language with subtyping.

Components:	C	$::=$	$\mathbf{0} \mid C \parallel C \mid \nu(n:T)\,.\,C \mid n[O] \mid n\langle t\rangle$
Objects:	O	$::=$	$l = M, \ldots, l = M$
Methods:	M	$::=$	$\varsigma(n:T)\,.\,\lambda(x:T,\ldots,x:T)\,.\,\langle t\rangle$
Threads:	t	$::=$	$v \mid \mathsf{stop} \mid \mathsf{let}\ x:T = e\ \mathsf{in}\ t$
Expressions:	e	$::=$	$t \mid \mathsf{if}\ v = v\ \mathsf{then}\ e\ \mathsf{else}\ e \mid v.l(v,\ldots,v) \mid n.l \Leftarrow M \mid$ $\mathsf{new}[O] \mid \mathsf{new}\langle t\rangle \mid \mathsf{currentthread}$
Values:	v	$::=$	$x \mid n$
Types:	T	$::=$	$\mathsf{thread} \mid \mathsf{none} \mid [l:L,\ldots,l:L]$
Method types:	L	$::=$	$(T,\ldots,T) \to T$

We assume grammars for variables x, y, names n, p and method identifiers l.
In objects and object types, we require method identifiers l to be unique, and viewed up to reordering.

Figure 1: Syntax of the concurrent object calculus

We provide the calculus with an operational semantics, and a trace semantics, and then show that the trace semantics is fully abstract for may testing. This is the first full abstraction result for a concurrent object-based language.

2 Concurrent objects

In this section, we will present the syntax, static semantics and dynamic semantics of our concurrent object calculus. This is a variant of Gordon and Hankin's concurrent object calculus with named rather than anonymous threads.

2.1 Syntax

The syntax for the concurrent object calculus we will use in this paper is given in Figure 1. We make use of a number of distinct syntactic categories of identifiers, namely, object and thread *Names*, ranged over by n and p (the latter is typically used to indicate an object), *Variables*, ranged over by x, y, z, and *Method Identifiers*, ranged over by l. The operators let and λ act as binders for *Variables* and ς and ν act as binders for *Names*. *Method Identifiers* can not be bound. Note that, at the level of components, there is no facility for binding variables. We will work with terms up to α-conversion of both *Names* and *Variables* in the conventional way. We also make use of capture-free substitution of values for variables or names for names, again defined in the conventional way, and written $t[v/x]$ or $t[p/n]$ as appropriate.

In examples, we will often make use of base types such as integers and booleans: these are not part of our formal system, but will make examples easier to present. They could be comfortably included in the language without changing the theory significantly. We will also make use of some syntax sugar:

We will elide types from variable and name binders, where they can be reconstructed. We write $e;t$ as syntax sugar for $\mathsf{let}\ x = e\ \mathsf{in}\ t$ when x is a fresh variable. We use Abadi and Cardelli's definition of fields f as zero-argument methods:

- A field declaration $f = v$ in an object is syntax sugar for a method declaration $f = \varsigma(n : T) . \lambda() . \langle v \rangle$.
- A field type $f : T$ in an object type is syntax sugar for a method type $f : () \to T$.
- A field access expression $v.f$ is syntax sugar for a method call $v.f()$.
- A field update expression $n.f := v'$ is syntax sugar for a method update $n.f \Leftarrow (\varsigma(p : T) . \lambda() . \langle v \rangle)$.

In addition, we have restricted many subexpressions of an expression to be values rather than full expressions, for example in a method call $v.l(\vec{v})$ we require the object and the arguments to be values rather than expressions $e.l(\vec{e})$. This makes the operational semantics much easier to define, and does not restrict the expressivity of the language, for example we can define $(e.l(\vec{e})) \equiv (\mathsf{let}\ x = e\ \mathsf{in}\ \mathsf{let}\ \vec{x} = \vec{e}\ \mathsf{in}\ x.l(\vec{x}))$. Similarly, the distinction between threads and expressions makes the operational semantics much simpler, but we can treat any expression as a thread by η-converting it: $\langle e \rangle \equiv \langle \mathsf{let}\ x = e\ \mathsf{in}\ x \rangle$.

For the remainder of this section, we will provide an informal description of the syntax:

A component C is a collection of named objects $n[O]$ and threads $n\langle t \rangle$. For example, one possible component consisting of an integer reference p and a thread n which increments the reference is:

$$
\begin{array}{l}
p[\mathsf{contents} = 5] \parallel \\
n\langle \mathsf{let}\ x = p.\mathsf{contents}\ \mathsf{in}\ p.\mathsf{contents} := x + 1 \rangle
\end{array}
$$

We also use the ν-notation of the π-calculus [18] to indicate which names are private, and not known to the outside world. By default, names are public, and have to be marked by ν in order to be considered private. For example, n is private, and p is public in:

$$
\begin{array}{l}
\nu(n : \mathsf{thread}) . (\\
\quad p[\mathsf{contents} = 5] \parallel \\
\quad n\langle \mathsf{let}\ x = p.\mathsf{contents}\ \mathsf{in}\ p.\mathsf{contents} := x + 1 \rangle \\
)
\end{array}
$$

An object $[O]$ consists of a set of named methods, for example an integer reference with set and get methods might be written:

$$
\begin{array}{l}
[\\
\quad \mathsf{contents} = 5, \\
\quad \mathsf{set} = \varsigma(\mathsf{this} : \mathsf{IntRef}) . \lambda(x : \mathsf{Int}) . \langle \mathsf{this.contents} := x; x \rangle, \\
\quad \mathsf{get} = \varsigma(\mathsf{this} : \mathsf{IntRef}) . \lambda() . \langle \mathsf{this.contents} \rangle \\
]
\end{array}
$$

Each method M consists of a self name as well as a list of parameters and a body. For example, the set method above has self name $(\mathsf{this} : \mathsf{IntRef})$, parameters $(x : \mathsf{Int})$, and body $(\mathsf{this.contents} := x)$. Readers familiar with Abadi and Cardelli's work will note that we are taking parameterized methods as primitive, rather than defining them as syntax sugar. This is necessary for our semantics, which is based on method calls with arguments and return values.

A thread $\langle t \rangle$ consists of a stack of let-expressions, terminated either by a return value:

$$\langle \mathsf{let}\ x_1 : T_1 = e_1\ \mathsf{in}\ \cdots \mathsf{let}\ x_n : T_n = e_n\ \mathsf{in}\ v \rangle$$

or by a deadlocked stop thread:

$$\langle \mathsf{let}\ x_1 : T_1 = e_1\ \mathsf{in}\ \cdots \mathsf{let}\ x_n : T_n = e_n\ \mathsf{in}\ \mathsf{stop} \rangle$$

Each expression is either itself a thread, or:

- an if expression $\mathsf{if}\ v_1 = v_2\ \mathsf{then}\ e_1\ \mathsf{else}\ e_2$,
- a method call $v.l(\vec{v})$,
- a method update $n.l \Leftarrow M$, on a named object
- a new object $\mathsf{new}[O]$,
- a new thread $\mathsf{new}\langle t \rangle$, or
- the current thread name $\mathsf{currentthread}$.

Each value is simply a name or a variable and we defer the discussion of types until Section 2.2.

2.2 Static semantics

The static semantics for our concurrent object calculus is given in Figures 2–6. Most of the rules are straightforward adaptations of those given by Abadi and Cardelli [1]. The main judgement is $\Delta \vdash C : \Theta$ which is read as 'the component C uses names Δ and defines names Θ'. For example, if we define $C_1(v)$, C_2 and IntRef as:

$$\begin{array}{l}
C_1(v) \equiv p[\\
\quad \mathsf{contents} = v,\\
\quad \mathsf{set} = \varsigma(\mathsf{this} : \mathsf{IntRef})\,.\,\lambda(x : \mathsf{Int})\,.\,\langle \mathsf{this.contents} := x; x \rangle,\\
\quad \mathsf{get} = \varsigma(\mathsf{this} : \mathsf{IntRef})\,.\,\lambda()\,.\,\langle \mathsf{this.contents} \rangle\\
]\\
C_2 \equiv n\langle\\
\quad \mathsf{let}\ x = p.\mathsf{get}()\ \mathsf{in}\ p.\mathsf{set}(x+1); \mathsf{stop}\\
\rangle\\
\mathsf{IntRef} \equiv [\\
\quad \mathsf{contents} : \mathsf{Int}, \mathsf{set} : (\mathsf{Int}) \rightarrow \mathsf{Int}, \mathsf{get} : () \rightarrow \mathsf{Int}\\
]
\end{array}$$

then we can deduce (if $v : \mathsf{Int}$):

$$\begin{array}{rcl}
n : \mathsf{thread} & \vdash & C_1(v) : (p : \mathsf{IntRef})\\
p : \mathsf{IntRef} & \vdash & C_2 : (n : \mathsf{thread})\\
& \vdash & (C_1(v) \parallel C_2) : (p : \mathsf{IntRef}, n : \mathsf{thread})\\
& \vdash & \nu(n : \mathsf{thread})\,.\,(C_1(v) \parallel C_2) : (p : \mathsf{IntRef})
\end{array}$$

We will now introduce an important requirement of our components, that they be *write closed*:

$$\frac{}{\Delta \vdash \mathbf{0} : ()} \qquad \frac{;\Delta, n : T \vdash [O] : T}{\Delta \vdash n[O] : (n : T)} \qquad \frac{;\Delta, n : \mathsf{thread} \vdash t : \mathsf{none}}{\Delta \vdash n\langle t \rangle : (n : \mathsf{thread})}$$

$$\frac{\Delta, \Theta_2 \vdash C_1 : \Theta_1 \quad \Delta, \Theta_1 \vdash C_2 : \Theta_2}{\Delta \vdash (C_1 \parallel C_2) : (\Theta_1, \Theta_2)} \qquad \frac{\Delta \vdash C : \Theta, n : T}{\Delta \vdash \nu(n : T) \,.\, C : \Theta}$$

Figure 2: Rules for judgement $\Delta \vdash C : \Theta$

$$\frac{\Gamma; \Delta \vdash M_1 : T.l_1 \quad \cdots \quad \Gamma; \Delta \vdash M_k : T.l_k}{\Gamma; \Delta \vdash [l_1 = M_1, \ldots, l_k = M_k] : T}$$

Figure 3: Rule for judgement $\Gamma; \Delta \vdash [O] : T$ (when $T = [l_1 : L_1, \ldots, l_k : L_k]$)

$$\frac{\Gamma, x_1 : T_1, \ldots, x_k : T_k; \Delta, n : T \vdash t : U}{\Gamma; \Delta \vdash \varsigma(n : T) \,.\, \lambda(x_1 : T_1, \ldots, x_k : T_k) \,.\, \langle t \rangle : T.l}$$

Figure 4: Rule for judgement $\Gamma; \Delta \vdash M : T.l$ (when $T = [\ldots, l : (T_1, \ldots, T_k) \to U, \ldots]$ and $T.l$ is the record l selected from T)

$$\frac{\begin{array}{c}\Gamma; \Delta \vdash v_1 : T_1 \quad \Gamma; \Delta \vdash v_2 : T_1 \\ \Gamma; \Delta \vdash e_1 : T_2 \quad \Gamma; \Delta \vdash e_2 : T_2\end{array}}{\Gamma; \Delta \vdash \mathsf{if}\ v_1 = v_2\ \mathsf{then}\ e_1\ \mathsf{else}\ e_2 : T_2}$$

$$\frac{\begin{array}{c}\Gamma; \Delta \vdash v : [\ldots, l : (T_1, \ldots, T_k) \to T, \ldots] \\ \Gamma; \Delta \vdash v_1 : T_1 \quad \cdots \quad \Gamma; \Delta \vdash v_k : T_k\end{array}}{\Gamma; \Delta \vdash v.l(v_1, \ldots, v_k) : T} \qquad \frac{\Gamma; \Delta \vdash n : T \quad \Gamma; \Delta \vdash M : T.l}{\Gamma; \Delta \vdash n.l \Leftarrow M : T}$$

$$\frac{\Gamma; \Delta \vdash [O] : T}{\Gamma; \Delta \vdash \mathsf{new}[O] : T} \qquad \frac{\Gamma; \Delta \vdash t : T}{\Gamma; \Delta \vdash \mathsf{new}\langle t \rangle : \mathsf{thread}} \qquad \frac{}{\Gamma; \Delta \vdash \mathsf{currentthread} : \mathsf{thread}}$$

$$\frac{\Gamma; \Delta \vdash e : T_1 \quad \Gamma, x : T_1; \Delta \vdash t : T_2}{\Gamma; \Delta \vdash \mathsf{let}\ x : T_1 = e\ \mathsf{in}\ t : T_2} \qquad \frac{}{\Gamma; \Delta \vdash \mathsf{stop} : T} \qquad \frac{}{\Gamma, x : T, \Gamma'; \Delta \vdash x : T} \qquad \frac{}{\Gamma; \Delta, n : T, \Delta' \vdash n : T}$$

Figure 5: Rules for judgement $\Gamma; \Delta \vdash e : T$

Variable contexts: $\Gamma \ ::= \ x : T, \ldots, x : T$ Name contexts: $\Delta, \Theta, \Sigma, \Phi \ ::= \ n : T, \ldots, n : T$

In variable contexts, variables must be unique, and are viewed up to reordering.
In name contexts, names must be unique, types must not be none, and are viewed up to reordering.

Figure 6: Syntax of name and variable contexts

Whenever $\Delta \vdash C : \Theta$ contains a subexpression of the form $n.l \Leftarrow M$ with n free, then n appears in Θ.

This is intended to capture the common software engineering requirement that components should not export mutable fields, instead they should export suitable get and set methods. For example, the configurations C_1 and C_2 above are write closed, since the only updates are to this, but the following component which writes directly to p.contents is not write closed:

$$C_2' \equiv n\langle \mathsf{let}\ x = p.\mathsf{contents}\ \mathsf{in}\ p.\mathsf{contents} := x+1; \mathsf{stop}\rangle$$

For the remainder of the paper we will require components to be write closed. This makes developing a fully abstract semantics much simpler, since we do not need to model method update directly.

2.3 Dynamic semantics

The dynamic semantics for our concurrent object calculus is given in Figures 7–10.

We define three relations between components:

- $\equiv$, structural congruence, represents the least congruence on components which includes the axioms in Figure 7.
- $C \xrightarrow{\tau} C'$ when C can reduce to C' by the interaction of a thread and an object (either a method call or a method update).
- $C \xrightarrow{\beta} C'$ when C can reduce to C' by a thread acting independently of any other threads or objects.

We write $C \rightarrow C'$ when either $C \xrightarrow{\tau} C'$ or $C \xrightarrow{\beta} C'$; we write $C \Rightarrow C'$ when $C \rightarrow^* C'$.

The important property of β-reductions is that they do not introduce race conditions (and hence nondeterminism), where τ-reductions may introduce race conditions. This is discussed further in Appendix B.1.

For example, recalling the definition of $C_1(v)$ from Section 2.2 we have:

$$\begin{aligned}
& C_1(5) \parallel n\langle \mathsf{let}\ x = p.\mathsf{get}()\ \mathsf{in}\ p.\mathsf{set}(x+1); \mathsf{stop}\rangle \\
& \quad \xrightarrow{\tau} C_1(5) \parallel n\langle \mathsf{let}\ x = p.\mathsf{contents}\ \mathsf{in}\ p.\mathsf{set}(x+1); \mathsf{stop}\rangle \\
& \quad \xrightarrow{\tau} C_1(5) \parallel n\langle \mathsf{let}\ x = 5\ \mathsf{in}\ p.\mathsf{set}(x+1); \mathsf{stop}\rangle \\
& \quad \xrightarrow{\beta}{}^* C_1(5) \parallel n\langle p.\mathsf{set}(6); \mathsf{stop}\rangle \\
& \quad \xrightarrow{\tau} C_1(5) \parallel n\langle p.\mathsf{contents} := 6; 6; \mathsf{stop}\rangle \\
& \quad \xrightarrow{\tau} C_1(6) \parallel n\langle p; 6; \mathsf{stop}\rangle \\
& \quad \xrightarrow{\beta}{}^* C_1(6) \parallel n\langle \mathsf{stop}\rangle
\end{aligned}$$

as expected.

Proposition 2.1 (Subject Reduction) *If* $\Delta \vdash C : \Theta$ *and* $C \Rightarrow C'$ *then* $\Delta \vdash C' : \Theta$

Proof: Straightforward. □

$$\mathbf{0} \parallel C \equiv C \qquad (C_1 \parallel C_2) \parallel C_3 \equiv C_1 \parallel (C_2 \parallel C_3) \qquad C_1 \parallel C_2 \equiv C_2 \parallel C_1$$
$$C_1 \parallel \nu(n:T).C_2 \equiv \nu(n:T).(C_1 \parallel C_2) \qquad \nu(n_1:T_1).\nu(n_2:T_2).C \equiv \nu(n_2:T_2).\nu(n_1:T_1).C$$

Figure 7: Axioms for structural congruence (where n is not free in C_1)

$$\begin{array}{rcll}
n\langle \mathsf{let}\ x:T = v\ \mathsf{in}\ t\rangle & \xrightarrow{\beta} & n\langle t[v/x]\rangle & \\
n\langle \mathsf{let}\ x:T = (\mathsf{let}\ x_1:T_1 = e_1\ \mathsf{in}\ e_2)\ \mathsf{in}\ t\rangle & \xrightarrow{\beta} & n\langle \mathsf{let}\ x_1:T_1 = e_1\ \mathsf{in}\ (\mathsf{let}\ x:T = e_2\ \mathsf{in}\ t)\rangle & \\
n\langle \mathsf{let}\ x:T = (\mathsf{if}\ v = v\ \mathsf{then}\ e_1\ \mathsf{else}\ e_2)\ \mathsf{in}\ t\rangle & \xrightarrow{\beta} & n\langle \mathsf{let}\ x:T = e_1\ \mathsf{in}\ t\rangle & \\
n\langle \mathsf{let}\ x:T = (\mathsf{if}\ v_1 = v_2\ \mathsf{then}\ e_1\ \mathsf{else}\ e_2)\ \mathsf{in}\ t\rangle & \xrightarrow{\beta} & n\langle \mathsf{let}\ x:T = e_2\ \mathsf{in}\ t\rangle & (v_1 \neq v_2) \\
n\langle \mathsf{let}\ x:T = \mathsf{new}[O]\ \mathsf{in}\ t\rangle & \xrightarrow{\beta} & \nu(p:T).(p[O] \parallel n\langle \mathsf{let}\ x:T = p\ \mathsf{in}\ t\rangle) & (p \notin O \text{ or } t) \\
n\langle \mathsf{let}\ x:T = \mathsf{new}\langle f\rangle\ \mathsf{in}\ t\rangle & \xrightarrow{\beta} & \nu(p:T).(p\langle f\rangle \parallel n\langle \mathsf{let}\ x:T = p\ \mathsf{in}\ t\rangle) & (p \notin t \text{ or } f) \\
n\langle \mathsf{let}\ x:T = \mathsf{currentthread}\ \mathsf{in}\ t\rangle & \xrightarrow{\beta} & n\langle \mathsf{let}\ x:T = n\ \mathsf{in}\ t\rangle & \\
n\langle \mathsf{let}\ x:T = \mathsf{stop}\ \mathsf{in}\ t\rangle & \xrightarrow{\beta} & n\langle \mathsf{stop}\rangle & \\
p[O] \parallel n\langle \mathsf{let}\ x:T = p.l(\vec{v})\ \mathsf{in}\ t\rangle & \xrightarrow{\tau} & p[O] \parallel n\langle \mathsf{let}\ x:T = O.l(p)(\vec{v})\ \mathsf{in}\ t\rangle & \\
p[O] \parallel n\langle \mathsf{let}\ x:T = p.l \Leftarrow M\ \mathsf{in}\ t\rangle & \xrightarrow{\tau} & p[O.l \Leftarrow M] \parallel n\langle \mathsf{let}\ x:T = p\ \mathsf{in}\ t\rangle &
\end{array}$$

Figure 8: Axioms for reduction precongruence

$$\frac{C \equiv\ \xrightarrow{\beta}\ \equiv C'}{C \xrightarrow{\beta} C'} \qquad \frac{C \xrightarrow{\beta} C'}{C \parallel C'' \xrightarrow{\beta} C' \parallel C''} \qquad \frac{C \xrightarrow{\beta} C'}{\nu(n:T).C \xrightarrow{\beta} \nu(n:T).C'}$$
$$\frac{C \equiv\ \xrightarrow{\tau}\ \equiv C'}{C \xrightarrow{\tau} C'} \qquad \frac{C \xrightarrow{\tau} C'}{C \parallel C'' \xrightarrow{\tau} C' \parallel C''} \qquad \frac{C \xrightarrow{\tau} C'}{\nu(n:T).C \xrightarrow{\tau} \nu(n:T).C'}$$

Figure 9: Rules for reduction precongruence

$$(\vec{l} = \vec{M}, l = M).l(p)(\vec{v}) = t[p/n, \vec{v}/\vec{x}] \qquad (\vec{l} = \vec{M}, l = M').l \Leftarrow M = (\vec{l} = \vec{M}, l = M)$$

Figure 10: Definition of $O.l(p)(\vec{v})$ and $O.l \Leftarrow M$ where $M = \varsigma(n:T).\lambda(\vec{x}:\vec{T}).\langle t\rangle$

2.4 Testing preorder

We will now define the testing semantics for our concurrent object calculus. We will do this by defining a notion of *barb* for a component, and let a successful component be one which communicates on that barb. This is similar to the use of barbs in process algebra [20].

Let the type barb be defined:

$$\mathsf{barb} = [\mathsf{succ} : () \to \mathsf{none}]$$

for some fresh method name succ. We say that a component *strongly barbs* on $b : \mathsf{barb}$ written $C{\downarrow_b}$ if and only if:

$$C \equiv \nu(\vec{n} : \vec{T})\,.\,(C' \parallel n\langle \mathsf{let}\ x : \mathsf{none} = b.\mathsf{succ}()\ \mathsf{in}\ t\rangle)$$

for $b \notin \vec{n}$ and *barbs* on $b : \mathsf{barb}$ written $C{\Downarrow_b}$ if and only if:

$$C \Rightarrow C'{\downarrow_b}$$

For components C_1 and C_2 such that $\Delta \vdash C_1 : \Theta$ and $\Delta \vdash C_2 : \Theta$, we define the may testing preorder $\Delta \models C_1 \sqsubseteq_{\mathrm{may}} C_2 : \Theta$ if and only if:

$$\text{for any } \Delta', \Theta, b : \mathsf{barb} \vdash C : \Delta \text{ if } (C_1 \parallel C){\Downarrow_b} \text{ then } (C_2 \parallel C){\Downarrow_b}$$

This is a straightforward adaptation of the standard [9] definition of may testing for concurrent systems.

3 Trace semantics

The trace semantics for the concurrent object calculus is given by a labelled transition system (lts) with judgements:

$$(\Delta \vdash C : \Theta) \xrightarrow{\alpha} (\Delta' \vdash C' : \Theta')$$

The lts is given for components extended by introducing two new expressions:

$$e ::= \cdots \mid \mathsf{block} \mid \mathsf{return}\,(v : T)$$

These new threads are included purely to assist in the description of the lts and are intended to represent a command for a thread to wait for some unknown interaction with the environment and a command for a thread to report a value to the environment and then to go back to a blocked state. There are no reductions associated with these commands and they may be typed as:

$$\frac{}{\Gamma;\Delta \vdash \mathsf{block} : T} \qquad \frac{\Gamma;\Delta \vdash v : U}{\Gamma;\Delta \vdash \mathsf{return}\,(v : U) : T}$$

where T and U are any types. The lts for our concurrent object language are given in Figures 11–14.

For example if we define:

$$\Theta \equiv (p : \mathsf{IntRef})$$
$$\Theta' \equiv (p : \mathsf{IntRef}, n : \mathsf{thread})$$

$$(\Delta, n : \mathsf{thread} \vdash C : \Theta) \xrightarrow{n\langle \mathsf{call}\, p.l(\vec{v})\rangle ?} (\Delta \vdash C \parallel n\langle \mathsf{let}\, x : T = p.l(\vec{v})\ \mathsf{in}\ \mathsf{return}\,(x : T)\rangle : (n : \mathsf{thread}, \Theta))$$
$$\text{(when } ;\Delta, n : \mathsf{thread}, \Theta \vdash p.l(\vec{v}) : T \text{ and } p \in \Theta)$$

$$(\Delta \vdash C \parallel n\langle \mathsf{let}\, x : T = \mathsf{block}\ \mathsf{in}\ t\rangle : \Theta) \xrightarrow{n\langle \mathsf{call}\, p.l(\vec{v})\rangle ?} (\Delta \vdash C \parallel n\langle \mathsf{let}\, y : U = p.l(\vec{v})\ \mathsf{in}\ \mathsf{let}\, x : T = \mathsf{return}\,(y : U)\ \mathsf{in}\ t\rangle : \Theta)$$
$$\text{(when } ;\Delta, \Theta \vdash p.l(\vec{v}) : U \text{ and } p \in \Theta)$$

$$(\Delta \vdash C \parallel n\langle \mathsf{let}\, x : T = \mathsf{block}\ \mathsf{in}\ t\rangle : \Theta) \xrightarrow{n\langle \mathsf{return}\, v\rangle ?} (\Delta \vdash C \parallel n\langle t[v/x]\rangle : \Theta)$$
$$\text{(when } ;\Delta, \Theta \vdash v : T)$$

$$(\Delta \vdash C \parallel n\langle \mathsf{let}\, x : T = p.l(\vec{v})\ \mathsf{in}\ t\rangle : \Theta) \xrightarrow{n\langle \mathsf{call}\, p.l(\vec{v})\rangle !} (\Delta \vdash C \parallel n\langle \mathsf{let}\, x : T = \mathsf{block}\ \mathsf{in}\ t\rangle : \Theta)$$
$$\text{(when } p \in \Delta)$$

$$(\Delta \vdash C \parallel n\langle \mathsf{let}\, x : T = \mathsf{return}\,(v : U)\ \mathsf{in}\ t\rangle : \Theta) \xrightarrow{n\langle \mathsf{return}\, v\rangle !} (\Delta \vdash C \parallel n\langle \mathsf{let}\, x : T = \mathsf{block}\ \mathsf{in}\ t\rangle : \Theta)$$

Figure 11: Axioms for labelled transition system $(\Delta \vdash C : \Theta) \xrightarrow{\alpha} (\Delta' \vdash C' : \Theta')$

$$\frac{(\Delta \vdash C : (\Theta, n : T)) \xrightarrow{a} (\Delta' \vdash C' : (\Theta', n : T))}{(\Delta \vdash \nu(n : T)\,.\,C : \Theta) \xrightarrow{a} (\Delta' \vdash \nu(n : T)\,.\,C' : \Theta')} (n \text{ is not free in } a)$$

$$\frac{(\Delta \vdash C : (\Theta, n : T)) \xrightarrow{\gamma !} (\Delta' \vdash C' : \Theta')}{(\Delta \vdash \nu(n : T)\,.\,C : \Theta) \xrightarrow{\nu(n:T).\gamma !} (\Delta' \vdash C' : \Theta')} (n \text{ is free in } \gamma)$$

$$\frac{(\Delta, n : T \vdash C : \Theta) \xrightarrow{\gamma ?} (\Delta' \vdash C' : \Theta')}{(\Delta \vdash C : \Theta) \xrightarrow{\nu(n:T).\gamma ?} (\Delta' \vdash C' : \Theta')} (n \text{ is free in } \gamma, T \text{ is not } \mathsf{none})$$

Figure 12: Rules for labelled transition system $(\Delta \vdash C : \Theta) \xrightarrow{\alpha} (\Delta' \vdash C' : \Theta')$

$$\frac{C \Rightarrow C'}{(\Delta \vdash C : \Theta) \stackrel{\varepsilon}{\Longrightarrow} (\Delta \vdash C' : \Theta)} \qquad \frac{(\Delta \vdash C : \Theta) \xrightarrow{a} (\Delta' \vdash C' : \Theta')}{(\Delta \vdash C : \Theta) \stackrel{a}{\Longrightarrow} (\Delta' \vdash C' : \Theta')}$$

$$\frac{(\Delta \vdash C : \Theta) \stackrel{s}{\Longrightarrow} (\Delta' \vdash C' : \Theta') \stackrel{s'}{\Longrightarrow} (\Delta'' \vdash C'' : \Theta'')}{(\Delta \vdash C : \Theta) \stackrel{ss'}{\Longrightarrow} (\Delta'' \vdash C'' : \Theta'')}$$

Figure 13: Rules for trace semantics $(\Delta \vdash C : \Theta) \stackrel{s}{\Longrightarrow} (\Delta' \vdash C' : \Theta')$

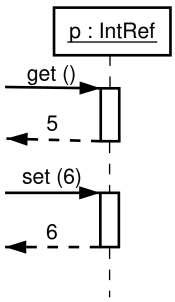
Basic labels:	γ	$::=$	$n\langle \mathsf{call}\, p.l(\vec{v})\rangle \mid n\langle \mathsf{return}\, v\rangle \mid \nu(n : T)\,.\,\gamma$
Visible labels:	a	$::=$	$\gamma ? \mid \gamma !$
Traces:	q, r, s	$::=$	$a \cdots a$

Figure 14: Syntax of labels and traces

then (where $C_1(v)$ is defined in Section 2.2) we have:

$$
\begin{array}{l}
(\vdash C_1(5) : \Theta) \\
\quad \xrightarrow{\nu(n{:}\mathsf{thread}).n\langle \mathsf{call}\, p.\mathsf{get}()\rangle ?} \\
(\vdash (C_1(5) \parallel n\langle \mathsf{let}\, x = p.\mathsf{get}() \;\mathsf{in}\; \mathsf{return}\, x\rangle) : \Theta') \\
\quad \Rightarrow \\
(\vdash (C_1(5) \parallel n\langle \mathsf{return}\, 5\rangle) : \Theta') \\
\quad \xrightarrow{n\langle \mathsf{return}\, 5\rangle !} \\
(\vdash (C_1(5) \parallel n\langle \mathsf{block}\rangle) : \Theta') \\
\quad \xrightarrow{n\langle \mathsf{call}\, p.\mathsf{set}(6)\rangle ?} \\
(\vdash (C_1(5) \parallel n\langle \mathsf{let}\, x = p.\mathsf{set}(6) \;\mathsf{in}\; \mathsf{return}\, x\rangle) : \Theta') \\
\quad \Rightarrow \\
(\vdash (C_1(6) \parallel n\langle \mathsf{return}\, 6\rangle) : \Theta') \\
\quad \xrightarrow{n\langle \mathsf{return}\, 6\rangle !} \\
(\vdash (C_1(5) \parallel n\langle \mathsf{block}\rangle) : \Theta')
\end{array}
$$

which corresponds to the interaction diagram:



For any component $(\Delta \vdash C : \Theta)$ we define its traces to be:

$$\mathsf{Traces}(\Delta \vdash C : \Theta) = \{ s \mid (\Delta \vdash C : \Theta) \stackrel{s}{\Longrightarrow} (\Delta' \vdash C' : \Theta') \}$$

We will now show that this trace semantics is fully abstract for may testing.

4 Soundness of traces for may testing

Having defined our trace semantics we must demonstrate that it provides a sound characterisation of our notion of equivalence, that is, may testing. Specifically we must show that whenever the traces of a well-typed component are contained in another's then the components must be related in the may testing preorder. We immediately see some difficulty in proving this directly as the traces are defined using terms over an extended syntax whereas testing is defined purely in the base language. However, the extensions made to the syntax represent *interaction points*, between a component and a putative testing component. Therefore, given an actual testing component we may *merge* the original component and the test together at these interaction points, thereby recovering the term in the base language which would have been reached had the component and test actually interacted. This operation of merging is defined below:

4.1 The merge operator

Define the partial *merge* operator $C_1 \bowtie C_2$ on components as the symmetric operator defined up to $\equiv$ where:

$$\begin{array}{rcl}
\mathbf{0} \bowtie C & = & C \\
(\nu(p:T).C_1) \bowtie C_2 & = & \nu(p:T).(C_1 \bowtie C_2) \\
(p[O] \parallel C_1) \bowtie C_2 & = & p[O] \parallel (C_1 \bowtie C_2) \\
(p\langle t\rangle \parallel C_1) \bowtie C_2 & = & p\langle t\rangle \parallel (C_1 \bowtie C_2) \\
(n\langle t_1\rangle \parallel C_1) \bowtie (n\langle t_2\rangle \parallel C_2) & = & n\langle t_1 \bowtie t_2\rangle \parallel (C_1 \bowtie C_2)
\end{array}$$

when $n \notin \mathsf{dom}\,(C_1, C_2)$ and $p \notin \mathsf{fn}\,(C_2)$.

We overload notation and define the partial merge operator $t_1 \bowtie t_2$ on threads as the symmetric operator where:

$$\begin{array}{rcl}
(\mathsf{let}\; x:T = \mathsf{block}\;\mathsf{in}\; t) \bowtie \mathsf{stop} & = & \mathsf{stop} \\
(\mathsf{let}\; x:T = \mathsf{block}\;\mathsf{in}\; t_1) \bowtie (\mathsf{let}\; y:U = \mathsf{return}\,(v:T)\;\mathsf{in}\; t_2) & = & (\mathsf{let}\; y:U = \mathsf{block}\;\mathsf{in}\; t_2) \bowtie (t_1[v/x]) \\
(\mathsf{let}\; x:T = \mathsf{block}\;\mathsf{in}\; t_1) \bowtie (\mathsf{let}\; y:U = e\;\mathsf{in}\; t_2) & = & \mathsf{let}\; y:U = e\;\mathsf{in}\;((\mathsf{let}\; x:T = \mathsf{block}\;\mathsf{in}\; t_1) \bowtie t_2)
\end{array}$$

when e is block/return free and $y \notin \mathsf{fv}\,(t_1)$.

Lemma 4.1 *If $\Delta \vdash (C_1 \parallel C_2) : \Theta$ then $(C_1 \bowtie C_2) \equiv (C_1 \parallel C_2)$.*

Proof: An induction on the definition of $C_1 \bowtie C_2$. □

Lemma 4.2 *If $C_1 \bowtie C_2 \equiv C$ and $C_1 {\downarrow_b}$ then $C {\downarrow_b}$.*

Proof: An induction on the definition of $C_1 \bowtie C_2$. □

4.2 Trace composition and decomposition

Given a trace s we write $\bar{s}$ for the complementary trace:

$$\bar{\varepsilon} = \varepsilon \qquad \overline{s_1 s_2} = \bar{s_1}\bar{s_2} \qquad \bar{\gamma?} = \gamma! \qquad \bar{\gamma!} = \gamma?$$

Proposition 4.3 (Trace composition/decomposition) *For any components $(\Delta, \Phi \vdash C_1 : \Theta, \Sigma)$ and $(\Theta, \Phi \vdash C_2 : \Delta, \Sigma)$ such that $C_1 \bowtie C_2 \equiv C$, we have:*

1. *If $(\Delta, \Phi \vdash C_1 : \Theta, \Sigma) \stackrel{s}{\Longrightarrow} (\Delta', \Phi \vdash C_1' : \Theta', \Sigma')$ and $(\Theta, \Phi \vdash C_2 : \Delta, \Sigma) \stackrel{\bar{s}}{\Longrightarrow} (\Theta', \Phi \vdash C_2' : \Delta', \Sigma')$ then $C \Rightarrow C'$ where $\nu(\Delta', \Theta', \Sigma' \setminus \Delta, \Theta, \Sigma).(C_1' \bowtie C_2') \equiv C'$.*

2. *If $C \Rightarrow C'$ then there exists some trace s such that $(\Delta, \Phi \vdash C_1 : \Theta, \Sigma) \stackrel{s}{\Longrightarrow} (\Delta', \Phi \vdash C_1' : \Theta', \Sigma')$ and $(\Theta, \Phi \vdash C_2 : \Delta, \Sigma) \stackrel{\bar{s}}{\Longrightarrow} (\Theta', \Phi \vdash C_2' : \Delta', \Sigma')$ where $\nu(\Delta', \Theta', \Sigma' \setminus \Delta, \Theta, \Sigma).(C_1' \bowtie C_2') \equiv C'$.*

Proof: Given in Appendix A. □

Corollary 4.4 *For any components* $(\Delta,\Phi \vdash C_1 : \Theta,\Sigma)$ *and* $(\Theta,\Phi \vdash C_2 : \Delta,\Sigma)$ *such that* $C_1 \mathbin{/\!\!\!\backslash} C_2 \equiv C$ *and* $C{\Downarrow_b}$ *then there exists some trace s such that* $(\Delta,\Phi \vdash C_1 : \Theta,\Sigma) \stackrel{s}{\Longrightarrow} (\Delta',\Phi \vdash C_1' : \Theta',\Sigma')$ *and* $(\Theta,\Phi \vdash C_2 : \Delta,\Sigma) \stackrel{\bar{s}}{\Longrightarrow} (\Theta',\Phi \vdash C_2' : \Delta',\Sigma')$ *where either* $C_1'{\downarrow_b}$ *or* $C_2'{\downarrow_b}$.

Proof: We know that $C{\Downarrow_b}$ which tells us that $C \Rightarrow C''$ for some C'' such that $C''{\downarrow_b}$. We use Proposition 4.3 Part 2, to obtain a trace s_1 such that

$$\begin{array}{l}(\Delta,\Phi \vdash C_1 : \Theta,\Sigma) \stackrel{s_1}{\Longrightarrow} (\Delta'',\Phi \vdash C_1'' : \Theta'',\Sigma'') \\ (\Theta,\Phi \vdash C_2 : \Delta,\Sigma) \stackrel{\bar{s_1}}{\Longrightarrow} (\Theta'',\Phi \vdash C_2'' : \Delta'',\Sigma'')\end{array}$$

where $\nu(\Delta'',\Theta'',\Sigma'' \setminus \Delta,\Theta,\Sigma)\,.\,(C_1'' \mathbin{/\!\!\!\backslash} C_2'') \equiv C''$. Given that $C''{\downarrow_b}$ we know that $(C_1'' \mathbin{/\!\!\!\backslash} C_2''){\downarrow_b}$ also. By the definition of $\mathbin{/\!\!\!\backslash}$ we see that one of the following (or their symmetric counterparts) must hold:

- $C_1''{\downarrow_b}$ and we are done, or
- $C_1'' \equiv \nu(\Delta_1)\,.\,(n\langle t_1\rangle \parallel C_1''')$ and $C_2'' \equiv \nu(\Delta_2)\,.\,(n\langle t_2\rangle \parallel C_2''')$ where $n\langle t_1 \mathbin{/\!\!\!\backslash} t_2\rangle{\downarrow_b}$. We now proceed by induction on the definition of $t_1 \mathbin{/\!\!\!\backslash} t_2$ to show that for all such C_1'' and C_2'', we can find s_2 where:

  $$\begin{array}{l}(\Delta'',\Phi \vdash C_1'' : \Theta'',\Sigma'') \stackrel{s_2}{\Longrightarrow} (\Delta',\Phi \vdash C_1' : \Theta',\Sigma') \\ (\Theta'',\Phi \vdash C_2'' : \Delta'',\Sigma'') \stackrel{\bar{s_2}}{\Longrightarrow} (\Theta',\Phi \vdash C_2' : \Delta',\Sigma')\end{array}$$

  and either $C_1'{\downarrow_b}$ or $C_2'{\downarrow_b}$. There are two cases (up to symmetry of $\mathbin{/\!\!\!\backslash}$):

  - If $t_1 = \mathsf{let}\ x : T = \mathsf{block}\ \mathsf{in}\ t_1'$ and $t_2 = \mathsf{let}\ y : U = b.\mathsf{succ}()\ \mathsf{in}\ t_2'$ then $C_2''{\downarrow_b}$.
  - If $t_1 = \mathsf{let}\ x : T = \mathsf{block}\ \mathsf{in}\ t_1'$ and $t_2 = \mathsf{let}\ y : U = \mathsf{return}\,(v : T)\ \mathsf{in}\ t_2'$ then we have:

    $$\begin{array}{l}(\Delta'',\Phi \vdash C_1'' : \Theta'',\Sigma'') \xrightarrow{\nu(\Delta_2').n\langle\mathsf{return}\,v\rangle?} (\Delta'',\Delta_2',\Phi \vdash \nu(\Delta_1)\,.\,(n\langle t_1'[v/x]\rangle \parallel C_1''') : \Theta',\Sigma') \\ (\Theta'',\Phi \vdash C_2'' : \Delta'',\Sigma'') \xrightarrow{\nu(\Delta_2').n\langle\mathsf{return}\,v\rangle!} (\Theta'',\Phi \vdash \nu(\Delta_2'')\,.\,(n\langle\mathsf{let}\ y : U = \mathsf{block}\ \mathsf{in}\ t_2'\rangle \parallel C_2''') : \Delta'',\Delta_2',\Sigma'')\end{array}$$

    where $\Delta_2 = (\Delta_2',\Delta_2'')$ and moreover:

    $$n\langle t_1 \mathbin{/\!\!\!\backslash} t_2\rangle \equiv n\langle(\mathsf{let}\ y : U = \mathsf{block}\ \mathsf{in}\ t_2') \mathbin{/\!\!\!\backslash} t_1[v/x]\rangle{\downarrow_b}$$

    so by inductive hypothesis:

    $$\begin{array}{l}(\Delta'',\Phi \vdash C_1'' : \Theta'',\Sigma'') \xrightarrow{\nu(\Delta_2').n\langle\mathsf{return}\,v\rangle?} \stackrel{s_2}{\Longrightarrow} (\Delta',\Phi \vdash C_1' : \Theta',\Sigma') \\ (\Theta'',\Phi \vdash C_2'' : \Delta'',\Sigma'') \xrightarrow{\nu(\Delta_2').n\langle\mathsf{return}\,v\rangle!} \stackrel{\bar{s_2}}{\Longrightarrow} (\Theta',\Phi \vdash C_2' : \Delta',\Sigma')\end{array}$$

    and either $C_1'{\downarrow_b}$ or $C_2'{\downarrow_b}$, as required. □

4.3 Proof of soundness

Theorem 4.5 (Soundness of traces for may testing) *If* $\mathsf{Traces}(\Delta \vdash C_1 : \Theta) \subseteq \mathsf{Traces}(\Delta \vdash C_2 : \Theta)$ *then* $\Delta \models C_1 \sqsubseteq_{may} C_2 : \Theta$

Proof: Suppose that $\mathsf{Traces}(\Delta \vdash C_1 : \Theta) \subseteq \mathsf{Traces}(\Delta \vdash C_2 : \Theta)$ and that we have $(\Theta, b : \mathsf{barb} \vdash C_0 : \Delta)$ such that $(C_1 \parallel C_0)\Downarrow_b$; we must show that $(C_1 \parallel C_0)\Downarrow_b$ also.

Now, since $(C_1 \parallel C_0)\Downarrow_b$, we can use Corollary 4.4 to get:

$$(\Delta, b : \mathsf{barb} \vdash C_1 : \Theta) \stackrel{s}{\Longrightarrow} (\Delta', b : \mathsf{barb} \vdash C_1' : \Theta', \Sigma')$$
$$(\Theta, b : \mathsf{barb} \vdash C_0 : \Delta) \stackrel{\bar{s}}{\Longrightarrow} (\Theta', b : \mathsf{barb} \vdash C_0' : \Delta', \Sigma')$$

and one of the following cases holds:

- **Case** $(C_1'\downarrow_b)$. Since $C_1'\downarrow_b$ we can find a label $\omega!$ of the form:

  $$\omega! = \nu(\vec{n} : \vec{T}) \,.\, n\langle \mathsf{call}\, b.\mathsf{succ}()\rangle!$$

  such that:
  $$(\Delta', b : \mathsf{barb} \vdash C_1' : \Theta', \Sigma') \stackrel{\omega!}{\longrightarrow}$$

  Since $\mathsf{Traces}(\Delta \vdash C_1 : \Theta) \subseteq \mathsf{Traces}(\Delta \vdash C_2 : \Theta)$ we have:

  $$(\Delta, b : \mathsf{barb} \vdash C_2 : \Theta) \stackrel{s}{\Longrightarrow} (\Delta', b : \mathsf{barb} \vdash C_2' : \Theta', \Sigma') \stackrel{\omega!}{\longrightarrow}$$

  and hence $C_2'\downarrow_b$. By Lemma 4.1 we know that $C_2 \parallel C_0 \equiv C_2 \mathbin{/\!\!\!\backslash\!\!\backslash} C_0$ and so by Proposition 4.3 we have: $(C_2 \parallel C_0) \Rightarrow C''$ where:

  $$\nu(\Delta', \Theta', \Sigma' \setminus \Delta, \Theta) \,.\, (C_2' \mathbin{/\!\!\!\backslash\!\!\backslash} C_0') \equiv C''$$

  By Lemma 4.2, since $C_2'\downarrow_b$ we have that $C''\downarrow_b$, and so $(C_2 \parallel C_0)\Downarrow_b$ as required.

- **Case** $(C_0'\downarrow_b)$. Similar to the above. □

5 Completeness of traces for may testing

We now turn to the question of whether trace inclusion captures the may testing preorder exactly. We have already shown that trace inclusion implies may testing inclusion, and so we must consider the converse—completeness.

A key step in demonstrating completeness of traces for may testing is to find, for each trace, a component which exhibits that trace; we call this problem *definability*. However, we only actually require definability for traces which originated from well-typed components. To identify these we present a type system for traces $\Delta \vdash s : \mathsf{trace}\ \Theta$ which captures exactly those we require.

Due to an amount of latency and asynchrony in the labelled transition system, to demonstrate definability, we found it necessary to define an information order $\Delta \vdash r \sqsubseteq s : \mathsf{trace}\ \Theta$ for typed traces which incorporates prefixing, input receptivity [12], and commutativity of certain actions.

In the next section we introduce the type system for traces and demonstrate that every trace from a well-typed component is in fact well-typed. In the section which follows this we introduce the information order on traces and prove the properties required of it.

$$\frac{}{\Delta \vdash \varepsilon : \mathsf{trace}\ \Theta}$$

$$\frac{\begin{array}{c} n \text{ is input-enabled in } \Delta \vdash s : \mathsf{trace}\ \Theta \\ \mathsf{dom}\ (\Delta') \subseteq \mathsf{fn}\ (n\langle \mathsf{call}\ p.l(\vec{v})\rangle) \\ ;\Theta,\Theta(s) \vdash p : [\ldots, l : (\vec{T}) \to T, \ldots] \\ ;\Delta,\Theta,\Delta(s),\Theta(s),\Delta' \vdash \vec{v} : \vec{T} \\ ;\Delta,\Theta,\Delta(s),\Theta(s),\Delta' \vdash n : \mathsf{thread} \end{array}}{\Delta \vdash s\,\nu(\Delta')\,.\,n\langle \mathsf{call}\ p.l(\vec{v})\rangle? : \mathsf{trace}\ \Theta} \qquad \frac{\begin{array}{c} n \text{ is output-enabled in } \Delta \vdash s : \mathsf{trace}\ \Theta \\ \mathsf{dom}\ (\Theta') \subseteq \mathsf{fn}\ (n\langle \mathsf{call}\ p.l(\vec{v})\rangle) \\ ;\Delta,\Delta(s) \vdash p : [\ldots, l : (\vec{T}) \to T, \ldots] \\ ;\Delta,\Theta,\Delta(s),\Theta(s),\Theta' \vdash \vec{v} : \vec{T} \\ ;\Delta,\Theta,\Delta(s),\Theta(s),\Theta' \vdash n : \mathsf{thread} \end{array}}{\Delta \vdash s\,\nu(\Theta')\,.\,n\langle \mathsf{call}\ p.l(\vec{v})\rangle! : \mathsf{trace}\ \Theta}$$

$$\frac{\begin{array}{c} \Delta \vdash s : \mathsf{trace}\ \Theta \\ \mathsf{pop}\,n\,(s) = \nu(\Delta')\,.\,n\langle \mathsf{call}\ p.l(\vec{v})\rangle? \\ \mathsf{dom}\ (\Theta') \subseteq \mathsf{fn}\ (v) \\ ;\Theta,\Theta(s) \vdash p : [\ldots, l : (\vec{T}) \to T, \ldots] \\ ;\Delta,\Theta,\Delta(s),\Theta(s),\Theta' \vdash v : T \end{array}}{\Delta \vdash s\,\nu(\Theta')\,.\,n\langle \mathsf{return}\ v\rangle! : \mathsf{trace}\ \Theta} \qquad \frac{\begin{array}{c} \Delta \vdash s : \mathsf{trace}\ \Theta \\ \mathsf{pop}\,n\,(s) = \nu(\Theta')\,.\,n\langle \mathsf{call}\ p.l(\vec{v})\rangle! \\ \mathsf{dom}\ (\Delta') \subseteq \mathsf{fn}\ (v) \\ ;\Delta,\Delta(s) \vdash p : [\ldots, l : (\vec{T}) \to T, \ldots] \\ ;\Delta,\Theta,\Delta(s),\Theta(s),\Delta' \vdash v : T \end{array}}{\Delta \vdash s\,\nu(\Delta')\,.\,n\langle \mathsf{return}\ v\rangle? : \mathsf{trace}\ \Theta}$$

Figure 15: Rules for judgement $\Delta \vdash s : \mathsf{trace}\ \Theta$

$$\frac{\Delta \vdash s : \mathsf{trace}\ \Theta \quad \mathsf{pop}\,n\,(s) = \gamma!}{n \text{ is input-enabled in } \Delta \vdash s : \mathsf{trace}\ \Theta} \qquad \frac{\Delta \vdash s : \mathsf{trace}\ \Theta \quad \mathsf{pop}\,n\,(s) = * \quad n \notin \Delta, \Delta(s)}{n \text{ is input-enabled in } \Delta \vdash s : \mathsf{trace}\ \Theta}$$

$$\frac{\Delta \vdash s : \mathsf{trace}\ \Theta \quad \mathsf{pop}\,n\,(s) = \gamma?}{n \text{ is output-enabled in } \Delta \vdash s : \mathsf{trace}\ \Theta} \qquad \frac{\Delta \vdash s : \mathsf{trace}\ \Theta \quad \mathsf{pop}\,n\,(s) = * \quad n \notin \Theta, \Theta(s)}{n \text{ is output-enabled in } \Delta \vdash s : \mathsf{trace}\ \Theta}$$

Figure 16: Rules for judgement n is input/output-enabled in $\Delta \vdash s : \mathsf{trace}\ \Theta$

5.1 Types for traces

The type rules for traces make use of some auxilliary notions which we define below:

We write $C \equiv C[D]$ to mean

$$C \equiv \nu(\Delta)\,.\,(D \parallel C')$$

for some Δ, C'.

Define the *thread* of an action as:

$$\mathsf{thread}\ (\nu(\Delta)\,.\,n\langle\cdots\rangle?) = \mathsf{thread}\ (\nu(\Theta)\,.\,n\langle\cdots\rangle!) = n$$

Define the *threads* of a trace as:

$$\mathsf{threads}\ (a_1 \cdots a_n) = \{\mathsf{thread}\ (a_1), \ldots, \mathsf{thread}\ (a_n)\}$$

For a given thread n and trace s, define n is *balanced in s* as:

- If $n \notin \mathsf{threads}\,(s)$ then n is balanced in s.
- If n is balanced in s_1 and s_2 then n is balanced in $s_1\,s_2$.
- If n is balanced in s then n is balanced in $\nu(\Delta)\,.\,n\langle \mathsf{call}\, p.l(\vec{n})\rangle ?\, s\, \nu(\Theta)\,.\,n\langle \mathsf{return}\, v\rangle !$.
- If n is balanced in s then n is balanced in $\nu(\Theta)\,.\,n\langle \mathsf{call}\, p.l(\vec{n})\rangle !\, s\, \nu(\Delta)\,.\,n\langle \mathsf{return}\, v\rangle ?$.

Define $\mathsf{pop}\, n\,(s)$ as:

- If n is balanced in s then $\mathsf{pop}\, n\,(s) = *$.
- If n is balanced in s and $a = \nu(\Delta)\,.\,n\langle \mathsf{call}\, p.l(\vec{v})\rangle ?$ then $\mathsf{pop}\, n\,(r\,a\,s) = a$.
- If n is balanced in s and $a = \nu(\Theta)\,.\,n\langle \mathsf{call}\, p.l(\vec{v})\rangle !$ then $\mathsf{pop}\, n\,(r\,a\,s) = a$.

Define $\Delta(s)$ to be the bound input names of s:

$$
\begin{array}{rcl}
\Delta(\varepsilon) & = & \varepsilon \\
\Delta(\nu(\vec{n} : \vec{T})\,.\,a!\,s) & = & \Delta(s) \\
\Delta(\nu(\vec{n} : \vec{T})\,.\,a?\,s) & = & \vec{n} : \vec{T}, \Delta(s)
\end{array}
$$

and $\Theta(s)$ to be the bound output names of s:

$$
\begin{array}{rcl}
\Theta(\varepsilon) & = & \varepsilon \\
\Theta(\nu(\vec{n} : \vec{T})\,.\,a?\,s) & = & \Theta(s) \\
\Theta(\nu(\vec{n} : \vec{T})\,.\,a!\,s) & = & \vec{n} : \vec{T}, \Theta(s)
\end{array}
$$

The type system for traces is given in Figures 15 and 16.

Lemma 5.1 (Trace Duality) *If* $\Delta \vdash s : \mathsf{trace}\ \Theta$ *then* $\Theta \vdash \bar{s} : \mathsf{trace}\ \Delta$

Proof: Follows by a straightforward induction on the derivation of $\Delta \vdash s : \mathsf{trace}\ \Theta$. □

It will be useful to prove two technical lemmas before we can prove that Trace Subject Reduction (Proposition 5.4) holds.

Lemma 5.2

1. *If n is balanced in s and:*

$$(\Delta \vdash C : \Theta) \stackrel{s}{\Longrightarrow} (\Delta' \vdash C'[n\langle \mathsf{let}\ x : T = \mathsf{block}\ \mathsf{in}\ t\rangle] : \Theta')$$

then $C \equiv C[n\langle \mathsf{let}\ x : T = \mathsf{block}\ \mathsf{in}\ t\rangle]$.

2. *If n is balanced in s and* $\vec{e}'$ *are block/return-free, and:*

$$(\Delta \vdash C : \Theta) \stackrel{s}{\Longrightarrow} (\Delta' \vdash C'[n\langle \mathsf{let}\ \vec{x}' : \vec{T}' = \vec{e}'\ \mathsf{in}\ \mathsf{let}\ y : U = \mathsf{return}\,(v : T)\ \mathsf{in}\ t\rangle] : \Theta')$$

then $C \equiv C[n\langle \mathsf{let}\ \vec{x} : \vec{T} = \vec{e}\ \mathsf{in}\ \mathsf{let}\ y : U = \mathsf{return}\,(v : T)\ \mathsf{in}\ t\rangle]$ *where* $\vec{e}$ *is block/return free.*

Proof: Easy induction on s. □

Lemma 5.3

1. *If C is block/return free and* $(\Delta \vdash C : \Theta) \overset{s}{\Longrightarrow} \xrightarrow{\nu(\Theta').n\langle \mathsf{return}\, v\rangle!}$ *then* $s = s_1\ \nu(\Delta') . n\langle \mathsf{call}\, p.l(\vec{v})\rangle?\, s_2$ *where n is balanced in s_2 .*

2. *If C is block/return free and* $(\Delta \vdash C : \Theta) \overset{s}{\Longrightarrow} \xrightarrow{\nu(\Delta').n\langle \mathsf{return}\, v\rangle?}$ *then* $s = s_1\ \nu(\Theta') . n\langle \mathsf{call}\, p.l(\vec{v})\rangle!\, s_2$ *where n is balanced in s_2 .*

Proof: We prove these properties simultaneously by an induction on the length of s. We only show the argument for Part 1 as Part 2 can be shown in a similar manner. By analysis of the rules of the lts, we have:

$$(\Delta \vdash C : \Theta) \overset{s}{\Longrightarrow} (\Delta'' \vdash C''[n\langle \mathsf{let}\, x : T = \mathsf{return}\,(v : U)\ \mathsf{in}\ t\rangle] : \Theta'') \xrightarrow{\nu(\Theta').n\langle \mathsf{return}\, v\rangle!}$$

Now, partition s into $s_3\ s_2$ picking s_2 to be the longest suffix of s in which n is balanced. We then use Lemma 5.2 to get that:

$$(\Delta \vdash C : \Theta) \overset{s_3}{\Longrightarrow} (\Delta'' \vdash C''[n\langle \mathsf{let}\, \vec{x} : \vec{T} = \vec{e}\ \mathsf{in}\ \mathsf{let}\, x : T = \mathsf{return}\,(v' : U)\ \mathsf{in}\ t\rangle] : \Theta'') \overset{s_2}{\Longrightarrow} \xrightarrow{\nu(\Theta').n\langle \mathsf{return}\, v\rangle!}$$

We now proceed by analysis of s_3:

- s_3 is not of the form ε since C is block/return free.
- s_3 is not of the form $s_1\ a$ with $\mathsf{thread}\,(a) \neq n$, since s_2 is required to be the longest suffix of s in which n is balanced.
- s_3 is not of the form $s_1\ \gamma!$ since $n\langle \mathsf{let}\, \vec{x} : \vec{T} = \vec{e}\ \mathsf{in}\ \mathsf{let}\, x : T = \mathsf{return}\,(v' : U)\ \mathsf{in}\ t\rangle$ is not of the form $n\langle \mathsf{let}\, y : U = \mathsf{block}\ \mathsf{in}\ t'\rangle$.
- s_3 is not of the form $s_1\ \nu(\Delta'') . n\langle \mathsf{return}\, v'\rangle?$ since otherwise, by applying Part 2 of the inductive hypothesis we can partition s_1 into $s_1'\ \nu(\Theta'') . n\langle \mathsf{call}\, p'.l'(\vec{v}')\rangle!\ s_2'$ where n is balanced in s_2', hence n is balanced in $\nu(\Theta'') . n\langle \mathsf{call}\, p'.l'(\vec{v}')\rangle!\ s_2' s_1\ \nu(\Delta'') . n\langle \mathsf{return}\, v'\rangle?\ s_2$, contradicting the requirement that s_2 is the longest such suffix of s.
- So, by a process of elimination, s_3 is of the form $s_1\ \nu(\Delta') . n\langle \mathsf{call}\, p.l(\vec{v})\rangle?$ as required. □

Proposition 5.4 (Trace Subject Reduction) *If $\Delta \vdash C : \Theta$ is block/return free and* $(\Delta \vdash C : \Theta) \overset{s}{\Longrightarrow} (\Delta' \vdash C' : \Theta')$ *then* $\Delta \vdash s : \mathsf{trace}\ \Theta$ *and* $\Delta' \vdash C' : \Theta'$.

Proof: We proceed by induction on the derivation of $(\Delta \vdash C : \Theta) \overset{s}{\Longrightarrow} (\Delta' \vdash C' : \Theta')$.

It is relatively easy to check that $\Delta' \vdash C' : \Theta'$ where $\overset{s}{\Longrightarrow}$ is given by a single axiom instance. We use the inductive hypothesis and Proposition 2.1 to deal with the more general case. We now show $\Delta \vdash s : \mathsf{trace}\ \Theta$. The base case in which s is empty is trivial. Suppose instead that s is non-empty: we perform a case-analsis on the last action of s.

Case $s = s'\ \nu(\Delta') \,.\, n\langle \mathsf{call}\, p.l(\vec{v})\rangle?$. We know that

$$(\Delta \vdash C : \Theta) \stackrel{s'}{\Longrightarrow} (\Delta, \Delta(s') \vdash C' : \Theta, \Theta(s')) \xrightarrow{\nu(\Delta').n\langle \mathsf{call}\, p.l(\vec{v})\rangle?}$$

so we have that either

$$C' \equiv \nu(\Delta') \,.\, \nu(\Delta'') \,.\, n\langle \mathsf{let}\ x : T = \mathsf{block\ in}\ t\rangle \parallel C''$$

or $n \in \Delta, \Delta(s')$ and n is a fresh thread to s'. We can apply the inductive hypothesis to s' to see that $\Delta \vdash s' : \mathsf{trace}\ \Theta$ and we consider $\mathsf{pop}\, n(s')$: if $n \in \Delta, \Delta(s')$ and n is fresh thread to s' then $\mathsf{pop}\, n(s')$ is necessarily $*$. Otherwise we know that $C' \equiv \nu(\Delta') \,.\, \nu(\Delta'') \,.\, n\langle \mathsf{let}\ x : T = \mathsf{block\ in}\ t\rangle \parallel C''$ and therefore the last action which could have occurred at n must have been an output, that is, $\mathsf{pop}\, n(s') = \gamma!$. In both cases we see that

$$n \text{ is input enabled in } \Delta \vdash s' : \mathsf{trace}\ \Theta \tag{1}$$

We know that $(\Delta, \Delta(s') \vdash C' : \Theta, \Theta(s')) \xrightarrow{\nu(\Delta').n\langle \mathsf{call}\, p.l(\vec{v})\rangle?}$ and we know that the side-conditions on the transition rule for $\nu(\Delta') \,.\, \gamma?$ actions guarantees that

$$\mathsf{dom}\ (\Delta') \subseteq \mathsf{fn}\ (\vec{v}) \tag{2}$$

We also know that the side-conditions on rule for call-input actions guarantees that

$$;\Delta, \Delta(s'), \Theta, \Theta(s'), \Delta' \vdash p.l(\vec{v}) : T \text{ and } p \in \Theta, \Theta(s')$$

We use this to see that

$$;\Theta, \Theta(s'), \Delta' \vdash p : [\ldots l : (\vec{T}) \to T] \tag{3}$$

and

$$;\Delta, \Delta(s'), \Theta, \Theta(s'), \Delta' \vdash \vec{v} : \vec{T} \tag{4}$$

Lastly, it is easy to see that

$$;\Delta, \Delta(s'), \Theta, \Theta(s'), \Delta' \vdash n : \mathsf{thread} \tag{5}$$

We collect the statements (1)–(5) together to see that they form the hypotheses of the type rule which allows us to conclude

$$\Delta \vdash s'\ \nu(\Delta') \,.\, n\langle \mathsf{call}\, p.l(\vec{v})\rangle? : \mathsf{trace}\ \Theta$$

as required.

Case $s = s'\ \nu(\Theta') \,.\, n\langle \mathsf{call}\, p.l(\vec{v})\rangle!$. Similar to previous case.

Case $s = s'\ \nu(\Theta') \,.\, n\langle \mathsf{return}\, v\rangle!$. We know that

$$(\Delta \vdash C : \Theta) \stackrel{s'}{\Longrightarrow} (\Delta, \Delta(s') \vdash C' : \Theta, \Theta(s')) \xrightarrow{\nu(\Theta').n\langle \mathsf{return}\, v\rangle!}$$

so we have that

$$C' \equiv C'[n\langle \mathsf{let}\ x : T = \mathsf{return}\,(v : U)\ \mathsf{in}\ t\rangle]$$

We can apply the inductive hypothesis to obtain

$$\Delta \vdash s' : \mathsf{trace}\,\Theta \tag{1}$$

and we notice that because C is block/return free we can apply Lemma 5.3 to get:

$$s' = s_1\ \nu(\Delta')\,.\,n\langle \mathsf{call}\, p.l(\vec{v})\rangle?\ s_2$$

where n is balanced in s_2. Given this, we see that

$$\mathsf{pop}\, n\,(s_1\ \nu(\Delta')\,.\,n\langle \mathsf{call}\, p.l(\vec{v})\rangle?\ s_2) = \nu(\Delta')\,.\,n\langle \mathsf{call}\, p.l(\vec{v})\rangle?$$

hence

$$\mathsf{pop}\, n\,(s') = \nu(\Delta')\,.\,n\langle \mathsf{call}\, p.l(\vec{v})\rangle? \tag{2}$$

Again, the side-conditions on the transition rule for $\nu(\Theta')\,.\,\gamma!$ guarantee that

$$\mathsf{dom}\ (\Theta') \subseteq \mathsf{fn}\ (v) \tag{3}$$

We also know, by (1) and the fact that prefixes of well-typed traces are also well-typed, that

$$\Delta \vdash s_1\ \nu(\Delta')\,.\,n\langle \mathsf{call}\, p.l(\vec{v})\rangle? : \mathsf{trace}\,\Theta$$

and we see that this must have been inferred using a hypothesis

$$;\Theta, \Theta(s_1) \vdash p : [\ldots l : (\vec{U}) \to U' \ldots]$$

which, by weakening, gives us

$$;\Theta, \Theta(s') \vdash p : [\ldots l : (\vec{U}) \to U' \ldots] \tag{4}$$

Lastly, because

$$(\Delta, \Delta(s') \vdash C' : \Theta, \Theta(s'))$$

and

$$C' \equiv C'[n\langle \mathsf{let}\ x : T = \mathsf{return}\,(v : U)\ \mathsf{in}\ t\rangle]$$

we see that

$$;\Delta, \Delta(s'), \Theta, \Theta(s'), \Theta' \vdash v : U$$

So, by Lemma 5.2 together with the typing side-conditions for call-input transitions, we have that $U = U'$, and so

$$;\Delta, \Delta(s), \Theta, \Theta(s), \Theta' \vdash v : U \tag{5}$$

We collect the statements (1)–(5) together to see that they form the hypotheses of the type rule which allows us to conclude

$$\Delta \vdash s'\ \nu(\Theta')\,.\,n\langle \mathsf{return}\, v\rangle! : \mathsf{trace}\,\Theta$$

as required.

Case $s = s'\ \nu(\Delta')\,.\,n\langle \mathsf{return}\, v\rangle?$. Similar to previous case. □

5.2 Information order on traces

The information preorder on traces $\Delta \vdash r \sqsubseteq s : \mathsf{trace}\ \Theta$ is generated by axioms (where in each case we require both sides of the inequation to be well-typed traces):

$$\Delta \vdash s \sqsubseteq s\,r : \mathsf{trace}\ \Theta$$
$$\Delta \vdash s\gamma? \sqsubseteq s : \mathsf{trace}\ \Theta$$
$$\Delta \vdash s\gamma_1?\gamma_2!\,r \sqsubseteq s\gamma_2!\gamma_1?\,r : \mathsf{trace}\ \Theta$$
$$\Delta \vdash s\nu(\Delta)\,.\,\gamma_1?\gamma_2?\,r \sqsubseteq s\nu(\Delta)\,.\,\gamma_2?\gamma_1?\,r : \mathsf{trace}\ \Theta$$
$$\Delta \vdash s\nu(\Theta)\,.\,\gamma_1!\gamma_2!\,r \sqsubseteq s\nu(\Theta)\,.\,\gamma_2!\gamma_1!\,r : \mathsf{trace}\ \Theta$$

Lemma 5.5 (Information Order Duality) *If* $\Delta \vdash r\gamma! \sqsubseteq s\gamma! : \mathsf{trace}\ \Theta$ *and* $\mathsf{fn}\ (\gamma) \cap \Theta(r) = \emptyset$ *and* $\gamma! \notin s, r$ *then* $\Theta \vdash \bar{s} \sqsubseteq \bar{r} : \mathsf{trace}\ \Delta$.

Proof: We write $\Delta \vdash r \sqsubseteq^n s : \mathsf{trace}\ \Theta$ if $\Delta \vdash r \sqsubseteq s : \mathsf{trace}\ \Theta$ can be derived using n instances of transitivity and no reflexivity. It is sufficient to show, by induction on n, that

$$\Delta \vdash r_1\gamma!r_2 \sqsubseteq^n s\gamma! : \mathsf{trace}\ \Theta \text{ implies } \Theta \vdash \bar{s} \sqsubseteq \bar{r}_1 : \mathsf{trace}\ \Delta$$

whenever $\mathsf{fn}\ (\gamma) \cap \Theta(r_1) = \emptyset$ and $\gamma! \notin s, r_1$. The base case, $n = 0$, asks that $\Delta \vdash r_1\gamma!r_2 \sqsubseteq s_1\gamma! : \mathsf{trace}\ \Theta$ be derived from axioms alone. The argument is similar to that used in the inductive case so we omit it here. Suppose then that $\Delta \vdash r_1\gamma!r_2 \sqsubseteq^{n+1} s\gamma! : \mathsf{trace}\ \Theta$, that is

$$\Delta \vdash r_1\gamma!r_2 \sqsubseteq^0 q \sqsubseteq^n s\gamma! : \mathsf{trace}\ \Theta$$

for some q. We examine each of the five axioms in turn (for brevity we will elide the type environments in the judgements $\Delta \vdash r \sqsubseteq s : \mathsf{trace}\ \Theta$):

(i) Suppose q is $r_1\gamma!r_2r$ so that

$$r_1\gamma!r_2 \sqsubseteq^0 r_1\gamma!r_2r \sqsubseteq^n s\gamma!.$$

We apply the inductive hypothesis to $q = r_1\gamma!r_2r$ to obtain $\bar{s} \sqsubseteq \bar{r}_1$ as required.

(ii) Suppose r_2 is $r_2'\gamma'?$ and q is $r_1\gamma!r_2'$ so that

$$r_1\gamma!r_2'\gamma'? \sqsubseteq^0 r_1\gamma!r_2' \sqsubseteq^n s\gamma!.$$

We apply the inductive hypothesis to finish.

(iii) (a) Suppose r_1 is $r_1'\gamma_1?\gamma_2!r_1''$ and q is $r_1'\gamma_2!\gamma_1?r_1''\gamma!r_2$ so that

$$r_1'\gamma_1?\gamma_2!r_1''\gamma!r_2 \sqsubseteq^0 r_1'\gamma_2!\gamma_1?r_1''\gamma!r_2 \sqsubseteq^n s\gamma!.$$

We apply the inductive hypothesis to see that

$$\bar{s} \sqsubseteq \bar{r}_1'\gamma_2?\gamma_1!\bar{r}_1'' \sqsubseteq \bar{r}_1'\gamma_1!\gamma_2?\bar{r}_1'' = \bar{r}_1$$

as required.

(b) Suppose r_2 is $r_2'\gamma_1?\gamma_2!r_2''$ and q is $r_1\gamma!r_2'\gamma_2!\gamma_1?r_2''$ so that

$$r_1\gamma!r_2'\gamma_1?\gamma_2!r_2'' \sqsubseteq^0 r_1\gamma!r_2'\gamma_2!\gamma_1?r_2'' \sqsubseteq^n s\gamma!.$$

We apply the inductive hypothesis to see $\bar{s} \sqsubseteq \bar{r}_1$ as required.

(c) Suppose r_1 is $r_1'\gamma'?$ and q is $r_1'\gamma!\gamma'?r_2$ so that

$$r_1'\gamma'?\gamma!r_2 \sqsubseteq^0 r_1'\gamma!\gamma'?r_2 \sqsubseteq^n s\gamma!.$$

We apply the inductive hypothesis to obtain $\bar{s} \sqsubseteq \bar{r}_1'$ and use the first axiom and transitivity to see $\bar{s} \sqsubseteq \bar{r}_1' \sqsubseteq \bar{r}_1'\gamma'! = \bar{r}_1$.

(iv) (a) Suppose r_1 is $r_1'\nu(\Delta).\gamma_1?\gamma_2?r_1''$ and q is $r_1'\nu(\Delta).\gamma_2?\gamma_1?r_1''\gamma!r_2$ so that

$$r_1'\nu(\Delta).\gamma_1?\gamma_2?r_1''\gamma!r_2 \sqsubseteq^0 r_1'\nu(\Delta).\gamma_2?\gamma_1?r_1''\gamma!r_2 \sqsubseteq^n s\gamma!.$$

We apply the inductive hypothesis to obtain $\bar{s} \sqsubseteq \bar{r}_1'\nu(\Delta).\gamma_2!\gamma_1!\bar{r}_1''$ and we note that

$$\bar{r}_1'\nu(\Delta).\gamma_2!\gamma_1!\bar{r}_1'' \sqsubseteq \bar{r}_1'\nu(\Delta).\gamma_1!\gamma_2!\bar{r}_1'' = \bar{r}_1$$

as required.

(b) Suppose r_2 is $r_2'\nu(\Delta).\gamma_1?\gamma_2?r_2''$ and q is $r_1\gamma!r_2'\nu(\Delta).\gamma_2?\gamma_1?r_2''$ so that

$$r_1\gamma!r_2'\nu(\Delta).\gamma_1?\gamma_2?r_2'' \sqsubseteq^0 r_1\gamma!r_2'\nu(\Delta).\gamma_2?\gamma_1?r_2'' \sqsubseteq^n s\gamma!.$$

We apply the inductive hypothesis to obtain $\bar{s} \sqsubseteq \bar{r}_1$ as required.

(v) (a) Suppose r_1 is $r_1'\nu(\Theta).\gamma_1!\gamma_2!r_1''$ and q is $r_1'\nu(\Theta).\gamma_2!\gamma_1!r_1''\gamma!r_2$, for which the proof follows as for Case (iv)(a).

(b) Suppose r_2 is $r_2'\nu(\Theta).\gamma_1!\gamma_2!r_2''$ and q is $r_1\gamma!r_2'\nu(\Theta).\gamma_2!\gamma_1!r_2''$, for which the proof follows as for Case (iv)(b).

(c) Suppose r_1 is $r_1'\nu(\Theta).\gamma'!$ and q is $r_1'\nu(\Theta).\gamma!\gamma'!r_2$ so that

$$r_1'\nu(\Theta).\gamma'!\gamma!r_2 \sqsubseteq^0 r_1'\nu(\Theta).\gamma!\gamma'!r_2 \sqsubseteq^n s\gamma!.$$

We know that $\mathsf{fn}(\gamma) \cap \Theta(r_1) = \emptyset$. This implies that Θ must be empty. Therefore we can apply the inductive hypothesis to obtain $\bar{s} \sqsubseteq \bar{r}_1'$ and then note $r_1' \sqsubseteq r_1'\nu(\Theta).\gamma'? = \bar{r}_1$ by the first axiom.

(d) Suppose r_2 is $\gamma''!r_2'$, γ is $\nu(\Theta).\gamma'$ and q is $r_1\nu(\Theta).\gamma''!\gamma'!r_2'$ so that

$$r_1\nu(\Theta).\gamma'\gamma''!r_2' \sqsubseteq^0 r_1\nu(\Theta).\gamma''!\gamma'!r_2' \sqsubseteq^n s\nu(\Theta).\gamma'!.$$

We first show a subsidiary result (as an induction on the derivation of $\sqsubseteq$), that:

$$\text{if } r_3\ \nu(n:T).\gamma_3!\ r_4\ \gamma_4!\ r_5 \sqsubseteq s_3\ \nu(n:T).\gamma_5!\ s_4 \text{ then } s_4 \neq \varepsilon \qquad (1)$$

from which it follows that Θ is empty. The inductive hypothesis tells us that $\bar{s} \sqsubseteq \bar{r}\nu(\Theta).\gamma''?$ and we note that $\bar{s} \sqsubseteq \bar{r}_1\nu(\Theta).\gamma''? \sqsubseteq \bar{r}_1$ follows from the second axiom. □

Proposition 5.6 (Information Order Closure) *If* $(\Delta \vdash C : \Theta) \stackrel{s}{\Longrightarrow}$ *and* $\Delta \vdash r \sqsubseteq s : \mathsf{trace}\ \Theta$ *then* $(\Delta \vdash C : \Theta) \stackrel{r}{\Longrightarrow}$.

Proof: Show that the following diagrams can be completed when $\mathsf{thread}\ (\gamma_1) \neq \mathsf{thread}\ (\gamma_2)$:

$$
\begin{array}{ccc}
\cdot & \xrightarrow{\gamma_2!} & \cdot \\
 & & \big\downarrow \\
 & & \cdot
\end{array}
\quad \text{as} \quad
\begin{array}{ccc}
\cdot & \xrightarrow{\gamma_2!} & \cdot \\
\big\downarrow & & \big\downarrow \\
\cdot & \xrightarrow{\gamma_2!} & \cdot
\end{array}
$$

$$
\begin{array}{ccc}
\cdot & \longrightarrow & \cdot \\
 & & \big\downarrow{\scriptstyle \gamma_1?} \\
 & & \cdot
\end{array}
\quad \text{as} \quad
\begin{array}{ccc}
\cdot & \longrightarrow & \cdot \\
{\scriptstyle \gamma_1?}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_1?} \\
\cdot & \longrightarrow & \cdot
\end{array}
$$

$$
\begin{array}{ccc}
\cdot & \xrightarrow{\gamma_2!} & \cdot \\
 & & \big\downarrow{\scriptstyle \gamma_1?} \\
 & & \cdot
\end{array}
\quad \text{as} \quad
\begin{array}{ccc}
\cdot & \xrightarrow{\gamma_2!} & \cdot \\
{\scriptstyle \gamma_1?}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_1?} \\
\cdot & \xrightarrow{\gamma_2!} & \cdot
\end{array}
$$

$$
\begin{array}{ccc}
\cdot & \xrightarrow{\nu(\Delta)\,.\,\gamma_2?} & \cdot \\
 & & \big\downarrow{\scriptstyle \gamma_1?} \\
 & & \cdot
\end{array}
\quad \text{as} \quad
\begin{array}{ccc}
\cdot & \xrightarrow{\nu(\Delta)\,.\,\gamma_2?} & \cdot \\
{\scriptstyle \nu(\Delta)\,.\,\gamma_1?}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_1?} \\
\cdot & \xrightarrow{\gamma_2?} & \cdot
\end{array}
$$

$$
\begin{array}{ccc}
\cdot & \xrightarrow{\nu(\Theta)\,.\,\gamma_2!} & \cdot \\
 & & \big\downarrow{\scriptstyle \gamma_1!} \\
 & & \cdot
\end{array}
\quad \text{as} \quad
\begin{array}{ccc}
\cdot & \xrightarrow{\nu(\Theta)\,.\,\gamma_2!} & \cdot \\
{\scriptstyle \nu(\Theta)\,.\,\gamma_1!}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_1!} \\
\cdot & \xrightarrow{\gamma_2!} & \cdot
\end{array}
$$

The result follows by an induction on the derivation of $\Delta \vdash r \sqsubseteq s : \mathsf{trace}\ \Theta$. □

5.3 Definability of traces

For a well-typed trace $\Delta \vdash s : \mathsf{trace}\ \Theta$ we give the definition of a component $\mathsf{Comp}\ (\Delta \vdash s : \mathsf{trace}\ \Theta)$ in Figure 17. It is this component that we will show to exhibit the trace s and only traces r such that $r \sqsubseteq s$.

The definition of $\mathsf{Comp}\ (\Delta \vdash s : \mathsf{trace}\ \Theta)$ is rather lengthy so we offer an indication of how it is constructed. Firstly, we construct two objects called Ref and State. The former contains a field holding a pointer to the latter. The State object provides type-indexed families of methods called out, inReturn, and inCall. We also provide object and thread definitions for all those references for which the type demands it, i.e. those in Θ. The object definitions provide methods according to the object types, where the method bodies simply indirectly re-route all calls to the appropriate State.inCall. The thread definitions make indirect calls to State.out. It it through these that traces are begun.

The bodies for the out, inReturn, and inCall methods depend on the next action in the trace we are providing a definition for. For instance, if the next action to be performed is an output

$$
\begin{array}{l}
\mathsf{Comp}\ (\Delta \vdash s : \mathsf{trace}\ \Theta) = \nu(\Theta(s), \mathsf{ref} : \mathsf{Ref}, \mathsf{state}_\varepsilon : \mathsf{State})\ .\ (\\
\quad \mathsf{ref}[\mathsf{val} = \mathsf{state}_\varepsilon] \parallel \\
\quad \mathsf{state}_\varepsilon[\mathsf{State}(\Delta \vdash \varepsilon \le s : \mathsf{trace}\ \Theta)] \parallel \\
\quad \prod\{p[l_i = \mathsf{ref.val.inCall}_{p.l_i:L_i} \mid i = 1 \dots n] \mid p : [l_i : L_i \mid i = 1 \dots n] \in \Theta, \Theta(s)\} \parallel \\
\quad \prod\{n\langle \mathsf{ref.val.out}_{\mathsf{none}}()\rangle \mid n : \mathsf{thread} \in \Theta, \Theta(s)\} \\
)
\end{array}
$$

$$\mathsf{Ref} = [\mathsf{val} : \mathsf{State}]$$

$$\mathsf{State} = [\mathsf{out}_T : () \to T, \mathsf{inReturn}_T : (T) \to T, \mathsf{inCall}_{p.l:L} : L]$$

$$
\begin{array}{l}
\mathsf{State}(\Delta \vdash r \le s : \mathsf{trace}\ \Theta) = (\\
\quad \mathsf{out}_T = \mathsf{Out}_T(\Delta \vdash r \le s : \mathsf{trace}\ \Theta), \\
\quad \mathsf{inReturn}_T = \mathsf{InReturn}_T(\Delta \vdash r \le s : \mathsf{trace}\ \Theta), \\
\quad \mathsf{inCall}_{p.l:L} = \mathsf{InCall}_{p.l:L}(\Delta \vdash r \le s : \mathsf{trace}\ \Theta) \\
)
\end{array}
$$

$$
\begin{array}{l}
\mathsf{Out}_T(\Delta \vdash r \le s : \mathsf{trace}\ \Theta) = \lambda()\ .\ (\\
\quad \text{when } ra \le s \text{ and } a = \nu(\Theta')\ .\ n\langle \mathsf{call}\, p.l(\vec{v})\rangle! \text{ and } ;\Delta, \Theta, \Delta(r), \Theta(r), \Theta' \vdash p.l(\vec{v}) : U : \\
\qquad \mathsf{if\ currentthread} = n \ \mathsf{then} \\
\qquad\quad \mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \le s : \mathsf{trace}\ \Theta)]; \\
\qquad\quad \mathsf{ref.val.inReturn}_U(p.l(\vec{v})); \\
\qquad\quad \mathsf{ref.val.out}_T() \\
\quad \text{when } ra \le s \text{ and } a = \nu(\Theta')\ .\ n\langle \mathsf{return}\, v\rangle! \text{ and } ;\Delta, \Theta, \Delta(r), \Theta(r), \Theta' \vdash v : T : \\
\qquad \mathsf{if\ currentthread} = n \ \mathsf{then} \\
\qquad\quad \mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \le s : \mathsf{trace}\ \Theta)]; \\
\qquad\quad v \\
\quad \text{otherwise :} \\
\qquad \mathsf{stop} \\
)
\end{array}
$$

$$
\begin{array}{l}
\mathsf{InReturn}_T(\Delta \vdash r \le s : \mathsf{trace}\ \Theta) = \lambda(x : T)\ .\ (\\
\quad \text{when } ra \le s \text{ and } a = \nu(\Delta')\ .\ n\langle \mathsf{return}\, v\rangle? \text{ and } ;\Delta, \Theta, \Delta(r), \Theta(r), \Delta' \vdash v : T : \\
\qquad \mathsf{if}\ \Delta, \Theta, \Delta(r), \Theta(r) \vdash (\mathsf{currentthread}, x) = \nu(\Delta')\ .\ (n, v) \ \mathsf{then} \\
\qquad\quad \mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \le s : \mathsf{trace}\ \Theta)]; \\
\qquad\quad v \\
\quad \text{otherwise :} \\
\qquad \mathsf{stop} \\
)
\end{array}
$$

$$
\begin{array}{l}
\mathsf{InCall}_{p.l:(\vec{T}) \to T}(\Delta \vdash r \le s : \mathsf{trace}\ \Theta) = \lambda(\vec{x} : \vec{T})\ .\ (\\
\quad \text{when } ra \le s \text{ and } a = \nu(\Delta')\ .\ n\langle \mathsf{call}\, p.l(\vec{v})\rangle? \text{ and } ;\Delta, \Theta, \Delta(r), \Theta(r), \Delta' \vdash \vec{v} : \vec{T} : \\
\qquad \mathsf{if}\ \Delta, \Theta, \Delta(r), \Theta(r) \vdash (\mathsf{currentthread}, \vec{x}) = \nu(\Delta')\ .\ (n, \vec{v}) \ \mathsf{then} \\
\qquad\quad \mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \le s : \mathsf{trace}\ \Theta)]; \\
\qquad\quad \mathsf{ref.val.out}_T() \\
\quad \text{otherwise :} \\
\qquad \mathsf{stop} \\
)
\end{array}
$$

Figure 17: Definition of $\mathsf{Comp}\ (\Delta \vdash s : \mathsf{trace}\ \Theta)$

$$
\begin{array}{rcl}
\mathsf{if}\ \Delta \vdash () = \nu() \,.\, ()\ \mathsf{then}\ t & = & t \\
\mathsf{if}\ \Delta \vdash (v, \vec{v}) = \nu(p : U, \vec{n} : \vec{T}) \,.\, (p, \vec{p})\ \mathsf{then}\ t & = & \mathsf{if}\ v \notin \Delta^{-1}(U)\ \mathsf{then} \\
 & & \quad (\mathsf{if}\ \Delta, p : U \vdash (\vec{v}) = \nu(\vec{n} : \vec{T}) \,.\, (\vec{p})\ \mathsf{then}\ t)[v/p]\ \mathsf{else\ stop} \\
\mathsf{if}\ \Delta \vdash (v, \vec{v}) = \nu(\vec{n} : \vec{T}) \,.\, (p, \vec{p})\ \mathsf{then}\ t & = & \mathsf{if}\ v = p\ \mathsf{then}\ (\mathsf{if}\ \Delta \vdash (\vec{v}) = \nu(\vec{n} : \vec{T}) \,.\, (\vec{p})\ \mathsf{then}\ t)\ \mathsf{else\ stop}
\end{array}
$$

Figure 18: Definition of $\mathsf{if}\ \Delta \vdash (\vec{v}) = \nu(\vec{n} : \vec{T}) \,.\, (\vec{p})\ \mathsf{then}\ t$ (when $p \notin \vec{n}$).

$$
\begin{array}{rcl}
\mathsf{if}\ v \notin ()^{-1}(U)\ \mathsf{then}\ t\ \mathsf{else\ stop} & = & t \\
\mathsf{if}\ v \notin (n : U, \Delta)^{-1}(U)\ \mathsf{then}\ t\ \mathsf{else\ stop} & = & \mathsf{if}\ v = n\ \mathsf{then\ stop\ else}\ (\mathsf{if}\ v \notin \Delta^{-1}(U)\ \mathsf{then}\ t\ \mathsf{else\ stop}) \\
\mathsf{if}\ v \notin (n : T, \Delta)^{-1}(U)\ \mathsf{then}\ t\ \mathsf{else\ stop} & = & \mathsf{if}\ v \notin \Delta^{-1}(U)\ \mathsf{then}\ t\ \mathsf{else\ stop}
\end{array}
$$

Figure 19: Definition of $\mathsf{if}\ v \notin \Delta^{-1}(U)\ \mathsf{then}\ t\ \mathsf{else\ stop}$ (when $T \neq U$).

$n\langle \mathsf{call}\, p.l(\vec{v})\rangle!$ then all of the bodies will be a stopped thread save for out which will have a method body which will check that the calling thread is n and, if so, update Ref to point to a new State object which will perform the next action in the trace. It will then indirectly call State.inReturn with the result of calling $p.l(\vec{v})$ (on dangling p) to listen for an input interaction (cf. the labelled transition rule for output, any subsequent action at this thread must be an input). Having successfully observed an input interaction, the line of interrogation in this thread is complete so it must reset itself by returning to a state in which it makes an indirect call to State.out. Similar definitions are given for each type of action.

We provide no synchronisation in the $\mathsf{Comp}\ (\Delta \vdash s : \mathsf{trace}\ \Theta)$ component so that there is no guarantee that the reductions will follow the precise sequence of calls needed to exhibit the trace. However, with respect to may testing, this is irrelevant as we are only looking for one possible successful sequence of execution. We do guarantee the existence of this in Proposition 5.9.

To be of use in the completeness proof though we need to know that the component defined is actually well-typed. This is the subject of the next two lemmas.

Lemma 5.7 *If* $\vec{x} : \vec{T}; \Delta \vdash \vec{v} : \vec{U}$, $\vec{x} : \vec{T}; \Delta, \Delta' \vdash \vec{p} : \vec{U}$ *and* $\vec{x} : \vec{T}; \Delta, \Delta' \vdash t : T$ *then* $\vec{x} : \vec{T}; \Delta \vdash \mathsf{if}\ \Delta \vdash (\vec{v}) = \nu(\Delta') \,.\, (\vec{p})\ \mathsf{then}\ t : T$.

Proof: Straightforward induction on the definition of $\mathsf{if}\ \Delta \vdash (\vec{v}) = \nu(\Delta') \,.\, (\vec{p})\ \mathsf{then}\ t$. □

Lemma 5.8 *If* $\Delta \vdash s : \mathsf{trace}\ \Theta$ *then* $\Delta \vdash \mathsf{Comp}\ (\Delta \vdash s : \mathsf{trace}\ \Theta) : \Theta$.

Proof: By examining the definition of $\mathsf{Comp}\ (\Delta \vdash s : \mathsf{trace}\ \Theta)$ we see that we are required to show that

(i) $\Delta, \Theta, \Theta(s), \mathsf{state}_\varepsilon : \mathsf{State} \vdash \mathsf{ref}[\mathsf{val} = \mathsf{state}_\varepsilon] : (\mathsf{ref} : \mathsf{Ref})$

(ii) $\Delta, \Theta, \Theta(s), \mathsf{ref} : \mathsf{Ref} \vdash \mathsf{state}_\varepsilon[\mathsf{State}(\Delta \vdash \varepsilon \leq s : \mathsf{trace}\ \Theta)] : (\mathsf{state}_\varepsilon : \mathsf{State})$

(iii) $\Delta, \Theta, \Theta(s) \setminus p, \mathsf{ref} : \mathsf{Ref}, \mathsf{state}_\varepsilon : \mathsf{State} \vdash p[l_i = \mathsf{ref.val.inCall}_{p.l_i:L_i} \mid i = 1 \ldots n] : (p : [l_i : L_i \mid i = 1 \ldots n])$ for each $p \in \Theta, \Theta(s)$

(iv) $\Delta, \Theta, \Theta(s) \setminus n, \mathsf{ref} : \mathsf{Ref}, \mathsf{state}_\varepsilon : \mathsf{State} \vdash n\langle \mathsf{ref.val.out_{none}}() \rangle : (n : \mathsf{thread})$ for each $n \in \Theta, \Theta(s)$.

It is easy to check that all but (ii) follow from the definitions of the types State and Ref. We show (ii) by establishing

$$;\Delta, \Theta, \Theta(s), \mathsf{ref} : \mathsf{Ref} \vdash [\mathsf{State}(\Delta \vdash r \leq s : \mathsf{trace}\ \Theta)] : \mathsf{State}$$

by induction on the length of s less the length of r. The base case (when $s = r$) follows as each method body of $\mathsf{State}(\Delta \vdash r \leq s : \mathsf{trace}\ \Theta)$ is stop and hence can be given any type. The inductive case relies on the following properties:

(a) $;\Delta, \Theta, \Theta(s), \mathsf{ref} : \mathsf{Ref} \vdash \mathsf{Out}_T(\Delta \vdash r \leq s : \mathsf{trace}\ \Theta) : () \to T$

(b) $;\Delta, \Theta, \Theta(s), \mathsf{ref} : \mathsf{Ref} \vdash \mathsf{InReturn}_T(\Delta \vdash r \leq s : \mathsf{trace}\ \Theta) : (T) \to T$

(c) $;\Delta, \Theta, \Theta(s), \mathsf{ref} : \mathsf{Ref} \vdash \mathsf{InCall}_{p.l:L}(\Delta \vdash r \leq s : \mathsf{trace}\ \Theta) : L$

We only show how to establish (a) here as the remaining two cases can be dealt with similarly. Suppose then that $ra \leq s$ with $a = \nu(\Theta') \,.\, n\langle \mathsf{return}\ v \rangle!$ and $;\Delta, \Theta, \Delta(r), \Theta(r), \Theta' \vdash v : T$ It is easy to see by the inductive hypothesis that

$$;\Delta, \Theta, \Theta(s), \mathsf{ref} : \mathsf{Ref} \vdash \mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)]; v : T$$

holds, and also that $;\Delta, \Theta, \Theta(s), \mathsf{ref} : \mathsf{Ref} \vdash \mathsf{currentthread} : \mathsf{thread}$ and

$$;\Delta, \Theta, \Theta(s), \mathsf{ref} : \mathsf{Ref} \vdash n : \mathsf{thread}.$$

This latter fact follows from $\Delta \vdash ra : \mathsf{trace}\ \Theta$ guaranteeing

$$;\Delta, \Theta, \Delta(r), \Theta(r), \mathsf{ref} : \mathsf{Ref} \vdash n : \mathsf{thread}.$$

We can now apply the previous Lemma to see that

$$\begin{array}{lll};\Delta, \Theta, \Theta(s), \mathsf{ref} : \mathsf{Ref} \vdash & \mathsf{if}\ \Delta \vdash (\mathsf{currentthread}) = \nu() \,.\, (n)\ \mathsf{then} & \\ & \quad \mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)]; v & : T\end{array}$$

which gives us that $;\Delta, \Theta, \Theta(s), \mathsf{ref} : \mathsf{Ref} \vdash \mathsf{Out}_T(\Delta \vdash r \leq s : \mathsf{trace}\ \Theta) : () \to T$ as required.

Alternatively, suppose that $ra \leq s$ with $a = \nu(\Theta') \,.\, n\langle \mathsf{call}\ p.l(\vec{v}) \rangle!$ and $;\Delta, \Delta(r), \Theta, \Theta(r), \Theta' \vdash p.l(\vec{v}) : U$. Given that $\mathsf{State.inReturn}_U : (U) \to U$, and that $\mathsf{State.out}_T : () \to T$ we can apply the inductive hypothesis and previous Lemma as above to see that

$$\begin{array}{lll};\Delta, \Theta, \Theta(s), \mathsf{ref} : \mathsf{Ref} \vdash & \mathsf{if}\ \mathsf{currentthread} = n\ \mathsf{then} & \\ & \quad \mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)]; & \\ & \quad \mathsf{ref.val.inReturn}_U(p.l(\vec{v})); & \\ & \quad \mathsf{ref.val.out}_T() & : T\end{array}$$

as required.

Otherwise the body of $\mathsf{Out}_T(\Delta \vdash r \leq s : \mathsf{trace}\ \Theta)$ is stop and this can be given any type. □

Proposition 5.9 (Definability) *For any* $\Delta \vdash s : \mathsf{trace}\ \Theta$
we have $(\Delta \vdash \mathsf{Comp}\ (\Delta \vdash s : \mathsf{trace}\ \Theta) : \Theta) \stackrel{r}{\Longrightarrow}$ *if and only if* $\Delta \vdash r \sqsubseteq s : \mathsf{trace}\ \Theta$.

Proof: Given in Appendix B. □

5.4 Proof of completeness

Theorem 5.10 (Completeness of traces for may testing) *If* $\Delta \models C_1 \sqsubsetsim_{may} C_2 : \Theta$ *then* $\mathsf{Traces}(\Delta \vdash C_1 : \Theta) \subseteq \mathsf{Traces}(\Delta \vdash C_2 : \Theta)$.

Proof: Choose any trace s_1 such that:

$$(\Delta \vdash C_1 : \Theta) \xRightarrow{s_1} (\Delta' \vdash C_1' : \Theta')$$

By Proposition 5.4 we have that $\Delta \vdash s_1 : \mathsf{trace}\,\Theta$, and so by Lemma 5.1 we have that $\Theta \vdash \bar{s}_1 : \mathsf{trace}\,\Delta$.

Pick a fresh $b : \mathsf{barb}$ and let $\omega!$ be:

$$\omega! = \nu(n : \mathsf{thread})\,.\,n\langle \mathsf{call}\,b.\mathsf{succ}()\rangle!$$

and let C_0 be:

$$C_0 = \mathsf{Comp}\,(\Theta, b : \mathsf{barb} \vdash \bar{s}_1\omega! : \mathsf{trace}\,\Delta)$$

Then by Proposition 5.9 we have:

$$(\Theta, b : \mathsf{barb} \vdash C_0 : \Delta) \xRightarrow{\bar{s}_1} (\Theta', b : \mathsf{barb} \vdash C_0' : \Delta') \xrightarrow{\omega!}$$

and so $C_0' \downarrow_b$. Thus, by Lemma 4.1, Proposition 4.3, and Lemma 4.2 we have $(C_1 \parallel C_0) \Downarrow_b$. We know that $\Delta \models C_1 \sqsubsetsim_{\mathsf{may}} C_2 : \Theta$, that $\Theta, b : \mathsf{barb} \vdash C_0 : \Delta$, and $(C_1 \parallel C_0) \Downarrow_b$ so this implies $(C_2 \parallel C_0) \Downarrow_b$. Thus, by Lemma 4.1 and Corollary 4.4 we can find s_2 such that:

$$(\Delta, b : \mathsf{barb} \vdash C_2 : \Theta) \xRightarrow{s_2} (\Delta'', \Phi'' \vdash C_2'' : \Theta'', \Sigma'')$$
$$(\Theta, b : \mathsf{barb} \vdash C_0 : \Delta) \xRightarrow{\bar{s}_2} (\Theta'', \Phi'' \vdash C_0'' : \Delta'', \Sigma'')$$

and either $C_0'' \downarrow_b$ or $C_2'' \downarrow_b$. Since b was chosen to be fresh, we must have that $C_0'' \downarrow_b$ and hence $(\Theta, b : \mathsf{barb} \vdash C_0 : \Delta) \xRightarrow{\bar{s}_2\omega!}$ so by Proposition 5.9: $\Theta, b : \mathsf{barb} \vdash \bar{s}_2\omega! \sqsubseteq \bar{s}_1\omega! : \mathsf{trace}\,\Delta$ and so by Lemma 5.5 and narrowing: $\Delta \vdash s_1 \sqsubseteq s_2 : \mathsf{trace}\,\Theta$. Thus, by Proposition 5.6 we have: $(\Delta \vdash C_2 : \Theta) \xRightarrow{s_1} (\Delta' \vdash C_2' : \Theta')$ as required. □

6 Restricted sub-languages

The proof techniques use to obtain full abstraction here are quite robust and can also be carried out for two restricted sub-languages:

1. The single-threaded sub-language is given by only allowing one name of type thread, and removing new thread creation from the expression language. The definability result for Proposition 5.9 does not use thread creation, so the proof of full abstraction goes through with only minor changes to the proof of Theorem 5.10.

2. The sub-language with only field update (and no method update) can be given the same trace semantics. The definability result for Proposition 5.9 only uses field update, and so the proof of full abstraction goes through unchanged.

Thus, not only do we have a full abstraction result for the concurrent object calculus, we can also specialise the results to become full abstraction result for other related languages.

One change which cannot easily be made is to remove the restriction that components be write closed, since method, and even field, updates are not generally externally observable. It is unlikely that traces which represent write interactions will be definable in the current sense. However, we do believe that the restriction to write closed components is a reasonable one, since it corresponds to existing 'best practice' for component design.

7 Conclusions and future work

In this paper we have presented the first fully abstract semantics for concurrent objects. The semantics is fairly simple, and corresponds loosely to some of the messages used in UML interaction diagrams. We do need to road test the trace semantics with some reasonably sized examples to demonstrate that the calculation of traces is tractable.

There are a number of issues left open:

- Our semantics has much of the flavour of game semantics [2, 13], and this connection should be investigated.
- The trace semantics characterise may testing, rather than the more common must testing or bisimulation equivalence.
- The object calculus presented here does not include subtyping. We believe that the techniques of [11] should be applicable to the provision of a fully abstract semantics even in the presence of subtyping.

A Proof of trace composition and decomposition

We have to prove that for any components $(\Delta, \Phi \vdash C_1 : \Theta, \Sigma)$ and $(\Theta, \Phi \vdash C_2 : \Delta, \Sigma)$ such that $C_1 \bowtie C_2 \equiv C$, we have:

1. *Composition*: If $(\Delta, \Phi \vdash C_1 : \Theta, \Sigma) \stackrel{s}{\Longrightarrow} (\Delta', \Phi \vdash C_1' : \Theta', \Sigma')$
   and $(\Theta, \Phi \vdash C_2 : \Delta, \Sigma) \stackrel{\bar{s}}{\Longrightarrow} (\Theta', \Phi \vdash C_2' : \Delta', \Sigma')$ then $C \Rightarrow C'$
   where $\nu(\Delta', \Theta', \Sigma' \setminus \Delta, \Theta, \Sigma) . (C_1' \bowtie C_2') \equiv C'$.
2. *Decomposition*: If $C \Rightarrow C'$ then there exists some trace s
   such that $(\Delta, \Phi \vdash C_1 : \Theta, \Sigma) \stackrel{s}{\Longrightarrow} (\Delta', \Phi \vdash C_1' : \Theta', \Sigma')$
   and $(\Theta, \Phi \vdash C_2 : \Delta, \Sigma) \stackrel{\bar{s}}{\Longrightarrow} (\Theta', \Phi \vdash C_2' : \Delta', \Sigma')$ where $\nu(\Delta', \Theta', \Sigma' \setminus \Delta, \Theta, \Sigma) . (C_1' \bowtie C_2') \equiv C'$.

A.1 Composition

We show four lemmas, from which Composition follows by a simple induction.

Lemma A.1

1. *If* $C_1 \mathbin{/\!\!\wedge\!\!\backslash} C_2 \equiv D \parallel E$ *then there exist components such that* $C_1 \equiv D_1 \parallel E_1$ *and* $C_2 \equiv D_2 \parallel E_2$ *with* $D \equiv D_1 \mathbin{/\!\!\wedge\!\!\backslash} D_2$ *and* $E \equiv E_1 \mathbin{/\!\!\wedge\!\!\backslash} E_2$.

2. *If* $C_1 \mathbin{/\!\!\wedge\!\!\backslash} C_2 \equiv \nu(\vec{n} : \vec{T})\,.\,C$ *then there exist components such that* $C_1 \equiv \nu(\vec{n}_1 : \vec{T}_1)\,.\,C_1'$ *and* $C_2 \equiv \nu(\vec{n}_2 : \vec{T}_2)\,.\,C_2'$ *with* $(\vec{n} : \vec{T}) = (\vec{n}_1 : \vec{T}_1, \vec{n}_2 : \vec{T}_2)$ *and* $C' \equiv C_1' \mathbin{/\!\!\wedge\!\!\backslash} C_2'$.

Proof: Proved by induction on the derivation of $C_1 \mathbin{/\!\!\wedge\!\!\backslash} C_2$. □

Lemma A.2 *If* $C_1 \mathbin{/\!\!\wedge\!\!\backslash} C_2 \equiv C$ *and* $C_1 \xrightarrow{\beta} C_1'$ *then* $C \xrightarrow{\beta} C'$ *where* $C_1' \mathbin{/\!\!\wedge\!\!\backslash} C_2 \equiv C'$.

Proof: An induction on the proof of $C_1 \xrightarrow{\beta} C_1'$, making use of Lemma A.1. □

Lemma A.3 *If* $C_1 \mathbin{/\!\!\wedge\!\!\backslash} C_2 \equiv C$ *and* $C_1 \xrightarrow{\tau} C_1'$ *then* $C \xrightarrow{\tau} C'$ *where* $C_1' \mathbin{/\!\!\wedge\!\!\backslash} C_2 \equiv C'$.

Proof: An induction on the proof of $C_1 \xrightarrow{\tau} C_1'$, making use of Lemma A.1. □

Lemma A.4 *If* $C_1 \mathbin{/\!\!\wedge\!\!\backslash} C_2 \equiv C$ *and* $(\Delta, \Phi \vdash C_1 : \Theta, \Sigma) \xrightarrow{\gamma?} (\Delta', \Phi \vdash C_1' : \Theta', \Sigma')$ *and* $(\Theta, \Phi \vdash C_2 : \Delta, \Sigma) \xrightarrow{\gamma!} (\Theta', \Phi \vdash C_2' : \Delta', \Sigma')$ *then* $\nu(\Delta', \Theta', \Sigma' \setminus \Delta, \Theta, \Sigma)\,.\,(C_1' \mathbin{/\!\!\wedge\!\!\backslash} C_2') \equiv C$.

Proof: A case analysis on γ.

- **Case** $(\gamma = \nu(\vec{n} : \vec{T})\,.\,n\langle \mathsf{call}\, p.l(\vec{v})\rangle$ and $n \notin \Sigma)$.

  Since $(\Delta, \Phi \vdash C_1 : \Theta, \Sigma) \xrightarrow{\gamma?} (\Delta', \Phi \vdash C_1' : \Theta', \Sigma')$ and $n \notin \Sigma$, we must have that:

  $$\begin{aligned} &C_1' \equiv C_1 \parallel n\langle \mathsf{let}\; y : T = p.l(\vec{x})\; \mathsf{in}\; \mathsf{return}\,(y : T)\rangle \\ &\Delta' = (\Delta, \vec{n} : \vec{T}) \setminus (n : \mathsf{thread}) \\ &\Theta' = \Theta \\ &\Sigma' = \Sigma, n : \mathsf{thread} \end{aligned}$$

  Since $(\Theta, \Phi \vdash C_2 : \Delta, \Sigma) \xrightarrow{\gamma!} (\Theta', \Phi \vdash C_2' : \Delta', \Sigma')$ we must have that:

  $$\begin{aligned} &C_2 \equiv \nu(\vec{n} : \vec{T})\,.\,\nu(\vec{p} : \vec{U})\,.\,(C_2'' \parallel n\langle \mathsf{let}\; x : T = p.l(\vec{x})\; \mathsf{in}\; t\rangle) \\ &C_2' \equiv \nu(\vec{p} : \vec{U})\,.\,(C_2'' \parallel n\langle \mathsf{let}\; x : T = \mathsf{block}\; \mathsf{in}\; t\rangle) \end{aligned}$$

  We can then show that:

  $$C_1 \mathbin{/\!\!\wedge\!\!\backslash} C_2 \equiv \nu(\vec{n} : \vec{T})\,.\,\nu(\vec{p} : \vec{U})\,.\,((C_1 \mathbin{/\!\!\wedge\!\!\backslash} C_2'') \parallel n\langle \mathsf{let}\; x : T = p.l(\vec{x})\; \mathsf{in}\; t\rangle)$$

  and that:

  $$C_1' \mathbin{/\!\!\wedge\!\!\backslash} C_2' \equiv \nu(\vec{p} : \vec{U})\,.\,((C_1 \mathbin{/\!\!\wedge\!\!\backslash} C_2'') \parallel n\langle \mathsf{let}\; x : U = p.l(\vec{x})\; \mathsf{in}\; t\rangle)$$

  and so:

  $$\nu(\Delta', \Theta', \Sigma' \setminus \Delta, \Theta, \Sigma)\,.\,(C_1' \mathbin{/\!\!\wedge\!\!\backslash} C_2') \equiv C$$

  as required.

- **Case** ($\gamma = \nu(\vec{n}:\vec{T}) \,.\, n\langle \mathsf{call}\, p.l(\vec{v})\rangle$ and $n \in \Sigma$).
  Similar to the previous case.
- **Case** ($\gamma = \nu(\vec{n}:\vec{T}) \,.\, n\langle \mathsf{return}\, v\rangle$).
  Since $(\Delta,\Phi \vdash C_1 : \Theta,\Sigma) \xrightarrow{\gamma?} (\Delta',\Phi \vdash C_1' : \Theta',\Sigma')$ we must have that:

$$
\begin{aligned}
C_1 &\equiv \nu(\vec{p}_1:\vec{U}_1) \,.\, (C_1'' \parallel n\langle \mathsf{let}\, x:T = \mathsf{block}\ \mathsf{in}\ t_1\rangle)\\
C_1' &\equiv \nu(\vec{p}_1:\vec{U}_1) \,.\, (C_1'' \parallel n\langle t_1[v/x]\rangle)\\
\Delta' &= \Delta, \vec{n}:\vec{T}\\
\Theta' &= \Theta\\
\Sigma' &= \Sigma
\end{aligned}
$$

  Since $(\Theta,\Phi \vdash C_2 : \Delta,\Sigma) \xrightarrow{\gamma!} (\Theta',\Phi \vdash C_2' : \Delta',\Sigma')$ we must have that:

$$
\begin{aligned}
C_2 &\equiv \nu(\vec{n}:\vec{T}) \,.\, \nu(\vec{p}_2:\vec{U}_2) \,.\, (C_2'' \parallel n\langle \mathsf{let}\, y:U = \mathsf{return}\,(v:T)\ \mathsf{in}\ t_2\rangle)\\
C_2' &\equiv \nu(\vec{p}_2:\vec{U}_2) \,.\, (C_2'' \parallel n\langle \mathsf{let}\, y:U = \mathsf{block}\ \mathsf{in}\ t_2\rangle)
\end{aligned}
$$

  We then show that:

$$C_1 \bowtie C_2 \equiv \nu(\vec{n}:\vec{T}) \,.\, \nu(\vec{p}_1:\vec{U}_1) \,.\, \nu(\vec{p}_2:\vec{U}_2) \,.\, ((C_1'' \bowtie C_2'') \parallel n\langle (\mathsf{let}\, y:U = \mathsf{block}\ \mathsf{in}\ t_2) \bowtie (t_1[v/x])\rangle)$$

  and that:

$$C_1' \bowtie C_2' \equiv \nu(\vec{p}_1:\vec{U}_1) \,.\, \nu(\vec{p}_2:\vec{U}_2) \,.\, ((C_1'' \bowtie C_2'') \parallel n\langle (\mathsf{let}\, y:U = \mathsf{block}\ \mathsf{in}\ t_2) \bowtie (t_1[v/x])\rangle)$$

  and so:

$$\nu(\Delta',\Theta',\Sigma' \setminus \Delta,\Theta,\Sigma) \,.\, (C_1' \bowtie C_2') \equiv C$$

  as required. □

Composition follows, by induction on the derivation of $(\Delta,\Phi \vdash C_1 : \Theta,\Sigma) \stackrel{s}{\Longrightarrow} (\Delta',\Phi \vdash C_1' : \Theta',\Sigma')$ and $(\Theta,\Phi \vdash C_2 : \Delta,\Sigma) \stackrel{\bar{s}}{\Longrightarrow} (\Theta',\Phi \vdash C_2' : \Delta',\Sigma')$, making use of Lemmas A.2, A.3 and A.4.

A.2 Decomposition

We show three lemmas, from which Decomposition follows.

Lemma A.5 *For any* $\Delta,\Phi \vdash C_1 : \Theta,\Sigma$ *and* $\Theta,\Phi \vdash C_2 : \Delta,\Sigma$ *if* $(C_1 \bowtie C_2) \equiv \nu(\vec{n}:\vec{T}) \,.\, (C \parallel n\langle \mathsf{let}\, x : T = e\ \mathsf{in}\ t\rangle)$ *then either we have:*

$$
\begin{aligned}
&(\Delta,\Phi \vdash C_1 : \Theta,\Sigma) \stackrel{s}{\Longrightarrow} (\Delta',\Phi \vdash \nu(\vec{n}_1:\vec{T}_1) \,.\, (C_1' \parallel n\langle \mathsf{let}\, x:T = e\ \mathsf{in}\ t_1\rangle) : \Theta',\Sigma')\\
&(\Theta,\Phi \vdash C_2 : \Delta,\Sigma) \stackrel{\bar{s}}{\Longrightarrow} (\Theta',\Phi \vdash C_2' : \Delta',\Sigma')
\end{aligned}
$$

where:

$$\nu(\Delta',\Theta',\Sigma' \setminus \Delta,\Theta,\Sigma) \,.\, \nu(\vec{n}_1:\vec{T}_1) \,.\, (C_1' \parallel n\langle t_1\rangle) \bowtie C_2' \equiv \nu(\vec{n}:\vec{T}) \,.\, (C \parallel n\langle t\rangle)$$

or symmetrically, swapping the roles of C_1 *and* C_2.

Proof: An induction on the derivation of:

$$(C_1 \mathbin{/\!\!/\!\!\backslash} C_2) \equiv \nu(\vec{n} : \vec{T}) \,.\, (C \parallel n\langle \mathsf{let}\ x : T = e\ \mathsf{in}\ t\rangle)$$

The interesting case is when:

$$\begin{array}{rcl} C_1 & \equiv & n\langle \mathsf{let}\ x_1 : T_1 = \mathsf{block}\ \mathsf{in}\ t_1\rangle \\ C_2 & \equiv & n\langle \mathsf{let}\ x_2 : T_2 = \mathsf{return}\,(v : T_1)\ \mathsf{in}\ t_2\rangle \end{array}$$

and:

$$n\langle t_1[v/x]\rangle \mathbin{/\!\!/\!\!\backslash} n\langle \mathsf{let}\ x_2 : T_2 = \mathsf{block}\ \mathsf{in}\ t_2\rangle \equiv \nu(\vec{n} : \vec{T}) \,.\, (C \parallel n\langle \mathsf{let}\ x : T = e\ \mathsf{in}\ t\rangle)$$

so by definition of the lts, and by induction we have:

$$(\Delta, \Phi \vdash C_1 : \Theta, \Sigma) \xrightarrow{n\langle \mathsf{return}\ v\rangle?} (\Delta, \Phi \vdash n\langle t_1[v/x]\rangle : \Theta, \Sigma)$$

$$(\Delta, \Phi \vdash n\langle t_1[v/x]\rangle : \Theta, \Sigma) \stackrel{s}{\Longrightarrow} (\Delta', \Phi \vdash \nu(\vec{n}_1 : \vec{T}_1) \,.\, (C_1' \parallel n\langle \mathsf{let}\ x : T = e\ \mathsf{in}\ t_1\rangle) : \Theta', \Sigma')$$

and

$$(\Delta, \Phi \vdash C_2 : \Theta, \Sigma) \xrightarrow{n\langle \mathsf{return}\ v\rangle!} (\Theta, \Phi \vdash n\langle \mathsf{let}\ x_2 : T_2 = \mathsf{block}\ \mathsf{in}\ t_2\rangle : \Delta, \Sigma)$$

$$(\Theta, \Phi \vdash n\langle \mathsf{let}\ x_2 : T_2 = \mathsf{block}\ \mathsf{in}\ t_2\rangle : \Delta, \Sigma) \stackrel{\bar{s}}{\Longrightarrow} (\Theta', \Phi \vdash C_2' : \Delta', \Sigma')$$

where

$$\nu(\Delta', \Theta', \Sigma' \setminus \Delta, \Theta, \Sigma) \,.\, \nu(\vec{n}_1 : \vec{T}_1) \,.\, (C_1' \parallel n\langle t_1\rangle) \mathbin{/\!\!/\!\!\backslash} C_2' \equiv \nu(\vec{n} : \vec{T}) \,.\, (C \parallel n\langle t\rangle)$$

or symmetrically, as required. □

Lemma A.6 *If* $C_1 \mathbin{/\!\!/\!\!\backslash} C_2 \equiv C$ *and* $C \xrightarrow{\beta} C'$ *then there exists some trace* s *such that* $(\Delta, \Phi \vdash C_1 : \Theta, \Sigma) \stackrel{s}{\Longrightarrow} (\Delta', \Phi \vdash C_1' : \Theta', \Sigma')$ *and* $(\Theta, \Phi \vdash C_2 : \Delta, \Sigma) \stackrel{\bar{s}}{\Longrightarrow} (\Theta', \Phi \vdash C_2' : \Delta', \Sigma')$ *where* $\nu(\Delta', \Theta', \Sigma' \setminus \Delta, \Theta, \Sigma) \,.\, (C_1' \mathbin{/\!\!/\!\!\backslash} C_2') \equiv C'$.

Proof: We must have that $C \xrightarrow{\beta} C'$ from:

$$\begin{array}{rcl} C & \equiv & \nu(\vec{n} : \vec{T}) \,.\, (D \parallel n\langle \mathsf{let}\ x : T = e\ \mathsf{in}\ t\rangle) \\ C' & \equiv & \nu(\vec{n} : \vec{T}, \vec{n}' : \vec{T}') \,.\, (D \parallel E \parallel n\langle \mathsf{let}\ \vec{x} : \vec{T} = \vec{e}\ \mathsf{in}\ t\rangle) \end{array}$$

where we have an axiom:

$$n\langle \mathsf{let}\ x : T = e\ \mathsf{in}\ t\rangle \xrightarrow{\beta} \nu(\vec{n}' : \vec{T}') \,.\, (E \parallel n\langle \mathsf{let}\ \vec{x} : \vec{T} = \vec{e}\ \mathsf{in}\ t\rangle$$

We then use Lemma A.5 to get (wlog):

$$(\Delta, \Phi \vdash C_1 : \Theta, \Sigma) \stackrel{s}{\Longrightarrow} (\Delta', \Phi \vdash \nu(\vec{n}_1 : \vec{T}_1) \,.\, (C_1'' \parallel n\langle \mathsf{let}\ x : T = e\ \mathsf{in}\ t_1\rangle) : \Theta', \Sigma')$$

$$(\Theta, \Phi \vdash C_2 : \Delta, \Sigma) \stackrel{\bar{s}}{\Longrightarrow} (\Theta', \Phi \vdash C_2' : \Delta', \Sigma')$$

where

$$\nu(\Delta', \Theta', \Sigma' \setminus \Delta, \Theta, \Sigma) \,.\, \nu(\vec{n}_1 : \vec{T}_1) \,.\, (C_1'' \parallel n\langle t_1\rangle) \mathbin{/\!\!/\!\!\backslash} C_2' \equiv \nu(\vec{n} : \vec{T}) \,.\, (D \parallel n\langle t\rangle)$$

and so we use the axiom to get:

$$(\Delta, \Phi \vdash C_1 : \Theta, \Sigma) \stackrel{s}{\Longrightarrow} (\Delta', \Phi \vdash C_1' : \Theta', \Sigma')$$

where we define:

$$C_1' \equiv \nu(\vec{n}_1 : \vec{T}_1, \vec{n}' : \vec{T}') \,.\, (C_1'' \parallel E \parallel n\langle \mathsf{let}\ \vec{x} : \vec{T} = \vec{e}\ \mathsf{in}\ t_1 \rangle)$$

and then verify that:

$$\nu(\Delta', \Theta', \Sigma' \setminus \Delta, \Theta, \Sigma) \,.\, (C_1' \mathbin{/\!\!\!\backslash\!\!\backslash} C_2') \equiv C'$$

as required. □

Lemma A.7 *If* $C_1 \mathbin{/\!\!\!\backslash\!\!\backslash} C_2 \equiv C$ *and* $C \stackrel{\tau}{\longrightarrow} C'$ *then there exists some trace s such that* $(\Delta, \Phi \vdash C_1 : \Theta, \Sigma) \stackrel{s}{\Longrightarrow} (\Delta', \Phi \vdash C_1' : \Theta', \Sigma')$ *and* $(\Theta, \Phi \vdash C_2 : \Delta, \Sigma) \stackrel{\bar{s}}{\Longrightarrow} (\Theta', \Phi \vdash C_2' : \Delta', \Sigma')$ *where* $\nu(\Delta', \Theta', \Sigma' \setminus \Delta, \Theta, \Sigma) \,.\, (C_1' \mathbin{/\!\!\!\backslash\!\!\backslash} C_2') \equiv C'$.

Proof: We must have that:

$$\begin{array}{rcl} C & \equiv & \nu(\vec{n} : \vec{T}) \,.\, (D \parallel p[O] \parallel n\langle \mathsf{let}\ x : T = e\ \mathsf{in}\ t \rangle) \\ C' & \equiv & \nu(\vec{n} : \vec{T}) \,.\, (D \parallel p[O'] \parallel n\langle \mathsf{let}\ x : T = e'\ \mathsf{in}\ t \rangle) \end{array}$$

where we have an axiom:

$$p[O] \parallel n\langle \mathsf{let}\ x : T = e\ \mathsf{in}\ t \rangle \stackrel{\tau}{\longrightarrow} p[O'] \parallel n\langle \mathsf{let}\ x : T = e'\ \mathsf{in}\ t \rangle$$

We then use Lemma A.5 to get (wlog):

$$\begin{array}{l} (\Delta, \Phi \vdash C_1 : \Theta, \Sigma) \stackrel{s}{\Longrightarrow} (\Delta', \Phi \vdash \nu(\vec{n}_1 : \vec{T}_1) \,.\, (C_1'' \parallel n\langle \mathsf{let}\ x : T = e\ \mathsf{in}\ t_1 \rangle) : \Theta', \Sigma') \\ (\Theta, \Phi \vdash C_2 : \Delta, \Sigma) \stackrel{\bar{s}}{\Longrightarrow} (\Theta', \Phi \vdash C_2'' : \Delta', \Sigma') \end{array}$$

where:

$$\nu(\Delta', \Theta', \Sigma' \setminus \Delta, \Theta, \Sigma) \,.\, \nu(\vec{n}_1 : \vec{T}_1) \,.\, (C_1'' \parallel n\langle t_1 \rangle) \mathbin{/\!\!\!\backslash\!\!\backslash} C_2'' \equiv \nu(\vec{n} : \vec{T}) \,.\, (D \parallel p[O] \parallel n\langle t \rangle)$$

We now have three cases:

- **Case** ($p \in \mathsf{dom}\ (C_1'')$).

  We must have that:

  $$C_1'' \equiv \nu(\vec{p} : \vec{U}) \,.\, (C_1''' \parallel p[O])$$

  and so we use the axiom to get:

  $$(\Delta, \Phi \vdash C_1 : \Theta, \Sigma) \stackrel{s}{\Longrightarrow} (\Delta', \Phi \vdash C_1' : \Theta', \Sigma')$$

  where we define:

  $$C_1' \equiv \nu(\vec{n}_1 : \vec{T}_1, \vec{p} : \vec{U}) \,.\, (C_1''' \parallel p[O'] \parallel n\langle \mathsf{let}\ x : T = e'\ \mathsf{in}\ t_1 \rangle)$$

  and then verify that:

  $$\nu(\Delta', \Theta', \Sigma' \setminus \Delta, \Theta, \Sigma) \,.\, (C_1' \mathbin{/\!\!\!\backslash\!\!\backslash} C_2'') \equiv C'$$

  as required.

- **Case** $(p \notin \mathsf{dom}\,(C_1''), n \in \mathsf{dom}\,(C_2''))$.

  We must have that:

  $$C_2'' \equiv \nu(\vec{p} : \vec{U}) \,.\, (C_2''' \parallel p[O] \parallel n\langle \mathsf{let}\; y : U = \mathsf{block}\;\mathsf{in}\; t_2 \rangle)$$

  Moreover, since C_1 is write-closed we must have that the axiom is:

  $$p[O] \parallel n\langle \mathsf{let}\; x : T = p.l(\vec{v})\;\mathsf{in}\; t\rangle \xrightarrow{\tau} p[O] \parallel n\langle \mathsf{let}\; x : T = O.l(p)(\vec{v})\;\mathsf{in}\; t\rangle$$

  in which case:

  $$(\Delta, \Phi \vdash C_1 : \Theta, \Sigma) \xRightarrow{s\,\nu(\vec{n}_1' : \vec{T}_1').n\langle \mathsf{call}\, p.l(\vec{v})\rangle!} (\Delta, \Phi \vdash C_1' : \Theta', \vec{n}_1' : \vec{T}_1', \Sigma')$$

  where we define:

  $$C_1' \equiv \nu(\vec{n}_1'' : \vec{T}_1'') \,.\, (C_1'' \parallel n\langle \mathsf{let}\; x : T = \mathsf{block}\;\mathsf{in}\; t_1 \rangle)$$

  and we partition $\{\vec{n}_1 : \vec{T}_1\}$ into $\{\vec{n}_1' : \vec{T}_1', \vec{n}_1'' : \vec{T}_1''\}$ such that $\{\vec{n}_1'\} \subseteq \mathsf{fn}\,(p.l(\vec{v}))$ and $\{\vec{n}_1''\} \cap \mathsf{fn}\,(p.l(\vec{v})) = \emptyset$.

  We also have:

  $$(\Delta, \Phi \vdash C_2 : \Theta, \Sigma) \xRightarrow{s\,\nu(\vec{n}_1' : \vec{T}_1').n\langle \mathsf{call}\, p.l(\vec{v})\rangle?} (\Delta, \vec{n}_1' : \vec{T}_1', \Phi \vdash C_2' : \Theta', \Sigma')$$

  where we define:

  $$C_2' \equiv \nu(\vec{p} : \vec{U}) \,.\, (C_2''' \parallel p[O] \parallel n\langle \mathsf{let}\; x : T = O.l(p)(\vec{v})\;\mathsf{in}\;\mathsf{let}\; y : U = \mathsf{return}\,(x : T)\;\mathsf{in}\; t_2 \rangle)$$

  and then verify that:

  $$\nu(\Delta', \vec{n}_1' : \vec{T}_1', \Theta', \Sigma' \setminus \Delta, \Theta, \Sigma) \,.\, (C_1' \mathbin{/\!\!\!\backslash\!\!\backslash} C_2') \equiv C'$$

  as required.

- **Case** $(p \notin \mathsf{dom}\,(C_1''), n \notin \mathsf{dom}\,(C_2''))$.

  Similar to the above. □

Decomposition now follows by induction on the number of reductions in $C_1 \mathbin{/\!\!\!\backslash\!\!\backslash} C_2 \Rightarrow C'$ and makes use of Lemmas A.6 and A.7.

B Proof of definability

We have to show that for any $\Delta \vdash s : \mathsf{trace}\;\Theta$ we have $(\Delta \vdash \mathsf{Comp}\,(\Delta \vdash s : \mathsf{trace}\;\Theta) : \Theta) \xRightarrow{r}$ if and only if $\Delta \vdash r \sqsubseteq s : \mathsf{trace}\;\Theta$.

There are two parts to this proof: 'if' and 'only if', which we will detail in the following sections. First though, for technical reasons, we extend the notion of β-reduction.

B.1 Technical preliminaries

In a component $\nu(\Delta)\,.\,(p[O] \parallel C)$, the object name p is *immutable* if:

- There are no occurrences of $p.l \Leftarrow M$ in O or C.
- In each method $\varsigma(n:T)\,.\,\lambda(\vec{x}:\vec{T})\,.\,\langle t\rangle$ in O, there are no occurrences of $n.l \Leftarrow M$ in t.

We can now extend the notion of β-reduction to include method calls on immutable objects:

$$p[O] \parallel n\langle \mathsf{let}\, x:T = p.l(\vec{v})\ \mathsf{in}\ t\rangle \xrightarrow{\beta} p[O] \parallel n\langle \mathsf{let}\, x:T = O.l(p)(\vec{v})\ \mathsf{in}\ t\rangle \qquad \text{(when } p \text{ is immutable)}(\dagger)$$

The important property of β-reductions is that they are confluent with all other transitions:

Proposition B.1 *If*

$$\begin{array}{ccc} (\Delta \vdash C : \Theta) & \xrightarrow{\beta} & (\Delta \vdash C' : \Theta) \\ {\scriptstyle\alpha}\downarrow & & \\ (\Delta' \vdash C'' : \Theta') & & \end{array}$$

then either $\alpha = \beta$ *and* $C' \equiv C''$ *or*

$$\begin{array}{ccc} (\Delta \vdash C : \Theta) & \xrightarrow{\beta} & (\Delta \vdash C' : \Theta) \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\alpha}\downarrow \\ (\Delta' \vdash C'' : \Theta') & \xrightarrow{\beta} & (\Delta' \vdash C''' : \Theta') \end{array}$$

Proof: A case analysis of the possible reductions of C. □

Corollary B.2 *If*

$$\begin{array}{ccc} (\Delta \vdash C : \Theta) & \xrightarrow{\beta}{}^{*} & (\Delta \vdash C' : \Theta) \\ {\scriptstyle s}\Downarrow & & \\ (\Delta' \vdash C'' : \Theta') & & \end{array}$$

then

$$\begin{array}{ccc} (\Delta \vdash C : \Theta) & \xrightarrow{\beta}{}^{*} & (\Delta \vdash C' : \Theta) \\ {\scriptstyle s}\Downarrow & & {\scriptstyle s}\Downarrow \\ (\Delta' \vdash C'' : \Theta') & \xrightarrow{\beta}{}^{*} & (\Delta' \vdash C''' : \Theta') \end{array}$$

B.2 The 'if' direction

We suppose that $\Delta \vdash r \sqsubseteq s : \mathsf{trace}\ \Theta$. We note that, due to Proposition 5.6, it suffices to show that: $(\Delta \vdash \mathsf{Comp}\ (\Delta \vdash s : \mathsf{trace}\ \Theta) : \Theta) \stackrel{s}{\Longrightarrow}$. We proceed by describing the different components which may be reached from $\mathsf{Comp}\ (\Delta \vdash s : \mathsf{trace}\ \Theta)$ after performing each visible action in s. We do this by giving in Figure 20 a definition for *a component for* $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$. The intended meaning is that a component for $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$ has already performed the prefix r of s and is still able to perform the remaining actions in s. Note that in any component for $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$, the only mutable object is ref: all other objects are immutable. This allows us to use the extended notion of β-reduction given by ($\dagger$) above.

Lemma B.3 *For any* $\Delta \vdash s : \mathsf{trace}\ \Theta$ *we have* $\mathsf{Comp}\ (\Delta \vdash s : \mathsf{trace}\ \Theta)$ *is a component for* $\Delta \vdash \varepsilon \leq s : \mathsf{trace}\ \Theta$.

A *component for* $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$ (resp. for $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$) is one of the form:

$$
\begin{array}{l}
\nu(\Theta(s) \setminus \Theta(q)) \,.\, \nu(\mathsf{ref} : \mathsf{Ref}) \,.\, \nu(\mathsf{state}_{r'} : \mathsf{State} \mid \Delta \vdash r' \leq r : \mathsf{trace}\ \Theta) \,.\, (\\
\quad \mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel \\
\quad \prod\{\mathsf{state}_{r'}[\mathsf{State}(\Delta \vdash r' \leq s : \mathsf{trace}\ \Theta)] \mid \Delta \vdash r' \leq r : \mathsf{trace}\ \Theta\} \parallel \\
\quad \prod\{p[l_i = \mathsf{ref.val.inCall}_{p.l_i:L_i} \mid i = 1 \ldots n] \mid p : [l_i : L_i \mid i = 1 \ldots n] \in \Theta, \Theta(s)\} \parallel \\
\quad \prod\{n\langle t_n \rangle \mid n : \mathsf{thread} \in \Theta, \Theta(s)\} \parallel \\
\quad \prod\{n\langle t_n \rangle \mid n : \mathsf{thread} \in \Delta, \Delta(s) \text{ and } n \in \mathsf{threads}\ (q)\} \\
)
\end{array}
$$

where t_n is a thread at n for $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$ (resp. for $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$).

A *thread at n for* $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$ is one of the following:

1. $\mathsf{let}\ x : T = \mathsf{ref.val.out}_T()\ \mathsf{in}\ t$
   where n is output-enabled in $\Delta \vdash r : \mathsf{trace}\ \Theta$ and t is a $\mathsf{return}\,(x : T)$ thread at n for $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$.
2. $\mathsf{let}\ x : T = \mathsf{block}\ \mathsf{in}\ t$
   where n is input-enabled in $\Delta \vdash r : \mathsf{trace}\ \Theta$ and t is a $\mathsf{return}\,(x : T)$ thread at n for $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$.

A $\mathsf{return}\,(v : T)$ *thread at n for* $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$ is one of the following:

1. v
   where n is balanced in r.
2. $\mathsf{ref.val.inReturn}_T(v); t$
   where $r = r_1\, a\, r_2$, $a = \nu(\Theta') \,.\, n\langle \mathsf{call}\, p.l(\vec{v}) \rangle!$, n is balanced in r_2, and t is a thread at n for $\Delta \vdash r_1 \leq s : \mathsf{trace}\ \Theta$.
3. $\mathsf{let}\ y : U = \mathsf{return}\,(v : T)\ \mathsf{in}\ t$
   where $r = r_1\, a\, r_2$, $a = \nu(\Theta') \,.\, n\langle \mathsf{call}\, p.l(\vec{v}) \rangle?$, n is balanced in r_2, and t is a $\mathsf{return}\,(y : U)$ thread at n for $\Delta \vdash r_1 \leq s : \mathsf{trace}\ \Theta$.

Figure 20: Definition of a component for $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$ and for $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$

A *thread at n for* $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$ is one of the following:

1. stop
2. a thread at n for $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$
   where $\mathsf{proj}\, n\ (q) = \mathsf{proj}\, n\ (r)$.
3. $\mathsf{let}\ x : T = p.l(\vec{v})\ \mathsf{in}\ t$
   where $\mathsf{proj}\, n\ (qa) = \mathsf{proj}\, n\ (r)$, $a = \nu(\Theta')\,.\,n\langle \mathsf{call}\, p.l(\vec{v})\rangle!$, and t is a $\mathsf{return}\,(x : T)$ thread at n for $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$.
4. $\mathsf{let}\ x : T = \mathsf{return}\,(v : U)\ \mathsf{in}\ t$
   where $\mathsf{proj}\, n\ (qa) = \mathsf{proj}\, n\ (r)$, $a = \nu(\Theta')\,.\,n\langle \mathsf{return}\, v\rangle!$, and t is a $\mathsf{return}\,(x : T)$ thread at n for $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$.
5. $\mathsf{let}\ y : U = \mathsf{ref.val.inCall}_{p.l:L}(\vec{v})\ \mathsf{in}\ \mathsf{let}\ x : T = \mathsf{return}\,(y : U)\ \mathsf{in}\ t$
   where $\mathsf{proj}\, n\ (q) = \mathsf{proj}\, n\ (ra)$, $a = \nu(\Delta')\,.\,n\langle \mathsf{call}\, p.l(\vec{v})\rangle?$, and t is a $\mathsf{return}\,(x : T)$ thread at n for $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$.
6. t
   where $\mathsf{proj}\, n\ (q) = \mathsf{proj}\, n\ (ra)$, $a = \nu(\Delta')\,.\,n\langle \mathsf{return}\, v\rangle?$, and t is a $\mathsf{return}\,(v : T)$ thread at n for $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$ for some T.
7. $\mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)];t$
   where $\mathsf{proj}\, n\ (q) = \mathsf{proj}\, n\ (ra)$, and t is a thread at n for $\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta$.
8. t
   where $n\langle t\rangle \xrightarrow{\beta} n\langle t'\rangle$ and t' is a thread at n for $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$

Figure 21: Definition of a thread for $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$

Proof: An inspection of the definition of Comp $(\Delta \vdash s : \mathsf{trace}\ \Theta)$. □

Lemma B.4 *If* $\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta$ *and* $\Delta' \vdash C : \Theta'$ *is a component for* $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$ *then* $(\Delta' \vdash C : \Theta') \stackrel{a}{\Longrightarrow} (\Delta'' \vdash C' : \Theta'')$ *where* C' *is a component for* $\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta$.

Proof: By considering the definition of $\Delta \vdash r : \mathsf{trace}\ \Theta$ we see that the following cases are exhaustive:

1. **Case** $a = \nu(\Theta''') \,.\, n\langle \mathsf{return}\, v\rangle!$ and $C \equiv \nu(\Theta''') \,.\, C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel n\langle \mathsf{let}\ y : U = \mathsf{ref.val.out}_U()\ \mathsf{in}\ \mathsf{let}\ x : T = \mathsf{return}\,(y : U)\ \mathsf{in}\ t\rangle]$

   We have:

$$
\begin{aligned}
&(\Delta' \vdash C : \Theta')\\
&\stackrel{\tau}{\longrightarrow} (\Delta' \vdash \nu(\Theta''') \,.\, C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel \\
&\qquad n\langle \mathsf{let}\ y : U = \mathsf{state}_r.\mathsf{out}_U()\ \mathsf{in}\ \mathsf{let}\ x : T = \mathsf{return}\,(y : U)\ \mathsf{in}\ t\rangle] : \Theta')\\
&\stackrel{\beta}{\longrightarrow}{}^{*} (\Delta' \vdash \nu(\Theta''') \,.\, C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel \\
&\qquad n\langle \mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)]; \mathsf{let}\ y : U = v\ \mathsf{in}\ \mathsf{let}\ x : T = \mathsf{return}\,(y : U)\ \mathsf{in}\ t\rangle] : \Theta')\\
&\stackrel{\tau}{\longrightarrow} (\Delta' \vdash \nu(\Theta''', \mathsf{state}_{ra} : \mathsf{State}) \,.\, C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_{ra}] \parallel \mathsf{state}_{ra}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)] \parallel \\
&\qquad n\langle \mathsf{let}\ y : U = v\ \mathsf{in}\ \mathsf{let}\ x : T = \mathsf{return}\,(y : U)\ \mathsf{in}\ t\rangle] : \Theta')\\
&\stackrel{\beta}{\longrightarrow}{}^{*} (\Delta' \vdash \nu(\Theta''', \mathsf{state}_{ra} : \mathsf{State}) \,.\, C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_{ra}] \parallel \mathsf{state}_{ra}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)] \parallel \\
&\qquad n\langle \mathsf{let}\ x : T = \mathsf{return}\,(v : U)\ \mathsf{in}\ t\rangle] : \Theta')\\
&\stackrel{a}{\longrightarrow} (\Delta' \vdash \nu(\mathsf{state}_{ra} : \mathsf{State}) \,.\, C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_{ra}] \parallel \mathsf{state}_{ra}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)] \parallel \\
&\qquad n\langle \mathsf{let}\ x : T = \mathsf{block}\ \mathsf{in}\ t\rangle] : \Theta', \Theta''')
\end{aligned}
$$

   which is a component for $\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta$ as required.

2. **Case** $a = \nu(\Theta''') \,.\, n\langle \mathsf{call}\, p.l(\vec{v})\rangle!$ and $C \equiv \nu(\Theta''') \,.\, C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel n\langle \mathsf{let}\ y : U = \mathsf{ref.val.out}_U()\ \mathsf{in}\ t\rangle]$

   We have:

$$
\begin{aligned}
&(\Delta' \vdash C : \Theta')\\
&\stackrel{\tau}{\longrightarrow} (\Delta' \vdash \nu(\Theta''') \,.\, C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel \\
&\qquad n\langle \mathsf{let}\ y : U = \mathsf{state}_r.\mathsf{out}_U()\ \mathsf{in}\ t\rangle] : \Theta')\\
&\stackrel{\beta}{\longrightarrow}{}^{*} (\Delta' \vdash \nu(\Theta''') \,.\, C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel \\
&\qquad n\langle \mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)];\\
&\qquad\qquad \mathsf{let}\ x : T = p.l(\vec{v})\ \mathsf{in}\ \mathsf{ref.val.inReturn}_T(x); \mathsf{let}\ y : U = \mathsf{ref.val.out}_U()\ \mathsf{in}\ t\rangle] : \Theta')\\
&\stackrel{\tau}{\longrightarrow} (\Delta' \vdash \nu(\Theta''', \mathsf{state}_{ra} : \mathsf{State}) \,.\, C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_{ra}] \parallel \mathsf{state}_{ra}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)] \parallel \\
&\qquad n\langle \mathsf{let}\ x : T = p.l(\vec{v})\ \mathsf{in}\ \mathsf{ref.val.inReturn}_T(x); \mathsf{let}\ y : U = \mathsf{ref.val.out}_U()\ \mathsf{in}\ t\rangle] : \Theta')\\
&\stackrel{a}{\longrightarrow} (\Delta' \vdash \nu(\mathsf{state}_{ra} : \mathsf{State}) \,.\, C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_{ra}] \parallel \mathsf{state}_{ra}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)] \parallel \\
&\qquad n\langle \mathsf{let}\ x : T = \mathsf{block}\ \mathsf{in}\ \mathsf{ref.val.inReturn}_T(x); \mathsf{let}\ y : U = \mathsf{ref.val.out}_U()\ \mathsf{in}\ t\rangle] : \Theta', \Theta''')
\end{aligned}
$$

   which is a component for $\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta$ as required.

3. **Case** $a = \nu(\Delta''') \,.\, n\langle \mathsf{return}\, v\rangle?$ and $C \equiv C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel n\langle \mathsf{let}\ x : T = \mathsf{block}\ \mathsf{in}\ \mathsf{ref.val.inReturn}_T(x); t\rangle]$

We have:

$$
\begin{aligned}
&(\Delta' \vdash C : \Theta')\\
&\quad\xrightarrow{a} (\Delta', \Delta''' \vdash C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel\\
&\qquad\quad n\langle \mathsf{let}\ x : T = v\ \mathsf{in}\ \mathsf{ref.val.inReturn}_T(x); t\rangle] : \Theta')\\
&\quad\xrightarrow{\beta}{}^{*} (\Delta', \Delta''' \vdash C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel\\
&\qquad\quad n\langle \mathsf{ref.val.inReturn}_T(v); t\rangle] : \Theta')\\
&\quad\xrightarrow{\tau} (\Delta', \Delta''' \vdash C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel\\
&\qquad\quad n\langle \mathsf{state}_r.\mathsf{inReturn}_T(v); t\rangle] : \Theta')\\
&\quad\xrightarrow{\beta}{}^{*} (\Delta', \Delta''' \vdash C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel\\
&\qquad\quad n\langle \mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)]; t\rangle] : \Theta')\\
&\quad\xrightarrow{\tau} (\Delta', \Delta''' \vdash C[\nu(\mathsf{state}_{ra} : \mathsf{State})\,.\,\mathsf{ref}[\mathsf{val} = \mathsf{state}_{ra}] \parallel \mathsf{state}_{ra}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)] \parallel\\
&\qquad\quad n\langle t\rangle] : \Theta')
\end{aligned}
$$

which is a component for $\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta$ as required.

4. **Case** $a = \nu(\Delta''')\,.\,n\langle \mathsf{call}\ p.l(\vec{v})\rangle?$ and $C \equiv C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel n\langle \mathsf{let}\ x : T = \mathsf{block}\ \mathsf{in}\ t\rangle]$

   We have:

$$
\begin{aligned}
&(\Delta' \vdash C : \Theta')\\
&\quad\xrightarrow{a} (\Delta', \Delta''' \vdash C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel\\
&\qquad\quad n\langle \mathsf{let}\ y : U = p.l(\vec{v})\ \mathsf{in}\ \mathsf{let}\ x : T = \mathsf{return}\,(y : U)\ \mathsf{in}\ t\rangle] : \Theta')\\
&\quad\xrightarrow{\beta}{}^{*} (\Delta', \Delta''' \vdash C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel\\
&\qquad\quad n\langle \mathsf{let}\ y : U = \mathsf{ref.val.inCall}_{p.l:L}(\vec{v})\ \mathsf{in}\ \mathsf{let}\ x : T = \mathsf{return}\,(y : U)\ \mathsf{in}\ t\rangle] : \Theta')\\
&\quad\xrightarrow{\tau} (\Delta', \Delta''' \vdash C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel\\
&\qquad\quad n\langle \mathsf{let}\ y : U = \mathsf{state}_r.\mathsf{inCall}_{p.l:L}(\vec{v})\ \mathsf{in}\ \mathsf{let}\ x : T = \mathsf{return}\,(y : U)\ \mathsf{in}\ t\rangle] : \Theta')\\
&\quad\xrightarrow{\beta}{}^{*} (\Delta', \Delta''' \vdash C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel\\
&\qquad\quad n\langle \mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)];\\
&\qquad\quad \mathsf{let}\ y : U = \mathsf{ref.val.out}_U()\ \mathsf{in}\ \mathsf{let}\ x : T = \mathsf{return}\,(y : U)\ \mathsf{in}\ t\rangle] : \Theta')\\
&\quad\xrightarrow{\tau} (\Delta', \Delta''' \vdash C[\nu()\,.\,\mathsf{ref}[\mathsf{val} = \mathsf{state}_{ra}] \parallel \mathsf{state}_{ra}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)] \parallel\\
&\qquad\quad n\langle \mathsf{let}\ y : U = \mathsf{ref.val.out}_U()\ \mathsf{in}\ \mathsf{let}\ x : T = \mathsf{return}\,(y : U)\ \mathsf{in}\ t\rangle] : \Theta')
\end{aligned}
$$

   which is a component for $\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta$ as required.

5. **Case** $a = \nu(\Delta''')\,.\,n\langle \mathsf{call}\ p.l(\vec{v})\rangle?$ and $C \equiv C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r]]$ where $n \notin \Theta'$.

   Similar to the previous case. □

The 'if' half of definability now follows, by induction on Lemma B.4, with Lemma B.3 as the base case.

B.3 The 'only if' direction

We suppose that $\Delta \vdash s : \mathsf{trace}\ \Theta$ and that $(\Delta \vdash \mathsf{Comp}\ (\Delta \vdash s : \mathsf{trace}\ \Theta) : \Theta) \stackrel{r}{\Longrightarrow}$ so we must demonstrate that $\Delta \vdash r \sqsubseteq s : \mathsf{trace}\ \Theta$. As above we make an auxilliary definition of *a component* *for* $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$ in Figures 20 and 21 with the intended meaning that a component for $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$ has performed the trace q and this is $\sqsubseteq$ related to some prefix of s. Note that, as prefix ordering $\leq$ on traces is contained in $\sqsubseteq$ and $\sqsubseteq$ is transitive, then we also have $q \sqsubseteq s$ for such components. Again, in any component for $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$, the only mutable object is ref: all other objects are immutable. This allows us to use the extended notion of β-reduction given by (†) above.

Lemma B.5 *For any* $\Delta \vdash s : \mathsf{trace}\ \Theta$ *we have* $\mathsf{Comp}\ (\Delta \vdash s : \mathsf{trace}\ \Theta)$ *is a component for* $\Delta \vdash \varepsilon \sqsubseteq \varepsilon \leq s : \mathsf{trace}\ \Theta$.

Proof: An inspection of the definition of $\mathsf{Comp}\ (\Delta \vdash s : \mathsf{trace}\ \Theta)$. □

Lemma B.6 *If* C *is a component for* $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$ *and* $C \xrightarrow{\beta} C'$ *then* C' *is a component for* $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$.

Proof: An inspection of the definition of a component for $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$. □

Lemma B.7 *If* C *is a component for* $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$ *and* $C \xrightarrow{\tau} C'$ *then* $C' \xrightarrow{\beta}{}^{*} C''$ *where* C'' *is a component for* $\Delta \vdash q \sqsubseteq r' \leq s : \mathsf{trace}\ \Theta$.

Proof: The following cases are exhaustive:

1. **Case** $C \equiv C[n\langle \mathsf{let}\ x : T = \mathsf{ref.val.inCall}_{p.l:L}(\vec{v})\ \mathsf{in}\ t\rangle] \xrightarrow{\tau} C[n\langle \mathsf{let}\ x : T = \mathsf{state}_r.\mathsf{inCall}_{p.l:L}(\vec{v})\ \mathsf{in}\ t\rangle] \equiv C'$
   where $\mathsf{proj}\, n\ (q) = \mathsf{proj}\, n\ (ra)$, $a = \nu(\Delta') \,.\, n\langle \mathsf{call}\ p.l(\vec{v})\rangle?$, and t is a return $(x : T)$ thread at n for $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$.

   If (up to α-converting Θ′) $\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta$ then we have:

   $$C' \xrightarrow{\beta}{}^{*} C[n\langle \mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)]; \mathsf{let}\ x : T = \mathsf{ref.val.out}_U()\ \mathsf{in}\ t\rangle]$$

   which is a component for $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$ as required.

   If $\Delta \vdash ra \not\leq s : \mathsf{trace}\ \Theta$ then we have:

   $$C' \xrightarrow{\beta}{}^{*} C[n\langle \mathsf{stop}\rangle]$$

   which is a component for $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$ as required.

2. **Case** $C \equiv C[n\langle \mathsf{ref.val.inReturn}_T(v); t\rangle] \xrightarrow{\tau} C[n\langle \mathsf{state}_r.\mathsf{inReturn}_T(v); t\rangle] \equiv C'$
   where $\mathsf{proj}\, n\ (q) = \mathsf{proj}\, n\ (ra)$, $a = \nu(\Delta') \,.\, n\langle \mathsf{return}\ v\rangle?$, and t is a thread at n for $\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta$.

   If (up to α-converting Θ′) $\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta$ then we have:

   $$C' \xrightarrow{\beta}{}^{*} C[n\langle \mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)]; t\rangle]$$

   which is a component for $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$ as required.

   If $\Delta \vdash ra \not\leq s : \mathsf{trace}\ \Theta$ then we have:

   $$C' \xrightarrow{\beta}{}^{*} C[n\langle \mathsf{stop}\rangle]$$

   which is a component for $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$ as required.

3. **Case** $C \equiv C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_r] \parallel n\langle \mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)]; t\rangle]$
$\xrightarrow{\tau} \nu(\mathsf{state}_{ra} : \mathsf{State}) . C[\mathsf{ref}[\mathsf{val} = \mathsf{state}_{ra}] \parallel \mathsf{state}_{ra}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)] \parallel n\langle t\rangle] \equiv C'$
where t is a thread at n for $\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta$.

   By definition, C' is a component for $\Delta \vdash q \sqsubseteq ra \leq s : \mathsf{trace}\ \Theta$.

4. **Case** $C \equiv C[n\langle \mathsf{let}\ x : T = \mathsf{ref.val.out}_T()\ \mathsf{in}\ t\rangle] \xrightarrow{\tau} C[n\langle \mathsf{let}\ x : T = \mathsf{state}_r.\mathsf{out}_T()\ \mathsf{in}\ t\rangle] \equiv C'$
where $\mathsf{proj}\, n\ (q) = \mathsf{proj}\, n\ (r)$, n is output-enabled in $\Delta \vdash r : \mathsf{trace}\ \Theta$ and t is a $\mathsf{return}\,(x : T)$ thread at n for $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$.

   If $\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta$ and $a = \nu(\Theta') . n\langle \mathsf{call}\, p.l(\vec{v})\rangle!$ then:

$$C' \xrightarrow{\beta}{}^{*} \quad C[n\langle \mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)]; \\ \mathsf{ref.val.inReturn}_U(p.l(\vec{v})); \mathsf{let}\ x : T = \mathsf{ref.val.out}_T()\ \mathsf{in}\ t\rangle]$$

   which is a component for $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$ as required.

   If $\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta$ and $a = \nu(\Theta') . n\langle \mathsf{return}\, v\rangle!$ then we must have that $r = r_1\, \nu(\Theta') . n\langle \mathsf{call}\, p.l(\vec{v})\rangle?\, r_2$ where n is balanced in r_2. Thus, since t is a $\mathsf{return}\,(x : T)$ thread at n for $\Delta \vdash r \leq s : \mathsf{trace}\ \Theta$ we must have that:

$$t = \mathsf{let}\ y : U = \mathsf{return}\,(x : T)\ \mathsf{in}\ t'$$

   where t' is a $\mathsf{return}\,(y : U)$ thread at n for $\Delta \vdash r_1 \leq s : \mathsf{trace}\ \Theta$, so t' is also a $\mathsf{return}\,(y : U)$ thread at n for $\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta$, so $\mathsf{let}\ x : T = v\ \mathsf{in}\ t$ is a thread at n for $\Delta \vdash q \sqsubseteq ra \leq s : \mathsf{trace}\ \Theta$. Then:

$$C' \xrightarrow{\beta}{}^{*} C[n\langle \mathsf{ref.val} := \mathsf{new}[\mathsf{State}(\Delta \vdash ra \leq s : \mathsf{trace}\ \Theta)]; \mathsf{let}\ x : T = v\ \mathsf{in}\ t\rangle]$$

   which is a component for $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$ as required.

   Otherwise:

$$C' \xrightarrow{\beta}{}^{*} C[n\langle \mathsf{stop}\rangle]$$

   which is a component for $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$ as required. □

Lemma B.8 *If* $\Delta' \vdash C : \Theta'$ *is a component for* $\Delta \vdash q \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$ *and* $(\Delta' \vdash C : \Theta') \xrightarrow{a} (\Delta'' \vdash C' : \Theta'')$ *then* $C' \xrightarrow{\beta}{}^{*} C''$ *where* C'' *is a component for* $\Delta \vdash q\,a \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$.

Proof: The following cases are exhaustive:

1. **Case** $(\Delta' \vdash C : \Theta') \xrightarrow{\nu(\Delta''').n\langle \mathsf{call}\, p.l(\vec{v})\rangle?} (\Delta', \Delta''' \vdash C \parallel n\langle \mathsf{let}\ x : T = p.l(\vec{v})\ \mathsf{in}\ \mathsf{return}\,(x : T)\rangle : \Theta')$
where $n \notin \Theta'$.

   We have:

$$C' \xrightarrow{\beta}{}^{*} C \parallel n\langle \mathsf{let}\ x : T = \mathsf{ref.val.inCall}_{p.l:L}(\vec{v})\ \mathsf{in}\ \mathsf{return}\,(x : T)\rangle$$

   which is a component for $\Delta \vdash q\,a \sqsubseteq r \leq s : \mathsf{trace}\ \Theta$ as required.

2. **Case** $(\Delta' \vdash C[n\langle \mathsf{let}\ x : T = \mathsf{block}\ \mathsf{in}\ t\rangle] : \Theta') \xrightarrow{\nu(\Delta''').n\langle \mathsf{call}\, p.l(\vec{v})\rangle ?} (\Delta', \Delta''' \vdash C[n\langle \mathsf{let}\ y : U = p.l(\vec{v})\ \mathsf{in}\ \mathsf{let}\ x : T = \mathsf{return}\,(y : U)\ \mathsf{in}\ t\rangle] : \Theta')$
   where $\mathsf{proj}\, n\ (q) = \mathsf{proj}\, n\ (r)$, n is input-enabled in $\Delta \vdash r : \mathsf{trace}\ \Theta$ and t is a $\mathsf{return}\,(x : T)$ thread at n for $\Delta \vdash r \le s : \mathsf{trace}\ \Theta$.

   We have:

   $$C' \xrightarrow{\beta}{}^{*} C[n\langle \mathsf{let}\ y : U = \mathsf{ref.val.inCall}_{p.l:L}(\vec{v})\ \mathsf{in}\ \mathsf{let}\ x : T = \mathsf{return}\,(y : U)\ \mathsf{in}\ t\rangle]$$

   which is a component for $\Delta \vdash q\, a \sqsubseteq r \le s : \mathsf{trace}\ \Theta$ as required.

3. **Case** $(\Delta' \vdash C[n\langle \mathsf{let}\ x : T = \mathsf{block}\ \mathsf{in}\ t\rangle] : \Theta') \xrightarrow{\nu(\Delta''').n\langle \mathsf{return}\, v\rangle ?} (\Delta', \Delta''' \vdash C[n\langle \mathsf{let}\ x : T = v\ \mathsf{in}\ t\rangle] : \Theta')$
   where $\mathsf{proj}\, n\ (q) = \mathsf{proj}\, n\ (r)$, n is input-enabled in $\Delta \vdash r : \mathsf{trace}\ \Theta$ and t is a $\mathsf{return}\,(x : T)$ thread at n for $\Delta \vdash r \le s : \mathsf{trace}\ \Theta$.

   We have:

   $$C' \xrightarrow{\beta}{}^{*} C[n\langle t[v/x]\rangle]$$

   which is a component for $\Delta \vdash q\, a \sqsubseteq r \le s : \mathsf{trace}\ \Theta$ as required.

4. **Case** $(\Delta' \vdash \nu(\Theta''') \,.\, C[n\langle \mathsf{let}\ x : T = p.l(\vec{v})\ \mathsf{in}\ t\rangle] : \Theta') \xrightarrow{\nu(\Theta''').n\langle \mathsf{call}\, p.l(\vec{v})\rangle !} (\Delta' \vdash C[n\langle \mathsf{let}\ x : T = \mathsf{block}\ \mathsf{in}\ t\rangle] : \Theta', \Theta''')$
   where $\mathsf{proj}\, n\ (q\, a) = \mathsf{proj}\, n\ (r)$, and t is a $\mathsf{return}\,(x : T)$ thread at n for $\Delta \vdash r \le s : \mathsf{trace}\ \Theta$.

   We have C' is a component for $\Delta \vdash q\, a \sqsubseteq r \le s : \mathsf{trace}\ \Theta$ as required.

5. **Case** $(\Delta' \vdash \nu(\Theta''') \,.\, C[n\langle \mathsf{let}\ x : T = \mathsf{return}\,(v : U)\ \mathsf{in}\ t\rangle] : \Theta') \xrightarrow{\nu(\Theta''').n\langle \mathsf{return}\, v\rangle !} (\Delta' \vdash C[n\langle \mathsf{let}\ x : T = \mathsf{block}\ \mathsf{in}\ t\rangle] : \Theta', \Theta''')$
   where $\mathsf{proj}\, n\ (q\, a) = \mathsf{proj}\, n\ (r)$, and t is a $\mathsf{return}\,(x : T)$ thread at n for $\Delta \vdash r \le s : \mathsf{trace}\ \Theta$.

   We have C' is a component for $\Delta \vdash q\, a \sqsubseteq r \le s : \mathsf{trace}\ \Theta$ as required. □

The 'only if' half of definability now follows, by induction on Lemmas B.6, B.7, and B.8, with Lemma B.5 as the base case, making appropriate use of Corollary B.2.

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