A certain necessary condition of possible blow up for Navier-Stokes equations

Gregory A. Seregin

March 2, 2014

Cauchy Problem

- Navier-Stokes equations

$$\partial_t v(x,t) + v(x,t)\cdot\nabla v(x,t) - \Delta v(x,t) + \nabla q(x,t) = 0$$
$$\mathrm{div}\, v(x,t) = 0$$

for $(x,t) \in \mathbb{R}^3 \times]0,\infty[$

Cauchy Problem

- Navier-Stokes equations

$$\partial_t v(x,t) + v(x,t)\cdot\nabla v(x,t) - \Delta v(x,t) + \nabla q(x,t) = 0$$
$$\operatorname{div} v(x,t) = 0$$

  for $(x,t) \in \mathbb{R}^3 \times]0,\infty[$

- initial condition

$$v(\cdot,0) = v_0(\cdot) \in C^\infty_{0,0}(\mathbb{R}^3)$$

Blowup Time

- T is a blowup time if

$$M(t) = \sup_{x \in \mathbb{R}^3} |v(x,t)| \to \infty$$

as $t \to T - 0$

Blowup Time

- T is a blowup time if

$$M(t) = \sup_{x \in \mathbb{R}^3} |v(x,t)| \to \infty$$

  as $t \to T - 0$

- Question is whether or not $T = \infty$

Leray's Necessary Conditions

$$\|v(\cdot,t)\|_\infty \geq \frac{C}{\sqrt{T-t}}$$

Other norms blow up:

$$\|v(\cdot,t)\|_p = \Big(\int_{\mathbb{R}^3} |v(x,t)|^p dx\Big)^{\frac{1}{p}} \geq \frac{C(p)}{(T-t)^{\frac{p-3}{2p}}}$$

$$\|\nabla v(\cdot,t)\|_2 \geq \frac{C}{(T-t)^{\frac{1}{4}}}$$

Limit Case $p = 3$

$$\exists t_k \to T: \quad \|v(\cdot, t_k)\|_3 \to \infty$$

Question:

$$\|v(\cdot, t)\|_3 \to \infty \qquad \text{as} \quad t \to T?$$

$$\Updownarrow$$

if there exists $t_k \to T$ s.t. $\sup_k \|v(\cdot, t_k)\|_3 =: M < \infty$, then T is not a blow up time

Negative Issue

Assume that there exists a function f such that $f(\varepsilon) \to \infty$ as $\varepsilon \to 0$ and

$$\|v(\cdot, t)\|_3 \geq f(T - t)$$

Indeed, if

$$\|v(\cdot, 0)\|_3 \geq f(T),$$

then for $v^\lambda(y, s) = \lambda v(\lambda y, \lambda^2 s)$

$$\|v(\cdot, 0)\|_3 = \|v^\lambda(\cdot, 0)\|_3 \geq f(T/\lambda^2) \to \infty$$

as $\lambda \to \infty$

Outline

- I. Scaling with a parameter proportional to the distance (in time) to blowup

Outline

- I. Scaling with a parameter proportional to the distance (in time) to blowup
- II. Decomposition and local energy estimates of a scaled solution

Outline

- I. Scaling with a parameter proportional to the distance (in time) to blowup
- II. Decomposition and local energy estimates of a scaled solution
- III. Limiting procedure giving a non-trivial local energy solution (in the sense of Lemarie-Rieusset) on some time interval

Outline

- I. Scaling with a parameter proportional to the distance (in time) to blowup
- II. Decomposition and local energy estimates of a scaled solution
- III. Limiting procedure giving a non-trivial local energy solution (in the sense of Lemarie-Rieusset) on some time interval
- IV. Decay at spatial infinity by ε-regularity

Outline

- I. Scaling with a parameter proportional to the distance (in time) to blowup
- II. Decomposition and local energy estimates of a scaled solution
- III. Limiting procedure giving a non-trivial local energy solution (in the sense of Lemarie-Rieusset) on some time interval
- IV. Decay at spatial infinity by ε-regularity
- V. Backward uniqueness

I1. Blowup L_3-Profile

$$M := \sup_k \|v(\cdot, t_k)\|_3 < \infty$$

$$\Downarrow$$

$$\|v(\cdot, T)\|_3 < \infty$$

I2. Scaling

$$u^{(k)}(y,s) = \lambda_k v(x,t), \quad p^{(k)}(y,s) = \lambda_k^2 q(x,t),$$

where $(y,s) \in \mathbb{R}^3 \times]-\lambda_k^{-2}T, 0[$, $x = \lambda_k y, \qquad t = T + \lambda_k^2 s,$

$$\lambda_k = \sqrt{\frac{T - t_k}{S}}$$

and a positive parameter $S < 10$ will be defined later;

$$\sup_{k \in \mathbb{N}} \|u^{(k)}(\cdot, -S)\|_3 = M < \infty$$

III1. Decomposition $u^{(k)} = v^{(k)} + w^{(k)}$

$$\partial_t w^{(k)} - \Delta w^{(k)} = -\nabla r^{(k)}, \quad \mathrm{div}\, w^{(k)} = 0 \quad \mathrm{in} \quad \mathbb{R}^3 \times]-S, 0[,$$

$$w^{(k)}(\cdot, -S) = u^{(k)}(\cdot, -S)$$

and

$$\sup_k \{ \| w^{(k)} \|_{L_5(\mathbb{R}^3 \times]-S,0[} + \| w^{(k)} \|_{L_{3,\infty}(\mathbb{R}^3 \times]-S,0[} \} \leq c(M) < \infty$$

II2. Decomposition $u^{(k)} = v^{(k)} + w^{(k)}$ II

$$\partial_t v^{(k)} + \mathrm{div}(v^{(k)} + w^{(k)}) \otimes (v^{(k)} + w^{(k)}) - \Delta v^{(k)} = -\nabla p^{(k)}$$

$$\mathrm{div}\, v^{(k)} = 0 \quad \text{in} \quad \mathbb{R}^3 \times]-S, 0[$$

$$v^{(k)}(\cdot, -S) = 0$$

II3. Pressure

$$p^{(k)}(x,t) = -\frac{1}{3}|u^{(k)}(x,t)|^2 + \frac{1}{4\pi}\int\limits_{\mathbb{R}^3} K(x-y) : u^{(k)}(y,t) \otimes u^{(k)}(y,t)\,dy$$

II4. Pressure Decomposition

For $x_0 \in \mathbb{R}^3$ and for $x \in B(x_0, 3/2)$, we let

$$p_{x_0}^{(k)}(x,t) \equiv p^{(k)}(x,t) - c_{x_0}^{(k)}(t) = p_{x_0}^{1(k)}(x,t) + p_{x_0}^{2(k)}(x,t)$$

where

$$p_{x_0}^{1(k)}(x,t) = -\frac{1}{3}|u^{(k)}(x,t)|^2 + \frac{1}{4\pi}\int\limits_{B(x_0,2)} K(x-y) : u^{(k)}(y,t) \otimes u^{(k)}(y,t)dy$$

$$p_{x_0}^{2(k)}(x,t) = \frac{1}{4\pi}\int\limits_{\mathbb{R}^3 \setminus B(x_0,2)} (K(x-y) - K(x_0-y)) : u^{(k)}(y,t) \otimes u^{(k)}(y,t)dy$$

$$c_{x_0}^{(k)}(t) = \frac{1}{4\pi}\int\limits_{\mathbb{R}^3 \setminus B(x_0,2)} K(x_0 - y) : u^{(k)}(y,t) \otimes u^{(k)}(y,t)dy$$

II5. Local Energy

$$\alpha(s) = \alpha(s; k, S) = \|v^{(k)}(\cdot, s)\|^2_{2,\mathrm{unif}}$$

$$\beta(s) = \beta(s; k, S) = \sup_{x \in R^3} \int_{-S}^{s} \int_{B(x,1)} |\nabla v^{(k)}|^2 dy d\tau, \qquad \text{where}$$

$$\|g\|_{L_{2,\mathrm{unif}}} = \sup_{x_0 \in \mathbb{R}^3} \|g\|_{L_2(B(x_0,1))}.$$

More quantities:

$$\gamma(s) = \gamma(s; k, S) = \sup_{x \in R^3} \int_{-S}^{s} \int_{B(x,1)} |v^{(k)}(y, \tau)|^3 dy d\tau$$

$$\delta(s) = \delta(s; k, S) = \sup_{x \in R^3} \int_{-S}^{s} \int_{B(x,3/2)} |p^{(k)}(y, \tau) - c_x^{(k)}(\tau)|^{\frac{3}{2}} dy\, d\tau$$

II6. Pressure estimates

$$\|p^{1(k)}_{x_0}(\cdot,t)\|_{L_{\frac{3}{2}}(B(x_0,3/2))} \leq c(M)(\|v^{(k)}(\cdot,t)\|^2_{L_3(B(x_0,2))} + 1)$$

and

$$\sup_{B(x_0,3/2)} |p^{2(k)}_{x_0}(x,t)| \leq c(M)(\|v^{(k)}(\cdot,t)\|^2_{L_{2,\mathrm{unif}}} + 1)$$

II7. Energy Estimate and Choice of S

$$\alpha(s)+\beta(s)\leq c(M)\Big[(s+S)^{\frac{1}{5}}+$$

$$+\int\limits_{-S}^{s}\Big(\alpha(\tau)(1+\|w^{(k)}(\cdot,\tau)\|^5_{L_{5,\mathrm{unif}}})+\alpha^3(\tau)\Big)\,d\tau\Big]$$

$\forall s\in[-S,0[$ and $c(M)$ is independent of k, s, and S.
There is a positive constant $S(M)$:

$$\alpha(s)\leq\frac{1}{10}$$

and

$$\alpha(s)\leq c(M)(s+S)^{\frac{1}{5}}$$

for any $s\in]-S(M),0[$

III1. Limiting Procedure

$$w^{(k)} \to w$$

$$w(x,t) = \frac{1}{(4\pi(s+S))^{\frac{3}{2}}} \int\limits_{\mathbb{R}^3} \exp\Big(-\frac{|x-y|^2}{4(s+S)}\Big) w_0(y)dy$$

where w_0 is the weak $L_3(\mathbb{R}^3)$-limit of the sequence $u^{(k)}(\cdot,-S)$;

$$\sup_{-S<s<0} \sup_{x_0\in\mathbb{R}^3} \|w(\cdot,s)\|^2_{L_2(B(x_0,1))} +$$

$$+ \sup_{x_0\in\mathbb{R}^3} \int\limits_{-S}^{0} \int\limits_{B(x_0,1)} |\nabla w(y,s)|^2 dyds \leq c(M) < \infty$$

$$w \in C([-S,0];L_3(\mathbb{R}^3)) \cap L_5(\mathbb{R}^3\times]-S,0[)$$

III2. Limit function u is a local energy solution

$$u^{(k)} \to u, \qquad p^{(k)} \to p$$

$$\sup_{-S<s<0} \sup_{x_0\in\mathbb{R}^3} \|u(\cdot,s)\|^2_{L_2(B(x_0,1))} + \sup_{x_0\in\mathbb{R}^3} \int\limits_{-S}^{0} \int\limits_{B(x_0,1)} |\nabla u(y,s)|^2 dyds < \infty;$$

$$p \in L_{\frac{3}{2}}(-S,0; L_{\frac{3}{2},\mathrm{loc}}(\mathbb{R}^3);$$

the function

$$s \mapsto \int\limits_{\mathbb{R}^3} u(y,s)\cdot w(y)dy$$

is continuous on $[-S,0]$ for any compactly supported $w \in L_2(\mathbb{R}^3)$;

III3. Limit function u is a local energy solution

$$\partial_t u + u \cdot \nabla u - \Delta u = -\nabla p, \quad \mathrm{div}\, u = 0$$

in $\mathbb{R}^3 \times]-S, 0[$ in the sense of distributions;
for any $x_0 \in \mathbb{R}^3$, there exists a function $c_{x_0} \in L_{\frac{3}{2}}(-S, 0)$ such that

$$p(x,s) - c_{x_0}(s) = p^1_{x_0}(x,s) + p^2_{x_0}(x,s)$$

for all $x \in B(x_0, 3/2)$, where

$$p^1_{x_0}(x,s) = -\frac{1}{3}|u(x,s)|^2 + \frac{1}{4\pi} \int\limits_{B(x_0,2)} K(x-y) : u(y,s) \otimes u(y,s) dy,$$

$$p^2_{x_0}(x,s) = \frac{1}{4\pi} \int\limits_{\mathbb{R}^3 \setminus B(x_0,2)} (K(x-y) - K(x_0-y)) : u(y,s) \otimes u(y,s) dy;$$

III4. Limit function u is a local energy solution

for any $s \in]-S,0[$ and for $\varphi \in C_0^\infty(\mathbb{R}^3 \times]-S,S[)$,

$$\int\limits_{\mathbb{R}^3} \varphi^2(y,s)|u(y,s)|^2 dy + 2\int\limits_{-S}^{s}\int\limits_{\mathbb{R}^3} \varphi^2 |\nabla u|^2 dy d\tau \leq$$

$$\leq \int\limits_{-S}^{s}\int\limits_{\mathbb{R}^3} \Big(|u|^2(\Delta\varphi^2 + \partial\varphi^2) + u\cdot\nabla\varphi^2(|u|^2+2p)\Big) dy d\tau$$

III5. Limit function u is a local energy solution

$$\sup_{x_0\in\mathbb{R}^3}\|v(\cdot,s)\|^2_{L_2(B(x_0,1))}\leq c(M)(s+S)^{\frac{1}{5}}$$

for all $s\in[-S,0]$.

$$\Downarrow$$

$$v(\cdot,s)\to 0\qquad\mathrm{in}\quad L_{2,\mathrm{loc}}(\mathbb{R}^3)$$

as $s\downarrow -S$.

$$\Downarrow$$

$$u(\cdot,s)\to w_0\qquad\mathrm{in}\quad L_{2,\mathrm{loc}}(\mathbb{R}^3).$$

as $s\downarrow -S$.

IV1 Limit function is not trivial

$$\frac{1}{a^2}\int\limits_{Q(a)}(|u^{(k)}|^3+|\widetilde{p}^{(k)}|^{\frac{3}{2}})dy\,ds=\frac{1}{(a\lambda_k)^2}\int\limits_{Q(z_T,a\lambda_k)}(|v|^3+|q-b^{(k)}|^{\frac{3}{2}})dx\,dt$$

for all $0<a<a_*=\inf\{1,\sqrt{S/10},\sqrt{T/10}\}$ and for all $\lambda_k\leq 1$. Here, $z_T=(0,T)$, $\widetilde{p}^{(k)}\equiv\widetilde{p}_2^{(k)}$, and $b^{(k)}(t)=\lambda_k^{-2}c_2^{(k)}(s)$. Since the pair v and $q-b^{(k)}$ is a suitable weak solution to the Navier-Stokes equations in $Q(z_T,\lambda_k a_*)$, we find

$$\frac{1}{a^2}\int\limits_{Q(a)}(|u^{(k)}|^3+|\widetilde{p}^{(k)}|^{\frac{3}{2}})dy\,ds>\varepsilon \qquad (1)$$

for all $0<a<a_*$ with a positive universal constant ε

IV2 Limit function is not trivial

$$\frac{1}{a^2}\int\limits_{Q(a)}|u^{(k)}|^3 dy\, ds \to \frac{1}{a^2}\int\limits_{Q(a)}|u|^3 dy\, ds$$

for all $0 < a < a_*$ and

$$\sup_{k\in\mathbb{N}}\frac{1}{a_*^2}\int\limits_{Q(a_*)}(|u^{(k)}|^3 + |\widetilde{p}^{(k)}|^{\frac{3}{2}})dy\, ds = M_1 < \infty$$

IV3 Limit function is not trivial

$$\widetilde{p}^{(k)} = p_1^{(k)} + p_2^{(k)}$$

with

$$\|p_1^{(k)}(\cdot, s)\|_{\frac{3}{2}, B(a_*)} \leq c\|u^{(k)}(\cdot, s)\|^2_{3, B(a_*)}$$

$p_2^{(k)}$ is harmonic

$$\varepsilon \leq \frac{1}{a^2} \int\limits_{Q(a)} (|\widetilde{p}^{(k)}|^{\frac{3}{2}} + |u^{(k)}|^3) dy\, ds \leq$$

$$\leq c\frac{1}{a^2} \int\limits_{Q(a_*)} |u^{(k)}|^3 dy\, ds + cM_1 a a_*^2$$

for all $0 < a < a_*/2$

IV4. Spatial Decay

$$w_0 \in L_3$$

$$\Downarrow$$

$$\|w_0\|_{2,B(x_0,1)} \to 0$$

as $|x_0| \to \infty$

$$\Downarrow$$

There are positive numbers R, $T \in]a_*, S[$, and c_k with $k = 0, 1, ...$ such that

$$|\nabla^k u(x,t)| \leq c_k$$

for any $x \in \mathbb{R}^3 \setminus B(R/2)$ and for any $t \in]-T, 0[$

V1. Backward Uniqueness

$$v(\cdot, T) \in L_3(\mathbb{R}^3) \Rightarrow u(\cdot, 0) = 0$$

$$\Downarrow$$

$$\omega = \nabla \wedge u = 0$$

at $t = 0$.

$$\Downarrow \qquad \text{BackwardUniqueness}$$

$$\omega = 0$$

in $(\mathbb{R}^3 \setminus B(R/2)) \times]-T, 0[$

V2. Backward Uniqueness

technical part is to show

$$\omega = 0$$

in $\mathbb{R}^3 \times]-T, 0[$

$$\Downarrow$$

$u(\cdot, t)$ is a bounded harmonic function in $\mathbb{R}^3$ with

$$\|u(\cdot, t)\|_{L_2(B(x_0,1))} \to 0$$

as $|x_0| \to \infty$

$$\Downarrow$$

$$u = 0$$

in $\mathbb{R}^3 \times]-T, 0[$