Products of CW complexes
the full story

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For $n \in \mathbb{N}$, let

- D^n denote the closed ball of radius 1 about the origin in $\mathbb{R}^n$ (the *n-disc*),
- $\overset{\circ}{D}{}^n$ its interior (the open ball of radius 1 about the origin), and
- S^{n-1} its boundary (the $n-1$-*sphere*).

CW complexes

Definition

A Hausdorff space X is a *CW complex* if there exists a set of continuous functions $\varphi_\alpha^n : D^n \to X$ (*characteristic maps*), for α in an arbitrary index set and $n \in \mathbb{N}$ a function of α, such that:

1. $\varphi_\alpha^n \restriction \overset{\circ}{D}{}^n$ is a homeomorphism to its image, and X is the disjoint union as α varies of these homeomorphic images $\varphi_\alpha^n[\overset{\circ}{D}{}^n]$ ("cells").

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We often denote $\varphi_\alpha^n[\overset{\circ}{D}{}^n]$ by e_α^n.

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Convention

In this talk, $X \times Y$ is always taken to have the product topology, so "$X \times Y$ is a CW complex" means "the product topology on $X \times Y$ is the same as the weak topology".

Example (Dowker, 1952)

Let X be the "star" with a central vertex x_0 and countably many edges $e^1_{X,n}$ ($n \in \mathbb{N}$) emanating from it (and the countably many "other end" vertices of those edges).

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Consider the subset of $X \times Y$

$$H = \left\{ \left(\frac{1}{f(n)+1}, \frac{1}{f(n)+1} \right) \in e^1_{X,n} \times e^1_{Y,f} : n \in \mathbb{N}, f \in \mathbb{N}^{\mathbb{N}} \right\}$$

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Then $\left(\frac{1}{g(k)+1}, \frac{1}{g(k)+1} \right) \in U \times V \cap H$. So in the product topology, $(x_0, y_0) \in \bar{H}$.

More preliminaries: subcomplexes

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Definition

Let κ be a cardinal. We say that a CW complex X is *locally less than* κ if for all x in X there is a subcomplex A of X with fewer than κ many cells such that x is in the interior of A. We write *locally finite* for locally less than $\aleph_0$, and *locally countable* for locally less than $\aleph_1$.

Proposition

If κ is a regular uncountable cardinal, then a CW complex W is locally less than κ if and only if every connected component of W has fewer than κ many cells.

Proof sketch.

$\Leftarrow$ is trivial. For $\Rightarrow$, given any point w, recursively fill out to get an open (hence clopen) subcomplex containing w with fewer than κ many cells, using the fact that the cells are compact to control the number of cells along the way. $\square$

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If X and Y are both (locally) countable, then $X \times Y$ is a CW complex.

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Theorem (Y. Tanaka, 1982)

If neither X nor Y is locally countable, then $X \times Y$ is not a CW complex.

What was known, beyond ZFC

Theorem (Liu Y.-M., 1978)

Assuming CH, $X \times Y$ is a CW complex if and only if one of them is locally finite, or both are locally countable.

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Theorem (Y. Tanaka, 1982)

Assuming $\mathfrak{b} = \aleph_1$, $X \times Y$ is a CW complex if and only if one of them is locally finite, or both are locally countable.

Can we do better?

Question

Can we show, without assuming any extra set-theoretic axioms, that the product $X \times Y$ of CW complexes X and Y is a CW complex if and only if either

- one of them is locally finite, or
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Answer (follows from Tanaka's work)

No.

Can we nevertheless do better?

Refined question

Can we characterise exactly when the product of two CW complexes is a CW complex, without assuming any extra set-theoretic axioms?

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Answer (A. B.-T.)

Yes!

Pushing Dowker's example harder

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For this we need to talk about the cardinal $\mathfrak{b}$.

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$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}} \wedge \forall g \in \mathbb{N}^{\mathbb{N}} \exists f \in \mathcal{F} \neg(f \leq^* g)\}.$$

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$\aleph_1 \leq \mathfrak{b} \leq 2^{\aleph_0}$, and each of

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is consistent with ZFC.

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Consider the edge $e^1_{Y,f}$ of Y:

Let $k \in \mathbb{N}$ be such that $\frac{1}{f(k)+1} \in e^1_{Y,f} \cap V$ and $f(k) > g(k)$.

Then $\left(\frac{1}{f(k)+1}, \frac{1}{f(k)+1} \right) \in U \times V \cap H$. So in the product topology, $(x_0, y_0) \in \bar{H}$.

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A complete characterisation

Theorem (A.B.-T.)

Let X and Y be CW complexes. Then $X \times Y$ is a CW complex if and only if one of the following holds:

1. *X or Y is locally finite.*
2. *One of X and Y is locally countable, and the other is locally less than $\mathfrak{b}$.*

Proof

The forward direction was actually done by Tanaka (1982).

So it remains to show that if X and Y are CW complexes such that X is locally countable and Y is locally less than $\mathfrak{b}$, then $X \times Y$ is a CW complex.

By the Proposition earlier, we may assume that X has countably many cells and Y has fewer than $\mathfrak{b}$ many cells.

Topologies

Any compact subset of a CW complex X is contained in finitely many cells, and each closed cell $\bar{e}^n_\alpha$ is compact. So requiring X to have the weak topology is equivalent to requiring that the topology be *compactly generated*: a set is closed if and only if its intersection with every compact set is closed.

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We can also restrict to those compact sets which are continuous images of the space $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ (equivalently, of the space $\omega + 1$).

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A topological space Z is *sequential* if for every subset C of Z, C is closed if and only if C contains the limit of every convergent (countable) sequence from C.

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Any sequential space is compactly generated. Since D^n is sequential for every n, we have that CW complexes are sequential.

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Basic idea

The construction is essentially by simultaneous induction on cell number on the X side (after enumerating the cells of X in a reasonable order) and dimension on the Y side.

For each new cell e_α that you consider on the Y side, you get a function $f_\alpha : \mathbb{N} \to \mathbb{N}$ defining an open set on the X side avoiding H. Since there are fewer than $\mathfrak{b}$ many α, they can be eventually dominated by a single function f, with respect to which the e_α part of the neighbourhood can be chosen.

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Naïvely implemented, that doesn't work ($f_\alpha \leq^* f$ isn't enough), but with the right bookkeeping it does.