Semigroup factorization and relaxation rates of kinetic equations

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- Joint Gualdani, Mischler arXiv (H-theorem and Boltzmann eqn)
- Joint Mischler in progress (Fokker-Planck dynamics)
- Baranger-CM'05, CM'06, CM-Neumann'06, CM-Strain'07

Plan

The Boltzmann equation (Maxwell 1867, Boltzmann 1872)

$$\underbrace{\partial_t f}_{\text{time change}} + \underbrace{v \cdot \nabla_x f}_{\text{space change}} = \underbrace{Q(f,f)}_{\text{collision operator}} \qquad \text{on } f(t,x,v) \geq 0$$

- Transport term $v \cdot \nabla_x$: straight line along velocity v
- Collision operator $Q(f,f)$: bilinear, acting on v only, integral

$$Q(f,f)(v) = \int_{v_*} \int_{collisions} \Big[\underbrace{f(v')f(v'_*)}_{(v',v'_*)\to(v,v_*)} - \underbrace{f(v)f(v_*)}_{(v,v_*)\to\dots} \Big] B$$

- Balance-sheet of particles with velocity v due to collisions
- Here hard spheres: $B = |(v - v_*, \omega)|$

Structure of the Boltzmann equation (I)

- $Q(f,f)$ bilinear integral operator acting on v only (local in t and x), representing interactions between particles:

$$Q(f,f)(v) := \int_{v_* \in \mathbb{R}^3} \int_{\omega \in \mathbb{S}^2} [\underbrace{f(v'_*)f(v')}_{\text{“appearing”}} - \underbrace{f(v)f(v_*)}_{\text{“disappearing”}}] \underbrace{B(v - v_*, \omega)}_{\text{collision kernel } (\geq 0)} \, \mathrm{d}\omega \, \mathrm{d}v_*$$

- Velocity collision rule ($(d-1)$ free parameters $\to \omega$):

$$v' := v - (v - v_*, \omega)\omega, \qquad v'_* := v_* + (v - v_*, \omega)\omega$$

- One has (microscopic conservation laws)

$$v' + v'_* = v + v_*, \qquad |v'_*|^2 + |v'|^2 = |v|^2 + |v_*|^2$$

Structure of the Boltzmann equation (II)

- For $\omega \in \mathbb{S}^{d-1}$ the map $(v, v_*) \mapsto (v', v'_*)$ has Jacobian -1
- We deduce for a test function $\varphi(v)$

$$\int_{\mathbb{R}^d} Q(f,f)\varphi(v)\,\mathrm{d}v$$
$$= \frac{1}{4}\int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} [f'f'_* - ff_*]B(v-v_*,\omega)(\varphi+\varphi_*-\varphi'-\varphi'_*)\,\mathrm{d}\omega\,\mathrm{d}v_*\,\mathrm{d}v$$

- Choosing correctly φ we deduce

$$\int_{\mathbb{R}^d} Q(f,f)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}\mathrm{d}v = 0$$

- This implies formally

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} f\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}\mathrm{d}x\,\mathrm{d}v = 0$$

Structure of the Boltzmann equation (III)

- Choosing $\varphi = \log f$ we obtain the *H*-theorem

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{2d}} f \log f \,\mathrm{d}x\,\mathrm{d}v = -D(f) \leq 0$$

- The entropy production is

$$\begin{aligned} D(f) &= -\int_{\mathbb{R}^{2d}} Q(f,f) \log f \,\mathrm{d}x\,\mathrm{d}v \\ &= \int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} [f' f'_* - f f_*] \log \frac{f' f'_*}{f f_*} B(v - v_*, \omega) \,\mathrm{d}x\,\mathrm{d}v \geq 0 \end{aligned}$$

- Cancellation at $f f_* \equiv f' f'_*$: Maxwellian local equilibrium

$$M_f = \frac{\rho}{(2\pi T)^{d/2}} e^{-|v-u|^2/2T}, \quad \rho \geq 0,\ u \in \mathbb{R}^d,\ T > 0$$

- Time-irreversible equation and mathematical basis for studying relaxation to equilibrium (2-d law of thermodynamic)

Exponential H-theorem and Cercignani's conjecture (I)

Quantify H-Theorem for the Boltzmann equation

Old question... Truesdell and Muncaster 1980:

"*Much effort has been spent toward proof that place-dependent solutions exist for all time. [...]* **The main problem is really to discover and specify the circumstances that give rise to solutions which persist forever**. *Only after having done that can we expect to construct proofs that such solutions exist, are unique, and are regular.*"

Mathematically this amounts to prove for a priori smooth solutions

$$H(f_t|M) = \int_{\mathbb{T}^d \times \mathbb{R}^d} f_t \log \frac{f_t}{M} \, \mathrm{d}x \, \mathrm{d}v \xrightarrow{t \to +\infty} 0$$

with the correct timescale (hence requires constructive proofs)

Exponential H-theorem and Cercignani's conjecture (II)

- Cercignani's conjecture 1982

  "*The present contribution is intended as a step toward the solution of the first main problem of kinetic theory, as defined by Truesdell and Muncaster, i.e. 'to discover and specify the circumstances that give rise to solutions which persist forever'.*"

- Linearized semigroup (Hilbert, Grad, Ukai...) suggests exponential rate
- Conjecture: Linear inequality on the entropy production

$$D(f) = -\frac{\mathrm{d}H(f|M)}{\mathrm{d}t} \geq \lambda H(f|M), \quad \lambda > 0$$

- Kind of nonlinear spectral gap, cf. log-Sobolev inequalities...
- False (Bobylev-Cercignani, Wennberg) for physical B

The three main developements (I)

(I) Coercivity estimates in velocity for non-local collision operators

For the linearized collision operator:

- Hilbert'1909: compactness collision operator
- Carleman'1957, Grad'1960s: spectral gap hard spheres
- Wang-Chang-Uhlenbeck'1950s, Bobylev'1970s: eigenpairs for Maxwell molecules
- Baranger-CM'2005: bounds spectral gap hard spheres
- CM'2006: bounds coercivity for cutoff interactions
- CM-Strain'2007: bounds spectral gap "non-cutoff"

At nonlinear level (Cercignani's conjecture):
Carlen-Carvhalo, Toscani-Villani: $D(f) \geq \lambda_\varepsilon H(f|M)^{1+\varepsilon}$
with smoothness and moments... however polynomial time-scales

The three main developements (II)

(II) Hypocoercivity

- Mixing of conservative (skew-symmetric) transport operator with partially dissipative (symmetric) collision operator to produce relaxation towards the global equilibrium
- Hypoellipticity: Kolmogorov'34, Hörmander'67...
- Here focus on the long-time behavior not regularity, non-local
- Hérau-Nier'01, Desvillettes-Villani'02: kinetic Fokker-Planck in $L^2(M^{-1})$
- Desvillettes-Villani'05: polynomial relaxation rates for confined nonlinear Boltzmann equation, assuming regularity
- For linearized Boltzmann equation with periodic conditions:

$$\partial_t h + v \cdot \nabla_x h = Bh, \quad x \in \mathbb{T}^3, \ v \in \mathbb{R}^3,$$

CM-Neumann'06: proof-estimate spectral gap in terms of the spectral gap of B in velocity only in $L^2(M^{-1})$

The three main developements (III)

(III) Extension of the functional space in the linearized study

- Hence one has to turn to the study of semigroup decay properties (*vs.* functional inequality):
  importance of the Cauchy theory and the natural space for it
- Incompatibility of functional spaces: linearized study $L^2(M^{-1})$ and nonlinear evolution equation $L^1(poly)$
- First non-constructive proof of exponential decay in physical space Arkeryd-Esposito-Pulvirenti'87, Arkeryd'88: although the proof can be filled to my opinion, never been used and remained debatted...
- Constructive proof CM'06 in the spatially homogeneous case with sharp exponential rate: connection of entropy production estimates with new quantitative linearized estimates
- $\rightarrow$ Complete answer for Boltz. eqn in torus (hard spheres)...

The key new result

Cauchy theory in physical space (Gualdani-Mischler-CM)

Boltzmann equation for hard spheres in the torus $x \in \mathbb{T}^d$

• Locally well-posed around its global gaussian equilibrium $M = M(v) = Ce^{-|v|^2/2}$ in the space $L^1_v L^\infty_x (1 + |v|^k)$ for $k > 2$.

• Well-posed for weakly inhomogeneous initial data, in the sense: f_{in} is close in $L^1_v L^\infty_x (1 + |v|^k)$ to $g_{in} = g_{in}(v)$, where the smallness condition depends on $\|g_{in}\|_{L^1_4}$.

• Constructive rate of convergence $f_t \to M$ in $L^1_v L^\infty_x (1 + |v|^k)$ given by $Ce^{-\lambda t}$ where λ optimal if k large enough

Remarks:

- Variants with derivatives $W^{\sigma,1}_v W^{s,\infty}$ with $0 \le \sigma \le s$, $s > 6/p$
- Variants with other Lebesgue spaces...
- Variants with stretched exponential weights...
- Spectral decomposition of linearized semigroup in these spaces

Plan

Introduction

The factorization method

Relaxation rates for Fokker-Planck dynamics

(Hypo)dissipative and (hypo)coercive operators (I)

- $\Lambda - a$ dissipative in $X = (X, \|.\|_X)$ if

$$\forall f \in D(\Lambda),\ \forall \phi \in F(f), \quad \Re e\langle(\Lambda - a)\, f, \phi\rangle \leq 0$$

with

$$\phi \in F(f) \quad \Longleftrightarrow \quad \langle f, \phi\rangle = \|f\|_X^2 = \|\phi\|_{X^*}^2$$

- $\Lambda - a$ hypodissipative in X if

$\Lambda - a$ dissipative in $(X, |\cdot|_X)$ for an equivalent norm $|\cdot|_X \sim \|\cdot\|_X$

- In the case of a Hilbert space structure, this coincides with the notion of coercivity and hypocoercivity
- Notion of (maximal) *m*-(hypo)dissipativity and *m*-(hypo)coercivity if furthermore $\mathrm{Range}(\Lambda - a) = X$

(Hypo)dissipative and (hypo)coercive operators (II)

Lumer-Phillips Theorem

The m-dissipativity implies $\|e^{t\Lambda}\|_{B(X)} \leq e^{at}$

Hille-Yosida Theorem

The m-hypodissipativity implies $\|e^{t\Lambda}\|_{B(X)} \leq C_a e^{at}$, $C_a \geq 1$

- Add to these definition finite number of discrete eigenvalues

$$\|e^{t\Lambda}(1 - \Pi_{\Lambda,a})\|_{B(X)} \leq C_a \, e^{at}, \;\; C_a \geq 1$$

$$\text{or} \qquad \begin{cases} \Lambda - a \text{ is } m\text{-hypodissipative on invariant set } X_0 \\ X_0 \text{ closed}, \;\; \mathrm{codim} X_0 < \infty \end{cases}$$

- Stronger notions of self-adjoint and of sectorial operators: $\|R(x+iy)\| \leq C|x+iy-a|^{-1}$ for $y = \pm\mu(x-a)$, $x \leq a$, for some $\mu \in (0,+\infty)$

Spectral mapping theorem

- General problem in semigroup theory

$$\text{Prove that} \quad \Sigma(e^{tL}) = e^{\Sigma(tL)}$$

- When true, spectral localization implies semigroup decay
- In general hard problem for non-self-adjoint operators
- Hörmander operators type I: $X_0^* X_0 + X_1^* X_1$ self-adjoint, already hard problem for regularity and semigroup decay
- Hörmander operators type II: $X_0 + X_1^* X_1$ non self-adjoint but still symmetry structure, semigroup decay solved recently Hérau-Nier, Villani'2000s
- Here non-self-adjoint and non-symmetric situations, in Banach spaces, with possibly non-diffusive operators
- Idea: use theories in a small space E in order to get results in a larger Banach space $\mathcal{E}$

Assumptions in the small space E

Abstract method that can be used for different equations

Notation: half complex plane $\Delta_a := \{z \in \mathbb{C};\ \Re e\, z > a\}$

1. Localization of the spectrum
   - $\Sigma(L) \cap \Delta_a = \{\xi_1, \dots, \xi_k\}$
   - $\xi_1, \dots, \xi_k :=$ discrete eigenvalues
   - $\Pi_{L,\xi_j} :=$ eigenspace projector (finite dimensional eigenspace)
2. Growth estimate on the semigroup e^{tL}
   - $\Pi_{L,a} := \Pi_{L,\xi_1} + \cdots + \Pi_{L,\xi_k}$
   - $\|e^{tL}(1 - \Pi_{L,a})\|_{B(E)} \le C_a\, e^{a\,t}$

Idea of the result

$E \subset \mathcal{E}$ Banach spaces, $\mathcal{L}$ generator of C_0-semigroup s.t. $\mathcal{L}_{|E} = L$

- If $\mathcal{L}$ decomposes as $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with
  - $\mathcal{A} : \mathcal{E} \to E$ bounded (“regularizing” term)
  - $\mathcal{B} - a$ is dissipative (coercive term $\Rightarrow$ spectral localization)

- Then $\mathcal{L}$ inherits the spectral properties of L
  - $\Sigma(\mathcal{L}) \cap \Delta_a = \{\xi_1, \dots, \xi_k\}$, with ξ_j discrete eigenvalue
  - $\Pi_{\mathcal{L},\xi_j}|_E = \Pi_{L,\xi_j}$ = spectral projector

- And $e^{t\,\mathcal{L}}$ inherits the growth estimate of e^{tL}

$$\forall\, t \geq 0,\ \forall\, a' > a, \quad \|e^{t\mathcal{L}} - e^{tL}\Pi_{\mathcal{L},a}\|_{\mathcal{E}\to\mathcal{E}} \leq C_{a'}\, e^{a'\,t}$$

Remark: Quantitative *partial* spectral mapping theorem in $\mathcal{E}$

$$\Sigma(e^{\mathcal{L}\,t}) \cap \Delta_{e^{a'\,t}} = e^{\Sigma(\mathcal{L})\,t} \cap \Delta_{e^{a'\,t}}$$

Main difficulties and strategy (I)

Difficulties

- L and $\mathcal{L}$ may be non symmetric
- $E, \mathcal{E}$ may not have a Hilbert space structure
- we want constructive estimates

Factorization of resolvents for decomposing the semigroups and relating their decays

$$\begin{tikzcd}
\text{Semigroup } S_L(t) \arrow[r] & \text{Resolvent } (L-\xi)^{-1} \arrow[d, "\text{factorization}"] \\
\text{Semigroup } S_{\mathcal{L}}(t) & \text{Resolvent } (\mathcal{L}-\xi)^{-1} \arrow[l]
\end{tikzcd}$$

Main difficulties and strategy (II)

- The horizontal arrows:
  Quantitative dictionary between semigroup decay estimates and resolvent estimates
  - homogeneous case: complex integration for sectorial operators
  - inhomogeneous case: inversion of Laplace transforms = complex integration on vertical lines in $\mathbb{C}$

- The vertical arrow (factorization):
  - $\mathcal{L} = \mathcal{A} + \mathcal{B} \approx$ smooth + well known
  - $\mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{B}} - \mathcal{R}_L \, \mathcal{A} \, \mathcal{R}_{\mathcal{B}}$
    or more generally

$$\mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{B}} - \mathcal{R}_{\mathcal{B}}(\mathcal{A}\,\mathcal{R}_{\mathcal{B}}) + \cdots + (-1)^n \mathcal{R}_L \, (\mathcal{A}\,\mathcal{R}_{\mathcal{B}})^n$$

  - Additional difficulty of the discrete spectrum...

A model statement

$E \subset \mathcal{E}$ Banach spaces, L, $\mathcal{L}$ generators s.t. $\mathcal{L}_{|\mathcal{E}} = L$ with for $a < 0$:

(H1) **L is coercive:** $\leftarrow$ known

(i) $\Sigma(L) \cap \Delta_a = \Sigma_d(L) \cap \Delta_a = \{0\}$ (localization of the spectrum)

(ii) $L - a$ is dissipative on $\mathrm{Range}(I - \Pi_{L,0})$

(H2) **Decomposition of $\mathcal{L}$:** $\exists\, \mathcal{A}, \mathcal{B}$ s.t. $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and

(i) $\mathcal{B} - a$ is hypodissipative : $\|e^{\mathcal{B}t}(t)\|_{B(\mathcal{E})} \leq C_a\, e^{a\,t}$

(ii) $\mathcal{A} \in B(\mathcal{E})$

(iii) $T_n := (\mathcal{A}\,\mathcal{S}_{\mathcal{B}})^{(*n)}$ satisfies $\|T_n(t)\|_{B(\mathcal{E},E)} \leq C_a\, e^{a\,t}$ for $n \in \mathbb{N}^*$

Theorem

(i) $\Sigma(\mathcal{L}) \cap \Delta_a = \Sigma_d(\mathcal{L}) \cap \Delta_a = \{0\}$, $\Pi_{\mathcal{L},0}|_E = \Pi_{L,0}$

(ii) $\forall\, a' > a, \exists\, C_{a'} > 0$ s.t. $\forall\, t \geq 0$, $\quad \|e^{t\mathcal{L}} - \Pi_{\mathcal{L},0}\|_{B(\mathcal{E})} \leq C_{a'}\, e^{a'\,t}$

Proof of the theorem - Step 1: right inverse of $\mathcal{L}-\xi$

Define $\Omega := \Delta_a \backslash \{0\}$ (simpler example)

$$\mathcal{U}(\xi) := \sum_{k=0}^{n-1} (-1)^k \underbrace{\mathcal{R}_{\mathcal{B}}(\xi)}_{(\mathcal{B}-\xi)^{-1}} \left(\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(\xi)\right)^k + (-1)^n \underbrace{\mathcal{R}_L(\xi)}_{(L-\xi)^{-1}} \left(\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(\xi)\right)^n$$

For $\xi \in \Omega$:
- $(\mathcal{A}\mathcal{R}_{\mathcal{B}}(\xi))^n$ bounded from $\mathcal{E}$ to E
- $\mathcal{R}_{\mathcal{B}}(\xi)$ and $\mathcal{A}$ bounded in $\mathcal{E}$
- $\mathcal{R}_L(\xi)$ bounded in E

$\rightarrow$ hence $\mathcal{U}(\xi)$ bounded in $\mathcal{E}$ and right-inverse of $(\mathcal{L}-\xi)$

$$\begin{aligned}(\mathcal{L}-\xi)\,\mathcal{U}(\xi) \quad &= \quad \sum_{k=0}^{n-1} (-1)^k (\mathcal{A} + (\mathcal{B}-\xi))\,\mathcal{R}_{\mathcal{B}}(\xi)\,(\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(\xi))^k \\ &\quad\quad + (-1)^n (\mathcal{L}-\xi)\,\mathcal{R}_L(\xi)\,(\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(\xi))^n \\ &= \quad \mathrm{Id}_{\mathcal{E}} \qquad \text{(telescopic cancellations)}\end{aligned}$$

Step 2: $\mathcal{L} - \xi$ is one-to-one on Ω

- $\mathcal{L}$ generates a semigroup $\Rightarrow \exists\, \xi_0 \in \Delta_a$ s.t. $\mathcal{L} - \xi_0$ is invertible
- $\mathcal{R}_{\mathcal{L}}(\xi_0)$ exists and bounded by some C_R

  $\Rightarrow \mathcal{R}_{\mathcal{L}}(\xi)$ exists on $B(\xi_0, 1/C_R)$

  $\Rightarrow \mathcal{R}_{\mathcal{L}}(\xi) = \mathcal{U}(\xi)$ on $B(\xi_0, 1/C_R)$
- Then *a priori* bound on $\mathcal{U}(\xi)$

$$\begin{aligned}\|\mathcal{U}(\xi)\|_{B(\mathcal{E})} &\leq \|\mathcal{R}_{\mathcal{B}}(\xi)\|_{B(\mathcal{E})} + \|\mathcal{R}_L(\xi)\|_{B(E)}\, \|\mathcal{A}\|_{B(\mathcal{E},E)}\, \|\mathcal{R}_{\mathcal{B}}(\xi)\|_{B(\mathcal{E})} \\ &\leq C_R \qquad \text{on} \quad \Delta_a \backslash B(0,r)\end{aligned}$$

- Conclusion by a continuation argument:

$$\text{Existence of} \quad \mathcal{R}_{\mathcal{L}}(\xi) = \mathcal{U}(\xi) \quad \text{on } \Omega$$

  with the bound C_R

Step 3: the discrete spectrum (I)

- On Δ_a spectrum of $\mathcal{L}$ = poles of $\mathcal{R}_{\mathcal{L}}$ = poles of $\mathcal{R}_L = \{0\}$
- First inclusion clear on the algebraic eigenspaces

$$\mathrm{Range}(\Pi_{L,0}) \subset \mathrm{Range}(\Pi_{\mathcal{L},0})$$

- Eigenspaces and eigenprojectors: write the Laurent series

$$\mathcal{R}_L(\xi) = \sum_{\ell=1}^{\ell_0} \xi^{-\ell} R_{-\ell} + \sum_{\ell=0}^{\infty} \xi^{\ell} R_{\ell}, \qquad \mathcal{A}\,\mathcal{R}_{\mathcal{B}}(\xi) = \sum_{j=0}^{\infty} \xi^j \mathcal{C}_j$$

with $R_{-1} = \Pi_{L,0}$ and

$$\mathrm{Range}(R_{-\ell_0}), \dots, \mathrm{Range}(R_{-2}) \subset \mathrm{Range}(R_{-1})$$

Step 3: the discrete spectrum (II)

Then we have the following formula for the spectral projector

$$\begin{aligned}
\Pi_{\mathcal{L},0} &:= \frac{i}{2\pi}\int_{|z|=r}\mathcal{R}_{\mathcal{L}}(z)\,dz \\
&= \frac{1}{2i\pi}\int_{|z|=r}\mathcal{R}_L(z)\,\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(z)\,dz \\
&= R_{-1}\,\mathcal{C}_0 + R_{-2}\,\mathcal{C}_1 + \cdots + R_{-\ell_0}\,\mathcal{C}_{\ell_0-1} \\
R(\Pi_{\mathcal{L},0}) &\subset \textit{algebraic eigenspace of } L
\end{aligned}$$

Remark

Another proof can be done by assuming additionnally some invertibility of $B-\xi$ in E for $\xi\in\Delta_a$, however it is more convenient not to have to check this in the applications

Step 4: The representation formula (I)

Simpler case $\mathcal{R}_\mathcal{L} = \mathcal{R}_\mathcal{B} - \mathcal{R}_L \, \mathcal{A} \, \mathcal{R}_\mathcal{B}$: We want to establish and estimate the growth of (for $a' > a$)

$$\forall\, f_0 \in \mathcal{D}(\mathcal{L}),\ \forall\, t \geq 0, \quad e^{t\mathcal{L}} f_0 = \Pi_{\mathcal{L},0}\, f_0 + (\mathrm{Id} - \Pi_{\mathcal{L},0})\, e^{t\,\mathcal{B}} f_0 + T_1(t)\, f_0$$

$$\begin{aligned} T_1(t) &:= -\lim_{M\to\infty} \frac{1}{2i\pi} \int_{a'-iM}^{a'+iM} e^{zt}\, (\mathrm{Id} - \Pi_{\mathcal{L},0})\, \mathcal{R}_L(z)\, \mathcal{A}\, \mathcal{R}_\mathcal{B}(z)\, \mathrm{d}z \\ &= -[(\mathrm{Id} - \Pi_{\mathcal{L},0})\, S_L(t)] * (\mathcal{A}\, S_\mathcal{B}(t))\, f_0 \end{aligned}$$

with time convolution $\quad (S_1 * S_2)(t) := \int_0^t S_1(s) \circ S_2(t-s)\, ds$

<u>Remark</u>: $S_1 * S_2$ not a semigroup but has the good decay, and convolution behaves well w.r.t. the Laplace transform

Step 4: The representation formula (II)

Factorization of the resolvent

$$(*) \qquad \mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{B}} \sum_{k=0}^{n-1} (-1)^k \, (\mathcal{A}\,\mathcal{R}_{\mathcal{B}})^k + (-1)^n \, \mathcal{R}_L \, (\mathcal{A}\,\mathcal{R}_{\mathcal{B}})^n$$

Higher-order factorization of the semigroup

$$S_{\mathcal{L}}(t)\, f_0 = \Pi_{\mathcal{L},0}\, f_0 + (\mathrm{Id} - \Pi_{\mathcal{L},0})\, S_{\mathcal{L}}(t)\, f_0$$

$$S_{\mathcal{L}}(t)\, f_0 = \Pi_{\mathcal{L},0}\, f_0 + \sum_{k=0}^{n-1} (-1)^k \, (\mathrm{Id} - \Pi_{\mathcal{L},0})\, S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{*k}\, (t)\, f_0$$

$$+ (-1)^n \, [(\mathrm{Id} - \Pi_{L,0}) S_L] * (\mathcal{A} S_{\mathcal{B}})^{*n}\, (t)\, f_0$$

The RHS is bounded thanks to assumption (iii)

Related to Dyson-Phillips expansion for semigroups: quantitative version, with (non-local) general decomposition $\mathcal{L} = \mathcal{A} + \mathcal{B}$

“Shrinkage” of the functional spaces

It is also possible to go from $\mathcal{E}$ to E with a similar method by using the “left-inverse” decomposition of the resolvent

$$(**) \qquad \mathcal{R}_L = \sum_{k=0}^{n-1} (-1)^k \, (A\,\mathcal{R}_B)^k \mathcal{R}_B + (-1)^n \, (A\,\mathcal{R}_B)^n \, \mathcal{R}_{\mathcal{L}}$$

and corresponding factorization formula on the semigroups, by changing the assumption (iii) into estimates now on

(iii') $T'_n := (\mathcal{S}_{\mathcal{B}}\,\mathcal{A})^{(*n)}$ satisfies $\| T'_n(t)\|_{B(\mathcal{E},E)} \le C_a\, e^{a\,t}$ for $n \in \mathbb{N}^*$

Useful to increase regularity or Lebesgue integrability

Plan

The Fokker-Planck equation in L^1

$$\partial_t f = Lf := \nabla_v \cdot \left(\nabla_v f + \nabla_v \phi \, f\right), \quad v \in \mathbb{R}^d$$

$$\phi : \mathbb{R}^d \to \mathbb{R} \text{ satisfies } \begin{cases} \phi \in C^2 \\ e^{-\phi(v)} \, \mathrm{d}v \text{ satisfies Poincaré's inequality} \\ \phi(v) \sim |v|^\gamma, \gamma \geq 2 \end{cases}$$

Stationary measure $M(v) \, \mathrm{d}v = e^{-\phi(v)} \, \mathrm{d}v$ in $\mathbb{R}^d$

Then $\quad \|f_t - M \langle f_{in} \rangle\|_{L^1(\langle v \rangle^k)} \leq C \, e^{-\lambda_{\gamma,k} \, t} \, \|f_0 - M \langle f_{in} \rangle\|_{L^1(\langle v \rangle^k)}$

for any $k > 0$ with $\lambda_{\gamma,k} := \lambda_P$ when $\gamma > 2$ while it is given by

$$\lambda_{\gamma,k} := \min \{\lambda_P; 2k + 0\}$$

for the critical case $\gamma = 2$, which degenerates to zero as k goes to 0

Optimality of these L^1 spectral gaps?

The kinetic Fokker-Planck equation in L^1

$$\partial_t f = Lf := \nabla_v \cdot (\nabla_v f + \nabla_v \phi \, f) - v \cdot \nabla_x f, \quad x \in \mathbb{T}^d, \; v \in \mathbb{R}^d$$

$$\phi : \mathbb{R}^d \to \mathbb{R} \text{ satisfies } \begin{cases} \phi \in C^2 \\ e^{-\phi(v)} \, \mathrm{d}v \text{ satisfies Poincaré's inequality} \\ \phi(v) \sim |v|^\gamma, \gamma \geq 2 \end{cases}$$

Stationary measure $M(v) \, \mathrm{d}x \, \mathrm{d}v = e^{-\phi(v)} \, \mathrm{d}x \, \mathrm{d}v$ in $\mathbb{T}^d \times \mathbb{R}^d$

Then $\quad \| f_t - M \langle \langle f_{in} \rangle \rangle \|_{L^1(\langle v \rangle^k)} \leq C \, e^{-\lambda_{\gamma,k} \, t} \, \| f_0 - M \langle \langle f_{in} \rangle \rangle \|_{L^1(\langle v \rangle^k)}$

for any $k > 0$ with $\lambda_{\gamma,k} := \lambda_{KFP}$ when $\gamma > 2$ while it is given by

$$\lambda_{\gamma,k} := \min \{ \lambda_{KFP}; 2k + 0 \}$$

for the critical case $\gamma = 2$, which degenerates to zero as k goes to 0

Optimality of these L^1 spectral gaps?

Idea of the proof

- First ingredient: Poincaré's inequality in $L^2(M^{-1})$ in $\mathbb{R}^d$ or hypocoercivity estimate CM-Neumann'06 in $\mathbb{T}^d \times \mathbb{R}^d$
- Second ingredient: decomposition $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with

$$\mathcal{B}f = \mathcal{L}f - M\chi_R f, \quad \mathcal{A} = M\chi_R f, \quad \chi_R(v) = \mathbf{1}_{|v|\leq R},$$

and dissipativity estimate

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathcal{B}f \langle v \rangle^k \leq \underbrace{\text{diffusive term}}_{\leq 0} + \int f \langle v \rangle^k (\varphi_{\gamma,k} - M\chi_R)$$

using that $\varphi_{\gamma,k} \to -\lambda_{\gamma,k} < 0$ at $|v| >> 1$

- Third ingredient: Regularization estimates on $\mathcal{A}\mathcal{S}_{\mathcal{B}}(t)$: ultracontractivity in $\mathbb{R}^d$, hypoellipticity in $\mathbb{T}^d \times \mathbb{R}^d$ (following Hérau and Villani)

The (kinetic) Fokker-Planck equation in W_1

$$\begin{cases} \partial_t f = Lf := \nabla_v \cdot (\nabla_v f + v\, f) \quad v \in \mathbb{R}^d \\ \partial_t f = Lf := \nabla_v \cdot (\nabla_v f + v\, f) - v \cdot \nabla_x f, \quad x \in \mathbb{T}^d,\ v \in \mathbb{R}^d \end{cases}$$

with $\phi(v) = v^2/2$ for simplicity (in general $\gamma \geq 2$ required)

Stationary measure $M(v) = e^{-\phi(v)}$ in $\mathbb{R}^d$ or $\mathbb{T}^d \times \mathbb{R}^d$

Here f probability measure: $\langle\langle f \rangle\rangle = 1$

$$\text{Then} \qquad W_1\left(f_t, M\right) \leq C\, e^{-\lambda\, t}\, W_1\left(f_0, M\right)$$

for some rate $\lambda > 0$ and constant $C > 0$ explicit

Idea of the proof (I)

Introduce the following norm

$$\|\psi\|_{\mathcal{F}_\infty} := \max\left\{\|\psi\langle v\rangle^{-1}\|_{L^\infty_v},\ \varepsilon\|\nabla_v\psi\|_{L^\infty_v}\right\}$$

in the case $v \in \mathbb{R}^d$ with ε chosen in the proof, or

$$\|\psi\|_{\mathcal{F}_\infty} := \max\left\{\|\psi\langle v\rangle^{-1}\|_{L^\infty_{x,v}},\ \|\nabla_x\psi\|_{L^\infty_{x,v}},\ \varepsilon\|\nabla_v\psi\|_{L^\infty_{x,v}}\right\}$$

in the case $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$ with ε chosen in the proof, and check that

$$W_1(f,g) \leq \|f-g\|_{\mathcal{F}'_\infty} \leq \varepsilon^{-1} W_1(f,g)$$

Idea of the proof (II)

Then prove dissipativity of $\mathcal{B}$ for the norm $\mathcal{F}'_\infty$ by using:

- duality: estimate $\mathcal{B}^*$ for $\mathcal{F}_\infty$
- use energy estimate in L^p and then pass to the limit $p \to \infty$ in a uniform way
- use mollification of the truncation χ_R
- (coercive) energy estimates for $\psi \langle v \rangle^{-1}$ and $\nabla_x \psi$
- the estimate for the v-derivative produces zero-order terms and x-derivatives terms, controlled by the previous estimates by adjusting ε small enough

Finally use regularization estimates on $\mathcal{A}\mathcal{S}_\mathcal{B}(t)$ and $\mathcal{S}_\mathcal{B}(t)\mathcal{A}$

Remarks and open problems

- Used in other contexts:
  - Haff law and self-similar cooling process for granular gases: Mischler-CM'09
  - Wigner-Fokker-Planck equation: AGGMMS'12, Stürzer-Arnold'13
  - Smoluchowski equation: Cañizo-Lods'14...
  - Landau equation: Carrapatoso'14
- Possible to study singularities and pointwise estimates on the Green's function (fluid-kinetic) for the Boltzmann equation: First step (revisit Liu-Yu's results) K.-C. Wu
- Useful framework for the clustering problem in granular gases: Tristani'14, work in progress w/ Mischler & Rey
- Whole space with potential confinement w/ Mischler
- Situation with continuous spectrum under study...