On "Axiomatising Finite Concurrent Processes"

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Abstract

In his pioneering paper [14], Hennessy gave complete axiomatizations of Milner's observational congruence and of t-observational congruence which made use of an auxiliary operation to axiomatize parallel composition. Unfortunately, those axiomatizations turn out to be flawed due to the subtle interplay between Hennessy's auxiliary parallel operator and synchronization. The aim of this paper is to present correct versions of the equational characterizations given in [14]. Some of the problems which arise in giving operational semantics to the auxiliary operators used in [4, 6, 14] in the theory of congruences like Milner's observational congruence are also discussed.

Key words: Concurrent processes, observational congruence, t-observational congruence, equational logic.

1 Introduction

In his seminal paper [14], Matthew Hennessy has given complete axiomatizations of two behavioural congruences, namely those associated with Milner's *weak bisimulation equivalence* [19] and *t-observational equivalence* [14] (also known as *split-2 equivalence* [10] and *timed equivalence* [1]), over a simple language for concurrent processes. Paper [14] evolved from an early preprint, entitled "On the Relationship between Time and Interleaving", which dated back to 1981 and, in my opinion at least, did not receive the attention it deserved at the time of its first circulation.

Hennessy's "On the Relationship between Time and Interleaving" and its published version [14] have historically played an important role in the development of the theory of process algebras for at least two reasons. First, the equational characterization of observational congruence presented in these papers has been, to the best of my knowledge, the first one to use auxiliary operators in the axiomatization of CCS parallel composition [19]. At more or less the same time, J.A. Bergstra and J.W. Klop were working on a finite axiomatization of strong bisimulation equivalence over ACP which used two auxiliary operators [4], but extensions of their ideas to a setting involving internal actions were first presented in [6]. Secondly, Hennessy's papers present the first axiomatization known to the author of a non-interleaving behavioural equivalence and its laws have helped shape the form of many axiomatizations which followed. (See, *e.g.*, [7, 8, 18, 15].)

Unfortunately, however, there are subtle problems with the axiomatizations published in [14]. In particular, two of the axioms given by Hennessy for his auxiliary parallel operation are *unsound* due to the problems introduced by synchronization. In fact, the whole issue of giving semantics to the auxiliary operations used in [4, 6, 14] to axiomatize various parallel composition operators turns out to be rather subtle in the theory of behavioural congruences associated with weak bisimulation-like equivalences, such as observational congruence and t-observational congruence. The aim of this note is to present correct versions of the axiomatizations given in [14]. In passing, I shall also comment on some of the issues involved in giving suitable operational semantics for the auxiliary operations of ACP in the setting of observational congruence and related congruences. I hope that this will make this paper a useful reference for researchers interested in complete axiomatizations of behavioural congruences.

2 An axiomatization of Hennessy's t-observational congruence

I assume that the reader is familiar with [14] and the basic notions on process algebras and bisimulation equivalence. The uninitiated reader is referred to the textbooks [19, 3] for extensive motivations and background. As this is not an introductory paper, I shall feel free to refer the reader to the motivations, definitions and results given in [14]. Precise pointers to material in [14] will be given whenever necessary.

The language P used by Hennessy in [14] is a simple extension of finite, restriction and relabelling-free CCS. It is given by the grammar

$$p ::= \mathbf{0} \mid \mu.p \mid p + p \mid p \| p \mid p \not| \, p$$

where μ ranges over the set of *actions* Act. The set Act is assumed to have the form $\{\tau\} \cup \Lambda \cup \bar{\Lambda}$, where Λ is a given countable set of names, $\bar{\Lambda} = \{\bar{a} \mid a \in \Lambda\}$ is the set of complement names, and τ is a distinguished action. As usual, we assume that complementation is symmetric, *i.e.* $\bar{\bar{a}} = a$. We use VAct to denote $\Lambda \cup \bar{\Lambda}$, the set of *visible actions*, and a, b to range over it.

The operational semantics for the language P given by Hennessy in Section 2.1 of [14] is based upon the idea that visible actions have a beginning and an ending. Moreover, these distinct events may be observed and are denoted by $S(a)$ and $F(a)$ respectively. Let $\mathsf{E} = \{S(a), F(a) \mid a \in \mathsf{VAct}\} \cup \mathsf{Act}$; in the terminology of [14], this is the set of *events* and I shall use e to range over it. The operational semantics is given in terms of a set of next-state relations $\stackrel{e}{\Rightarrow}$, one for each $e \in \mathsf{E}$. As explained at length in [14], the relations $\stackrel{e}{\Rightarrow}$ are defined over the set of states S, a superlanguage of P obtained by adding new prefixing operators a_S to the formation rules for P. I shall use s, s', s_1, s_2 to range over the set of states S. The relations $\stackrel{e}{\Rightarrow}$ are defined to be the least ones over S which satisfy the rules in Figure 1. Comments on these rules may be found in Section 2.1 of [14].

The relation of *t-observational equivalence* $\approx_T$ is now defined as the largest symmetric relation on states which satisfies

$s_1 \approx_T s_2$ iff for every $e \in \mathsf{E}$, $s_1 \stackrel{e}{\Rightarrow} s_1'$ implies

- $e = \tau$ and $s_1' \approx_T s_2$, or
- $s_2 \stackrel{e}{\Rightarrow} s_2'$ for some s_2' such that $s_1' \approx_T s_2'$.

Following Hennessy, I shall only be interested in $\approx_T$ as it applies to the language of processes P. The equivalence $\approx_T$ is not a congruence over P for the usual reasons associated with the operators $+$ and $\not|$. For example, it is easy to see that $\mathbf{0} \approx_T \tau.\mathbf{0}$, but

$$\mathbf{0} \not| \, a.\mathbf{0} \approx_T \mathbf{0} \not\approx_T \tau.a.\mathbf{0} \approx_T \tau.\mathbf{0} \not| \, a.\mathbf{0}$$

One of the main results in [14] is a complete equational characterization of the largest congruence $\approx_T^C$ contained in $\approx_T$ over the set of processes. (See Theorem 2.1.2 in [14].) For ease of reference, Hennessy's equations for $\approx_T^C$ are collected in Figure 2.

$$\frac{}{a.p \overset{S(a)}{\Rightarrow} a_S.p} \qquad \frac{}{a_S.p \overset{F(a)}{\Rightarrow} p} \qquad \frac{}{\mu.p \overset{\mu}{\Rightarrow} p}$$

$$\frac{s_1 \overset{e}{\Rightarrow} s_1'}{s_1 + s_2 \overset{e}{\Rightarrow} s_1'} \qquad \frac{s_1 \overset{e}{\Rightarrow} s_1'}{s_2 + s_1 \overset{e}{\Rightarrow} s_1'}$$

$$\frac{s_1 \overset{e}{\Rightarrow} s_1'}{s_1 \| s_2 \overset{e}{\Rightarrow} s_1' \| s_2} \qquad \frac{s_1 \overset{e}{\Rightarrow} s_1'}{s_2 \| s_1 \overset{e}{\Rightarrow} s_2 \| s_1'} \qquad \frac{s_1 \overset{e}{\Rightarrow} s_1'}{s_1 \not| \, s_2 \overset{e}{\Rightarrow} s_1' \| s_2}$$

$$\frac{s_1 \overset{a}{\Rightarrow} s_1',\, s_2 \overset{\bar{a}}{\Rightarrow} s_2'}{s_1 \| s_2 \overset{\tau}{\Rightarrow} s_1' \| s_2'} \qquad \frac{s_1 \overset{a}{\Rightarrow} s_1',\, s_2 \overset{\bar{a}}{\Rightarrow} s_2'}{s_1 \not| \, s_2 \overset{\tau}{\Rightarrow} s_1' \| s_2'}$$

$$\frac{s_1 \overset{\tau}{\Rightarrow} s_1',\, s_1' \overset{e}{\Rightarrow} s_2}{s_1 \overset{e}{\Rightarrow} s_2} \qquad \frac{s_1 \overset{e}{\Rightarrow} s_1',\, s_1' \overset{\tau}{\Rightarrow} s_2}{s_1 \overset{e}{\Rightarrow} s_2}$$

Figure 1: Operational rules for $\overset{e}{\Rightarrow}$

Unfortunately, however, the axiomatization presented in Figure 2 is incorrect. This is due to the fact that axiom (B2), which plays a vital role in the reduction of terms to *simple forms* (see the proof of Proposition 2.2.3 in [14]), is unsound as the following example shows. (A similar example may be found on page 142 of [7].)

Example: Consider the terms $p \equiv (a.c.\mathbf{0} \not| \, b.\mathbf{0}) \not| \, \bar{b}.\bar{a}.\mathbf{0}$ and $q \equiv a.c.\mathbf{0} \not| \, (b.\mathbf{0} \| \bar{b}.\bar{a}.\mathbf{0})$. I claim that $p \not\approx^C_T q$. In fact, using the rules in Figure 1, it is easy to see that $b.\mathbf{0} \| \bar{b}.\bar{a}.\mathbf{0} \overset{\bar{a}}{\Rightarrow} \mathbf{0} \| \mathbf{0}$. This allows one to derive that $q \overset{c}{\Rightarrow} \mathbf{0} \| (\mathbf{0} \| \mathbf{0})$. On the other hand, p cannot initially perform a c action. □

The problem in the equational characterization of Hennessy's auxiliary operator $\not|$ derives from the fact that, although simpler than $\|$, $\not|$ still captures two conceptually distinct features of parallel composition. One of them is the asynchronous behaviour due to one of the parallel components; the other is synchronization between processes. In their work on ACP, Bergstra and Klop have used two auxiliary operators, namely *left-merge* $\lfloor\!\lfloor$ and *communication merge* $|$, to give a finite equational axiomatization of the parallel composition operation. Intuitively, the left-merge operation is used to capture the behaviour of parallel composition due to one of the parallel components and the communication merge is used to capture the behaviour deriving from synchronization. In the remainder of this section, I shall present an equational characterization of $\approx^C_T$ over P which will make a fundamental use of a noninterleaving variation on Bergstra & Klop's auxiliary operations[1]. All my attempts to find a sound and complete axiom system for P without the introduction of Bergstra & Klop's auxiliary operators have been to no avail.

Let $\mathsf{P}_{\mathsf{ext}}$ denote the language obtained by extending the grammar for P with the following formation rule:

[1]Bergstra & Klop's left-merge operation satisfies axiom (NLM2) in Figure 4, whilst the left-merge operation I shall use in this section does not.

$$
\begin{array}{lrcl}
\text{A1} & (x+y)+z & = & x+(y+z) \\
\text{A2} & x+y & = & y+x \\
\text{A3} & x+x & = & x \\
\text{A4} & x+\mathbf{0} & = & x \\
& & & \\
\text{B1} & (x+y) \mathbin{|\!\!/} z & = & x \mathbin{|\!\!/} z + y \mathbin{|\!\!/} z \\
\text{B2} & (x \mathbin{|\!\!/} y) \mathbin{|\!\!/} z & = & x \mathbin{|\!\!/} (y\|z) \\
\text{B3} & x \mathbin{|\!\!/} \mathbf{0} & = & x \\
\text{B4} & \mathbf{0} \mathbin{|\!\!/} x & = & \mathbf{0} \\
& & & \\
\text{I1} & x+\tau.x & = & \tau.x \\
\text{I2} & \mu.\tau.x & = & \mu.x \\
\text{NI3} & x \mathbin{|\!\!/} (y+\tau.z) & = & x \mathbin{|\!\!/} (y+\tau.z) + x \mathbin{|\!\!/} z \\
& & & \\
\text{X1} & x\|y & = & x \mathbin{|\!\!/} y + y \mathbin{|\!\!/} x \\
\text{NX2} & \tau.x \mathbin{|\!\!/} y & = & \tau.(x\|y) \\
\text{NX3} & x \mathbin{|\!\!/} \tau.y & = & x \mathbin{|\!\!/} y \\
& & & \\
\text{C} & a.x_1 \mathbin{|\!\!/} ((\bar{a}.x_2 \mathbin{|\!\!/} y)+z) & = & a.x_1 \mathbin{|\!\!/} ((\bar{a}.x_2 \mathbin{|\!\!/} y)+z) + \tau.(x_1\|x_2\|y)
\end{array}
$$

Figure 2: Hennessy's equations for $\approx_T^C$

if $p, q \in \mathsf{P_{ext}}$ then $p \mathbin{\lfloor\!\lfloor} q \in \mathsf{P_{ext}}$ and $p|q \in \mathsf{P_{ext}}$.

The set of states $\mathsf{S_{ext}}$ associated with the extended language $\mathsf{P_{ext}}$ is defined in exactly the same way as S. The operational semantics for the language of extended states $\mathsf{S_{ext}}$ is obtained by adding the following rules for the new operators to those in Figure 1:

$$
\frac{s_1 \overset{e}{\Rightarrow} s_1'}{s_1 \mathbin{\lfloor\!\lfloor} s_2 \overset{e}{\Rightarrow} s_1'\|s_2}
\qquad
\frac{s_1 \overset{a}{\Rightarrow} s_1',\ s_2 \overset{\bar{a}}{\Rightarrow} s_2'}{s_1|s_2 \overset{\tau}{\Rightarrow} s_1'\|s_2'}
$$

The notions of t-observational equivalence and t-observational congruence can now be conservatively extended to the language $\mathsf{P_{ext}}$ and, as in Lemmas 2.1.1 and 2.2.1 in [14], the following results hold:

Lemma 2.1 *For all* $p, q \in \mathsf{P_{ext}}$,

1. $p \approx_T^C q$ *iff* $p + a.\mathbf{0} \approx_T q + a.\mathbf{0}$ *for some* $a \in \mathsf{VAct}$ *not occurring in* p *and* q;
2. $p \approx_T q$ *if and only if* $p \approx_T^C q$ *or* $\tau.p \approx_T^C q$ *or* $p \approx_T^C \tau.q$.

A standard, useful corollary of the characterization given in statement (1) of the above lemma is that if $p \approx_T^C q$ then p and q must have matching τ-transitions. (See, *e.g.*, [14] on page 1010.)

I shall now address the problem of giving a sound and complete axiomatization of t-observational congruence over the language $\mathsf{P_{ext}}$, and hence over its sublanguage P. First of all, note that sound versions of equations (B1)–(B4) may be given by replacing Hennessy's $\mathbin{|\!\!/}$ with the left-merge operator. In particular, the following variation on equation (B2) holds in the quotient algebra $\mathsf{P_{ext}}/\approx_T^C$:

$$
(x \mathbin{\lfloor\!\lfloor} y) \mathbin{\lfloor\!\lfloor} z \quad = \quad x \mathbin{\lfloor\!\lfloor} (y\|z)
$$

The key to the soundness of the above equation is the fact that the left-merge operation does not allow for synchronization between its operands. For example, the reader can easily adapt the aforementioned example showing the unsoundness of axiom (B2) to prove that a version of the above equation in terms of the communication merge is not valid in $\mathsf{P}_{\mathsf{ext}}/\approx_T^C$, *i.e.* that there are processes $p, q, r \in \mathsf{P}_{\mathsf{ext}}$ such that

$$(p \mid q) \mid r \quad \not\approx_T^C \quad p \mid (q \| r)$$

Synchronization between processes is described by the communication merge operator. In fact, left-merge and communication merge together allow one to describe equationally the behaviour of parallel composition and of Hennessy's $\not|$ operator. The relevant equations are:

$$(x \not| y) \quad = \quad x \mathbin{\|\!\_} y + x|y \qquad (1)$$

$$x\|y \quad = \quad x \mathbin{\|\!\_} y + y \mathbin{\|\!\_} x + x|y \qquad (2)$$

Equation (1) has been given in [5] in a setting without internal actions, while equation (2) is the key to the finite axiomatizations of bisimulation congruences presented in many papers in the literature on ACP. (See, *e.g.*, [6].) Note that, in the presence of left-merge and communication merge, Hennessy's merge operator is no longer necessary to axiomatize CCS parallel composition.

The communication merge operator satisfies, among other laws, the following version of axiom (C):

$$(a.x \mathbin{\|\!\_} y) \mid (b.w \mathbin{\|\!\_} z) \quad = \quad \begin{cases} \tau.(x\|y\|w\|z) & \text{if } a = \bar{b} \\ \mathbf{0} & \text{otherwise} \end{cases}$$

where I have taken the liberty of omitting a cumbersome use of parentheses because parallel composition is commutative and associative modulo $\approx_T^C$. An equation expressing a fundamental property of the communication merge operator in the theory of t-observational congruence is the following:

$$\tau.x \mid y \quad = \quad x \mid y$$

This law was first presented in [6], where it was shown to be sound with respect to a graph model for Milner's observational congruence (or rooted τ-bisimulation equivalence, in Bergstra and Klop's terminology). It expresses the subtle interplay between internal τ-actions and synchronization in the theory of observational congruence-like relations.

The set $\mathcal{E}$ of equations which make up the axiomatization of t-observational congruence over $\mathsf{P}_{\mathsf{ext}}$ is given in Figure 3. The main result of this paper may now be stated.

Theorem 2.2 *For all $p, q \in \mathsf{P}_{\mathsf{ext}}$, $p \approx_T^C q$ if and only if $\mathcal{E} \vdash p = q$.*

I shall now sketch the steps involved in the proof of Theorem 2.2. The presentation will closely follow Section 2.2 in [14] and the interested reader is referred to that reference for many details.

The first step in the proof of Theorem 2.2 is to show that all the equations in $\mathcal{E}$ are indeed satisfied by $\approx_T^C$. This is the import of the following result, whose proof is straightforward, but rather tedious.

Proposition 2.3 (Soundness) *For all $p, q \in \mathsf{P}_{\mathsf{ext}}$, $\mathcal{E} \vdash p = q$ implies $p \approx_T^C q$.*

$$
\begin{array}{llcl}
\text{A1} & (x+y)+z & = & x+(y+z) \\
\text{A2} & x+y & = & y+x \\
\text{A3} & x+x & = & x \\
\text{A4} & x+\mathbf{0} & = & x \\
 & & & \\
\text{LM1} & (x+y) \mathbin{\lfloor\!\lfloor} z & = & x \mathbin{\lfloor\!\lfloor} z + y \mathbin{\lfloor\!\lfloor} z \\
\text{LM2} & (x \mathbin{\lfloor\!\lfloor} y) \mathbin{\lfloor\!\lfloor} z & = & x \mathbin{\lfloor\!\lfloor} (y \| z) \\
\text{LM3} & x \mathbin{\lfloor\!\lfloor} \mathbf{0} & = & x \\
\text{LM4} & \mathbf{0} \mathbin{\lfloor\!\lfloor} x & = & \mathbf{0} \\
 & & & \\
\text{I1} & x+\tau.x & = & \tau.x \\
\text{I2} & \mu.\tau.x & = & \mu.x \\
\text{ILM1} & x \mathbin{\lfloor\!\lfloor} (y+\tau.z) & = & x \mathbin{\lfloor\!\lfloor} (y+\tau.z) + x \mathbin{\lfloor\!\lfloor} z \\
\text{ILM2} & \tau.x \mathbin{\lfloor\!\lfloor} y & = & \tau.(x \| y) \\
\text{ILM3} & x \mathbin{\lfloor\!\lfloor} \tau.y & = & x \mathbin{\lfloor\!\lfloor} y \\
 & & & \\
\text{CM1} & (x+y) \mid z & = & x \mid z + y \mid z \\
\text{CM2} & x \mid y & = & y \mid x \\
\text{CM3} & x \mid \mathbf{0} & = & \mathbf{0} \\
\text{CM4} & (a.x \mathbin{\lfloor\!\lfloor} y) \mid (b.w \mathbin{\lfloor\!\lfloor} z) & = & \begin{cases} \tau.(x \| y \| w \| z) & \text{if } a = \bar{b} \\ \mathbf{0} & \text{otherwise} \end{cases} \\
\text{CM5} & \tau.x \mid y & = & x \mid y \\
 & & & \\
\text{PAR} & x \| y & = & x \mathbin{\lfloor\!\lfloor} y + y \mathbin{\lfloor\!\lfloor} x + x \mid y \\
\text{HM} & x \not{\lfloor} \, y & = & x \mathbin{\lfloor\!\lfloor} y + x \mid y
\end{array}
$$

Figure 3: Complete equations for $\approx_T^C$ over $\mathsf{P}_{\mathsf{ext}}$

The proof of the completeness of the equations in $\mathcal{E}$ with respect to t-observational congruence follows the general outline of that of Theorem 2.1.2 in [14]. As usual, I shall rely on the existence of normal forms for processes. These are very similar to Hennessy's simple forms. (See Definition 2.2.2 in [14].) As usual in the literature on process algebras, the notation $\sum\{p_i \mid i \in I\}$ is used as a shorthand for $p_{i_1} + \cdots + p_{i_n}$ where $I = \{i_1, \ldots, i_n\}$. If $I = \emptyset$ then $\sum\{p_i \mid i \in \emptyset\} \equiv \mathbf{0}$.

Definition 2.4 *The set of* normal forms NF *is the least subset of* $\mathsf{P}_{\mathsf{ext}}$ *such that*

$\sum\{a_i.p_i \mathbin{⌊\!⌊} p'_i \mid i \in I\} + \sum\{\tau.q_j \mid j \in J\} \in \mathsf{NF}$ *if* I, J *are finite index sets and each* $p_i, p'_i, q_j \in \mathsf{NF}$.

Proposition 2.5 (Normalization) *For every process* $p \in \mathsf{P}_{\mathsf{ext}}$, *there exists a normal form* $\hat{p} \in \mathsf{NF}$ *such that* $\mathcal{E} \vdash p = \hat{p}$.

Proof: The proof of this result is standard and many similar ones may be found in the literature. Detailed proofs for closely related languages may be found in, *e.g.*, [7, 18]. Equations (A3), (I1), (I2), (ILM1) and (ILM3) are not needed in the proof. □

Following Hennessy, the proof of completeness of the set of equations $\mathcal{E}$ relies on establishing so-called "derivation lemmas". As in [14], I shall only be interested in derivations with respect to τ-actions and $S(a)$-actions.

Lemma 2.6 (Derivation Lemma) *Let* $p \in \mathsf{P}_{\mathsf{ext}}$. *Then:*

1. $p \stackrel{\tau}{\Rightarrow} q$ *implies* $\mathcal{E} \vdash p = p + \tau.q$;
2. $p \stackrel{S(a)}{\Rightarrow} a_S.p_1 \| p_2$ *implies* $\mathcal{E} \vdash p = p + a.p_1 \mathbin{⌊\!⌊} p_2$.

Proof: By Proposition 2.5, it is sufficient to prove the above statements for normal forms. Assume then that p is of the form $\sum\{a_i.p_i \mathbin{⌊\!⌊} p'_i \mid i \in I\} + \sum\{\tau.q_j \mid j \in J\}$.

1. By induction on the length of the derivation $p \equiv \sum\{a_i.p_i \mathbin{⌊\!⌊} p'_i \mid i \in I\} + \sum\{\tau.q_j \mid j \in J\} \stackrel{\tau}{\Rightarrow} q$.

   Base case: $q \equiv q_j$ for some $j \in J$. Then $\mathcal{E} \vdash p = p + \tau.q$ follows immediately by using equations (A1)-(A4).

   Inductive step: $q_j \stackrel{\tau}{\Rightarrow} q$ for some $j \in J$. By the inductive hypothesis, it follows that $\mathcal{E} \vdash q_j = q_j + \tau.q$. By equations (I1) and (A1)-(A4), it is easy to derive that $\mathcal{E} \vdash \tau.q_j = \tau.q_j + \tau.q$, from which $\mathcal{E} \vdash p = p + \tau.q$ follows immediately.

2. This statement is proven exactly as Corollary 2.2.5 in [14].

□

The key to the proof of the completeness theorem is an important decomposition result proven by Hennessy in [14] for the language P. The extension of Hennessy's result to the language $\mathsf{P}_{\mathsf{ext}}$ is immediate and, in fact, his proof carries over unchanged to this language.

Proposition 2.7 (Hennessy) *For all* $p, p', q, q' \in \mathsf{P}_{\mathsf{ext}}$, $a_S.p \| p' \approx_T a_S.q \| q'$ *implies* $p \approx_T q$ *and* $p' \approx_T q'$.

Proof: See the proof of Proposition 2.2.8 in [14] and those of the lemmas leading up to it. □

The above results are all that is needed in the proof of the completeness result to follow.

Theorem 2.8 (Completeness) *For all* $p, q \in \mathsf{P}_{\mathsf{ext}}$, $p \approx_T^C q$ *implies* $\mathcal{E} \vdash p = q$.

Proof: This is just a reworking of Hennessy's proof of Theorem 2.2.9 in [14] using the results given above. The interested reader will have no difficulty in filling in the details following Hennessy's proof. □

2.1 An axiomatization of observational congruence

As mentioned in the introduction, Hennessy's axiomatization of Milner's observational congruence in [14] was the first one to use an auxiliary operator to give an equational characterization of parallel composition. For the sake of clarity and in order to support the discussion to follow, I shall now recapitulate the definitions of weak bisimulation equivalence and its associated congruence.

The relation of *weak bisimulation equivalence* $\approx$ is defined as the largest symmetric relation over P which satisfies

$p \approx q$ iff for every $\mu \in \mathsf{Act}$, $p \stackrel{\mu}{\Rightarrow} p'$ implies

- $\mu = \tau$ and $p' \approx_T q$, or
- $q \stackrel{\mu}{\Rightarrow} q'$ for some q' such that $p' \approx_T q'$.

As usual, $\approx$ is not a congruence over P. The largest congruence relation contained in $\approx$ will be denoted by $\approx^C$ and will be referred to as *observational congruence.*

The key to the axiomatization of observational congruence presented in Theorem 1.3.4 of [14] is a version of Milner's *interleaving law* in terms of Hennessy's $\not|$. This is the following conditional equation schema:

$$\text{(X2)} \qquad \frac{y = \sum \{\lambda_j.y_j \mid j \in J\} \; (J \text{ a finite index set})}{\mu.x \not| \; y = \mu.(x\|y) + \sum \left\{\tau.(x\|y_j) \mid \mu = \bar{\lambda}_j\right\}}$$

This equation schema plays a vital role in the reduction of process terms to the sumforms used by Hennessy and Milner in [16] and Hennessy in [14]. Unfortunately, however, it is *not* sound with respect to observational congruence as the following example shows.

Example: Consider the instance of the above equation obtained by taking $\mu \equiv a$, $x \equiv \mathbf{0}$ and $y \equiv \tau.\bar{a}.\mathbf{0}$. Then (X2) allows us to derive that $a.\mathbf{0} \not| \; \tau.\bar{a}.\mathbf{0} = a.(\mathbf{0}\|\tau.\bar{a}.\mathbf{0})$. However, this equality does not hold in the quotient algebra $\mathsf{P}/{\approx^C}$ as $a.\mathbf{0} \not| \; \tau.\bar{a}.\mathbf{0} \stackrel{\tau}{\Rightarrow} \mathbf{0}\|\mathbf{0}$, whilst obviously $a.(\mathbf{0}\|\tau.\bar{a}.\mathbf{0})$ has no comparable transition. □

The reader will have noticed that, once again, the unsoundness of axiom (X2) derives from the fact that Hennessy's $\not|$ allows for communication between its arguments. The above problem with the axiomatization presented in [14] can be solved by resorting to Bergstra and Klop's auxiliary operators. In fact, it is possible to conservatively extend observational congruence to the language $\mathsf{P}_{\mathsf{ext}}$ and give a sound and complete equational axiomatization of equality in the quotient algebra $\mathsf{P}_{\mathsf{ext}}/{\approx^C}$. In fact, all that is needed for this purpose is to add the following equations to those presented in Figure 3:

$$a.x \,\lfloor\!\lfloor\, y = a.(x\|y) \qquad (3)$$

$$a.(x + \tau.y) = a.(x + \tau.y) + a.y \qquad (4)$$

Equation (3) is the one that essentially expresses the fact that observational congruence induces an interleaving semantics on processes. Together with the other equations for left-merge and communication merge it allows for the derivation of Milner's expansion theorem. (See, *e.g.*, [6] for a detailed proof of this fact.) Equation (4) is Milner's "third τ-law". As it is well-known, see, *e.g.*, [14] on page 1010, this equation does not hold for $\approx_T^C$ because it strongly depends on the assumption of atomicity of action occurrences.

$$
\begin{array}{llcl}
\text{A1} & (x+y)+z & = & x+(y+z) \\
\text{A2} & x+y & = & y+x \\
\text{A3} & x+x & = & x \\
\text{A4} & x+\mathbf{0} & = & x \\
\\
\text{LM1} & (x+y) \mathbin{\lfloor\!\lfloor} z & = & x \mathbin{\lfloor\!\lfloor} z + y \mathbin{\lfloor\!\lfloor} z \\
\text{NLM2} & \mu.x \mathbin{\lfloor\!\lfloor} y & = & \mu.(x \| y) \\
\text{LM4} & \mathbf{0} \mathbin{\lfloor\!\lfloor} x & = & \mathbf{0} \\
\\
\text{I1} & x + \tau.x & = & \tau.x \\
\text{I2} & \mu.\tau.x & = & \mu.x \\
\text{I3} & a.(x+\tau.y) & = & a.(x+\tau.y) + a.y \\
\\
\text{CM1} & (x+y) \mid z & = & x \mid z + y \mid z \\
\text{CM2} & x \mid y & = & y \mid x \\
\text{CM3} & x \mid \mathbf{0} & = & \mathbf{0} \\
\text{NCM4} & a.x \mid b.y & = & \begin{cases} \tau.(x \| y) & \text{if } a = \bar{b} \\ \mathbf{0} & \text{otherwise} \end{cases} \\
\text{CM5} & \tau.x \mid y & = & x \mid y \\
\\
\text{PAR} & x \| y & = & x \mathbin{\lfloor\!\lfloor} y + y \mathbin{\lfloor\!\lfloor} x + x \mid y \\
\text{HM} & x \mathbin{|\!\!/} y & = & x \mathbin{\lfloor\!\lfloor} y + x \mid y
\end{array}
$$

Figure 4: Complete equations for $\approx^C$ over $\mathsf{P}_{\mathsf{ext}}$

In the presence of equation (3), several of the equations in Figure 3 are not necessary to give a complete equational characterization of observational congruence over the language $\mathsf{P}_{\mathsf{ext}}$. (Note, however, that those equations lead to more powerful axiomatic systems for what concerns provability of equivalences between *open terms* over sub-languages of $\mathsf{P}_{\mathsf{ext}}$. The interested reader is referred to [20, 13] for more on this issue.) Moreover, axiom (CM4) may be simplified to

$$
a.x \mid b.y \quad = \quad \begin{cases} \tau.(x \| y) & \text{if } a = \bar{b} \\ \mathbf{0} & \text{otherwise} \end{cases}
$$

A complete set of axioms for $\approx^C$ is given in Figure 4. Let $\mathcal{E}'$ denote the set of equations in Figure 4.

Theorem 2.9 *For all $p, q \in \mathsf{P}_{\mathsf{ext}}$, $p \approx^C q$ iff $\mathcal{E}' \vdash p = q$.*

Proof: This is just a reworking of many similar results in the literature, see *e.g.* [16, 6, 19], following the outline of the proof of Theorem 1.3.4 in [14] (page 1008). □

3 Remarks on the operational semantics of Bergstra and Klop's auxiliary operators

The reader familiar with the literature on bisimulation semantics for CCS will have already noted that the operational semantics for the language $\mathsf{P}_{\mathsf{ext}}$ given in the previous section is slightly non-standard. The rules in Figure 1 and those for Bergstra and Klop's auxiliary operators define the

$$\frac{}{a.p \stackrel{S(a)}{\longrightarrow} a_S.p} \qquad \frac{}{a_S.p \stackrel{F(a)}{\longrightarrow} p} \qquad \frac{}{\mu.p \stackrel{\mu}{\rightarrow} p}$$

$$\frac{s_1 \stackrel{e}{\rightarrow} s_1'}{s_1 + s_2 \stackrel{e}{\rightarrow} s_1'} \qquad \frac{s_1 \stackrel{e}{\rightarrow} s_1'}{s_2 + s_1 \stackrel{e}{\rightarrow} s_1'}$$

$$\frac{s_1 \stackrel{e}{\rightarrow} s_1'}{s_1 \| s_2 \stackrel{e}{\rightarrow} s_1' \| s_2} \qquad \frac{s_1 \stackrel{e}{\rightarrow} s_1'}{s_2 \| s_1 \stackrel{e}{\rightarrow} s_2 \| s_1'} \qquad \frac{s_1 \stackrel{e}{\rightarrow} s_1'}{s_1 \lfloor\!\lfloor s_2 \stackrel{e}{\rightarrow} s_1' \| s_2} \qquad \frac{s_1 \stackrel{e}{\rightarrow} s_1'}{s_1 \not| \, s_2 \stackrel{e}{\rightarrow} s_1' \| s_2}$$

$$\frac{s_1 \stackrel{a}{\rightarrow} s_1', s_2 \stackrel{\bar{a}}{\rightarrow} s_2'}{s_1 \| s_2 \stackrel{\tau}{\rightarrow} s_1' \| s_2'} \qquad \frac{s_1 \stackrel{a}{\rightarrow} s_1', s_2 \stackrel{\bar{a}}{\rightarrow} s_2'}{s_1 \mid s_2 \stackrel{\tau}{\rightarrow} s_1' \| s_2'} \qquad \frac{s_1 \stackrel{a}{\rightarrow} s_1', s_2 \stackrel{\bar{a}}{\rightarrow} s_2'}{s_1 \not| \, s_2 \stackrel{\tau}{\rightarrow} s_1' \| s_2'}$$

Figure 5: Operational rules for $\stackrel{e}{\rightarrow}$

so-called *weak transition relations* over the language $\mathsf{P}_{\mathsf{ext}}$ in one step, so to speak. This is in contrast with the developments in, *e.g.*, [19], where the operational semantics of CCS is defined first in terms of single step transition relations. These concrete transition relations are then used in the definition of the weak transition relations, which capture the intuition that τ-labelled transitions correspond to invisible events. For easy reference, the defining rules of the one step transition relations, $\stackrel{e}{\rightarrow}$, for the language $\mathsf{P}_{\mathsf{ext}}$ are collected in Figure 5. The associated transition relations which abstract from internal τ-transitions are then usually defined by:

$$s \stackrel{e}{\Rrightarrow} s' \Leftrightarrow \exists s_1, s_2 : \ s \stackrel{\tau}{\rightarrow}^\star s_1 \stackrel{e}{\rightarrow} s_2 \stackrel{\tau}{\rightarrow}^\star s'$$

where $\stackrel{\tau}{\rightarrow}^\star$ denotes the reflexive and transitive closure of the relation $\stackrel{\tau}{\rightarrow}$.

The process of abstraction from τ-labelled transitions is instead built in the definition of the transition relations $\stackrel{e}{\Rightarrow}$ by means of the rules

$$\frac{s_1 \stackrel{\tau}{\Rightarrow} s_1', s_1' \stackrel{e}{\Rightarrow} s_2}{s_1 \stackrel{e}{\Rightarrow} s_2} \qquad \frac{s_1 \stackrel{e}{\Rightarrow} s_1', s_1' \stackrel{\tau}{\Rightarrow} s_2}{s_1 \stackrel{e}{\Rightarrow} s_2}$$

It is easy to see that, for processes in $\mathsf{P}_{\mathsf{ext}}$ not containing occurrences of the communication merge and of Hennessy's $\not|$, the weak transition relations $\stackrel{e}{\Rightarrow}$ and $\stackrel{e}{\Rrightarrow}$ are in complete agreement, *i.e.* for all such s,

$$s \stackrel{e}{\Rightarrow} s' \Leftrightarrow s \stackrel{e}{\Rrightarrow} s'$$

In particular, this implies that observational congruence and t-observational congruence over the sublanguage of $\mathsf{P}_{\mathsf{ext}}$ consisting of these terms can be defined using either of these two transition relations.

This agreement does, however, break down for terms having the communication merge operator or Hennessy's $\not|$ as head operator. Consider, for example, the term $p \equiv \tau.a \mid \bar{a}.b.\mathbf{0}$. Then, using the defining rules for $\Rightarrow$, it is easy to derive that $p \stackrel{b}{\Rightarrow} \mathbf{0} \| \mathbf{0}$. However, p has no outgoing transition according to $\stackrel{\cdot}{\Rrightarrow}$, as rule

$$\frac{s_1 \stackrel{a}{\rightarrow} s_1', s_2 \stackrel{\bar{a}}{\rightarrow} s_2'}{s_1 \mid s_2 \stackrel{\tau}{\rightarrow} s_1' \| s_2'}$$

Behavioural Congruences	Suitable Transition Relations
Observational congruences satisfying (I1)	Define the semantics of the auxiliary operators by giving rules which give the weak transition relations in one step, as in Figure 1.
Branching bisimulation congruence [12]	Define the semantics of the auxiliary operators by giving rules which give the one step transition relation, as in Figure 5. See, *e.g.*, [3].
Testing and Failures congruences	As pointed out in [11], for these congruences the left-merge operator causes just as many problems as the other auxiliary operators. A possible solution, based on the use of the nondeterministic operators from CSP [17] in lieu of the CCS/ACP combination of sum and τ, and on the possibility of giving a suitable one step transition relation for the modified language, may be found in [2].

Figure 6: A menagerie of suitable semantics for the auxiliary operators

is not applicable to it. This fact has disastrous consequences in the theory of congruences which, like observational congruence and t-observational congruence, satisfy axiom

$$(\mathrm{I1}) \quad x + \tau.x \;=\; \tau.x$$

In fact, the communication merge operation and Hennessy's $\not|$ would not preserve any such congruence, if their operational semantics were given in terms of $\stackrel{\cdot}{\Rightarrow}$.

Example:(In terms of observational congruence) Consider the terms $\tau.a.\mathbf{0}$ and $a.\mathbf{0} + \tau.a.\mathbf{0}$. As $\approx^C$ satisfies axiom (I1), one has that $\tau.a.\mathbf{0} \approx^C a.\mathbf{0} + \tau.a.\mathbf{0}$. However, if the semantics of the communication merge operator were given in terms of the rules in Figure 5, it would be the case that

$$p \equiv \tau.a.\mathbf{0} \mid \bar{a}.b.\mathbf{0} \not\approx^C (a.\mathbf{0} + \tau.a.\mathbf{0}) \mid \bar{a}.b.\mathbf{0} \equiv q$$

In fact, $q \stackrel{b}{\Rightarrow} \mathbf{0}||\mathbf{0}$ whilst, as remarked above, p has no outgoing transitions with respect to $\stackrel{\cdot}{\Rightarrow}$. $\square$

The outcome of this discussion is that a suitable operational semantics for the communication merge operator and Hennessy's $\not|$ in the theory of congruences which, like those axiomatized in this paper, satisfy axiom (I1) can only be given by defining the weak transition relations in one step[2], as in Figure 1. I believe that this observation was already implicit in the denotational semantics for ACP in terms of process graphs presented by Bergstra and Klop in their seminal paper [6], but, probably because of the denotational nature of the semantics presented in that reference, it seems to have gone unnoticed in several papers in the literature. (A notable exception being [9], where an interesting operational semantics for ACP along the lines of that in Figure 1 has been presented.)

Indeed, contrary to what happens for the basic CCS combinators, the operational semantics of the auxiliary operators used in the axiomatization of parallel composition is highly sensitive to

[2]The point here is that one is really interested in the identifications induced by the chosen congruence over the basic language, *e.g.* CCS, used to write specifications of concurrent systems. Auxiliary operators are only added for axiomatization purposes and as an aid in algebraic manipulations of terms. Hence, one should like to add these operations *conservatively*, that is to say that their presence should not influence the equalities over the basic language. This, of course, requires that these new operations preserve the chosen behavioural congruence.

the kind of behavioural congruence one wants to impose on terms. A full discussion of this point would lead me too far from the main aim of this paper. Thus, I shall just end by giving a short "recipe book" for giving semantics to the auxiliary operators discussed in this paper in the setting of some of the best-known semantic theories for processes, with pointers to references where they are discussed in detail. These may be found in Figure 6. I hope that they will be a useful reference for researchers interested in complete axiomatizations of behavioural congruences.

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