QCD with chemical potential in a small hyperspherical box

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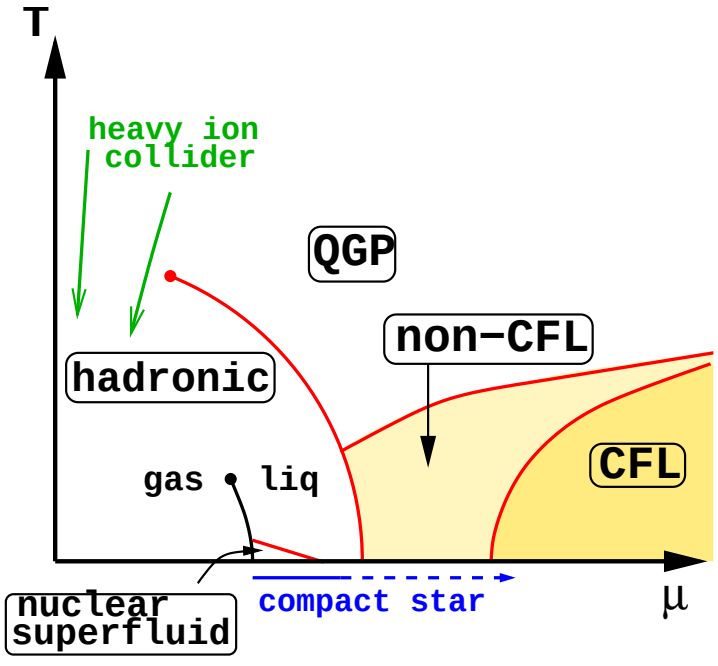
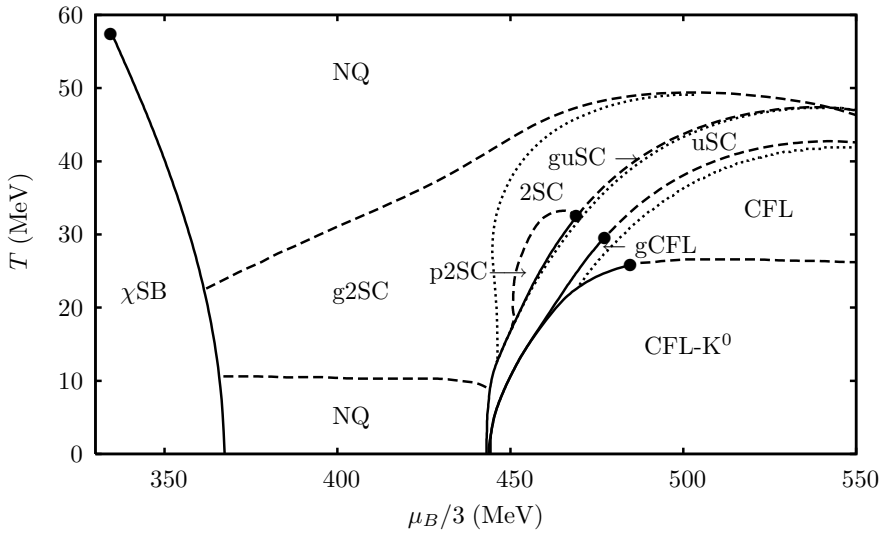
Outline

- Overview of QCD at non-zero quark chemical potential
- Formulation of QCD on $S^1 \times S^3$ using perturbation theory
- Results for various observables for $N = 3$ and $N = \infty$, preliminary $N = 2$ lattice results.

Conjectured phase diagram of QCD

Progress has been made towards obtaining the phase diagram of QCD at non-zero density using lattice simulations as well as models of QCD.





(Left) Conjectured phase diagram of QCD in the μ - T plane. (Right) Detailed possible phase diagram from an NJL model (from Alford arXiv:0907.0200).

What makes QCD at non-zero μ so difficult? The Sign Problem:

QCD at finite quark chemical potential μ has a complex action.

- The boltzmann weight e^{-S} is complex so it is not possible to perform lattice simulations which use importance sampling.
- The sign problem also complicates large N analysis: In the large N limit the saddle point approximation becomes valid, but the stationary point of a complex action with respect to the angles of the Polyakov line lies in the space where the angles are complex. Therefore the eigenvalues of the Polyakov line lie off the unit circle on an arc in the complex plane.

$\Longrightarrow$ Need to generalize our techniques to handle a complex action.

Successful techniques that deal with or avoid the sign problem

Lattice techniques valid for $\mu/T < 1$

- Taylor expansion
- Reweighting
- Imaginary μ + analytic continuation

Infinite volume perturbation theory

- chiral perturbation theory
- large μ perturbation theory

Using models

- 2-color QCD
- Random Matrix Theory
- Nambu-Jona-Lasinio Models
- AdS/CFT

New

- Complex Langevin
- Finite spatial volume perturbation theory (this talk)

Region of validity of 1-loop calculations

Properties of $SU(N)$ gauge theories on $S^1 \times S^3$

- Valid for $\min[R_{S^1}, R_{S^3}] \ll \Lambda_{QCD}^{-1}$
  - $\mathbb{R}^3 \times S^1$, small S^1:
    - ★ Good: Allows study at any N and in the limit of large 3-volume.
      YM/QCD: $m = 0$, $\mu = 0$: Gross, Pisarski, Yaffe (Rev.Mod.Phys.53:43,1981), $\mu \neq 0$: Korthals Altes, Pisarski and Sinkovics (hep-ph/9904305), $m \neq 0$: Meisinger and Ogilvie (hep-ph/0108026), QCD(Adj/AS/S): $m = 0$, $\mu = 0$: Unsal and Yaffe (hep-th/0608180), $m \neq 0$: Myers and Ogilvie (arXiv:0903.4638)
    - ★ Bad: Have to be in the limit of high temperatures (or small S^1)
  - $S^3 \times S^1$, small S^3:
    - ★ Good: Allows study at any temperature (or any S^1).
      YM: Aharony et al (hep-th/0310285), QCD(Adj/AS/S): $m = 0$: Hollowood and Naqvi (hep-th/0609203), Unsal (hep-th/0703025), $m \neq 0$: Hollowood and Myers (arXiv:0907.3665), QCD: $m \neq 0$, $\mu \neq 0$: Hands, Hollowood, Myers (arXiv:0910.0441)
    - ★ Bad: Must be in small 3-volume. Finite N studies are more complicated.

Partition Function of QCD

The partition function of QCD at finite temperature $T = 1/\beta$, for N_f quark flavors, each with a mass m_f and coupled to a chemical potential μ_f is:

$$Z_{QCD} = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\int_0^\beta d\tau \int d^3\mathbf{x}\mathcal{L}_{QCD}}$$

where ψ and $\bar{\psi}$ are the fundamental and anti-fundamental fermion fields, respectively, and A is the $SU(N)$ gauge field, $A_\mu = A_\mu^a T^a$.

The Lagrangian is

$$\mathcal{L}_{QCD} = \frac{1}{4g^2}\mathrm{Tr}_F\left(F_{\mu\nu}F_{\mu\nu}\right) + \sum_{f=1}^{N_f}\bar{\psi}_f\left(\not{D}_F(A) - \gamma_0\mu_f + m_f\right)\psi_f,$$

with covariant derivative

$$D_\mu(A) \equiv \partial_\mu - A_\mu.$$

1-loop Lagrangian

Introduce fluctuations around a background field: $A_0 = \alpha + g\mathscr{A}_0$, then gauge fix and retain the one-loop contributions:

$$\begin{aligned}\mathcal{L}_{QCD} = &-\frac{1}{2}\mathscr{A}_0^a(D_0^2(\alpha)+\Delta^{(s)})\mathscr{A}_0^a - \frac{1}{2}B_i^a(D_0^2(\alpha)+\Delta^{(v,T)})B_i^a \\ &-\frac{1}{2}C_i^a(D_0^2(\alpha)+\Delta^{(v,L)})C_i^a - \bar{c}(D_0^2(\alpha)+\Delta^{(s)})c \\ &+\sum_{f=1}^{N_f}\bar{\psi}_f(\not{D}_F(\alpha)-\gamma_0\mu_f+m_f)\psi_f\end{aligned}$$

where $$A_i = B_i + C_i.$$

- B_i = transverse: $\nabla_i B_i = 0$
- C_i = longitudinal: $C_i = \nabla_i f$

1-loop partition function

Performing the Gaussian integrals the almost-cancellation of the scalar field contributions simplifies the one loop effective partition function:

$$Z(\alpha) =$$
$$\det{}_{\ell=0}^{1/2}\left(D_0^2(\alpha)+\Delta^{(s)}\right)\det{}^{-1}\left(-D_0^2(\alpha)+\Delta^{(v,T)}\right)\det{}^{N_f}\left(\not{D}_F(\alpha)-\gamma_0\mu+m\right)$$

Eigenvalues ε_l and degeneracies d_l of Laplacians on S^3:

$$\Delta^{(type)}\Omega_{j,l,m_1,m_2}(\theta_1,...,\theta_3) = -\varepsilon_l^{(type)2}\Omega_{j,l,m_1,m_2}(\theta_1,...,\theta_3)$$

Example: scalars

$$\varepsilon_l^{(s)2} = l(l+2)/R^2$$

$$d_l^{(s)} = (l+1)^2$$

where $l = 0, 1, ...,$ and R is the radius of S^3.

1-loop partition function: S^1 contribution

The eigenvalues of the Dirac operator can be computed in frequency space in terms of the Matsubara frequencies:

$$D_0(\alpha) \to i\omega_n^- - \alpha,$$

where the Matsubara frequencies, for antiperiodic (thermal) boundary conditions are

$$\omega_n^- = (2n+1)\pi/\beta.$$

We define the Polyakov loop:

$$P = \mathscr{P} e^{\int_0^\beta \mathrm{d}t\, A_0(x)} = e^{\beta\alpha} = \mathrm{diag}\{e^{i\theta_1}, ..., e^{i\theta_N}\}$$

1-loop effective action

Simplification of the effective partition function leads gives the effective action

$$
\begin{aligned}
S(P) &= -\log Z(P) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - z_b(n\beta/R)\right) \mathrm{Tr}_A P^n \\
&\qquad + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} N_f z_f(n\beta/R, mR) \left[e^{n\beta\mu} \mathrm{Tr}_F P^n + e^{-n\beta\mu} \mathrm{Tr}_F P^{\dagger n} \right],
\end{aligned}
$$

where

$$
z_b(\beta/R) = \sum_{\ell=1}^{\infty} d_\ell^{(v,T)} e^{-\beta\varepsilon_\ell^{(v,T)}} = 2 \sum_{\ell=1}^{\infty} \ell(\ell+2) e^{-n\beta(\ell+1)/R}
$$

$$
z_f(\beta/R, mR) = \sum_{\ell=1}^{\infty} d_\ell^{(f)} e^{-\beta\varepsilon_\ell^{(f,m)}} = 2 \sum_{\ell=1}^{\infty} \ell(\ell+1) e^{-\beta\sqrt{(\ell+\frac{1}{2})^2 + m^2R^2}/R}
$$

For the pure Yang-Mills theory the weak-coupling analogue of the deconfinement transition temperature can be calculated in the large N limit: $T_d R \simeq 0.759$ or $\beta_d/R \simeq 1.317$ [Aharony et al (hep-th/0310285)].

Important observables

At finite N we can calculate observables by numerically performing the integrals over the gauge field angles θ_i.

$$\langle \mathcal{O} \rangle \equiv \frac{\int [\mathrm{d}\theta] e^{-S} \mathcal{O}}{Z}, \qquad Z = \int [\mathrm{d}\theta] e^{-S}.$$

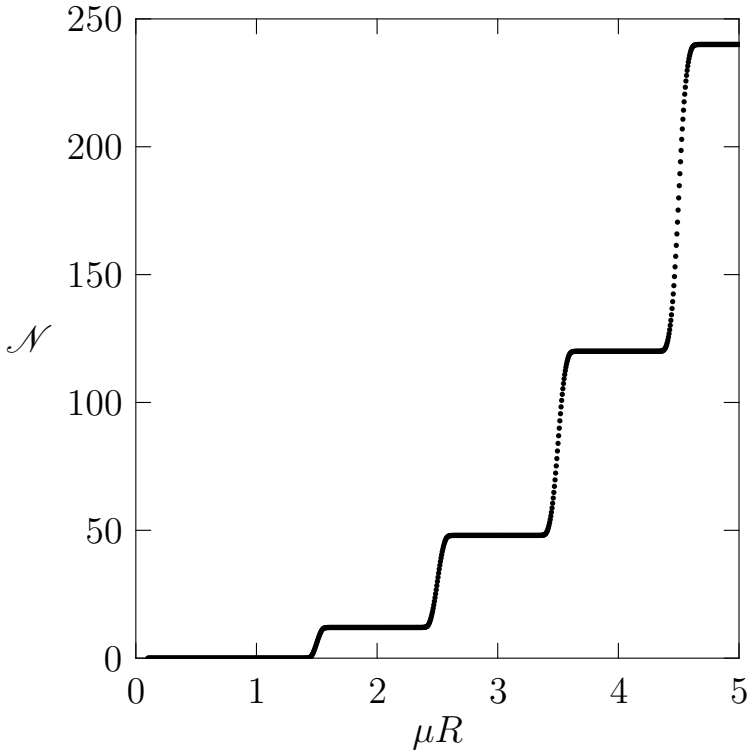
- Polyakov loop order parameters for phase transitions: $\langle \mathrm{Tr} P \rangle$, $\langle \mathrm{Tr} P^\dagger \rangle$.

$$\mathscr{P}_1 = \langle \mathrm{Tr} P \rangle$$

$$\mathscr{P}_{-1} = \langle \mathrm{Tr} P^\dagger \rangle$$

  - These should differ for $N > 2$.
- Average Phase: $\langle e^{i\phi} \rangle_{pq} \equiv Z/Z_{pq}$.
  - This tells us where the sign problem is severe.
- Average number $\mathcal{N} = \langle N_{quarks} - N_{antiquarks} \rangle$.
  - Gives the net number of quarks allowed at a given value of μ.
- Pressure: $\mathcal{P}$
- Energy: E

Average fermion number $\mathscr{N}$ ($N = 3$, $N_f = 1$, $mR = 0$)



- Average fermion number for QCD on $S^1 \times S^3$ with $m = 0$, $\beta/R = 30$ (low T).

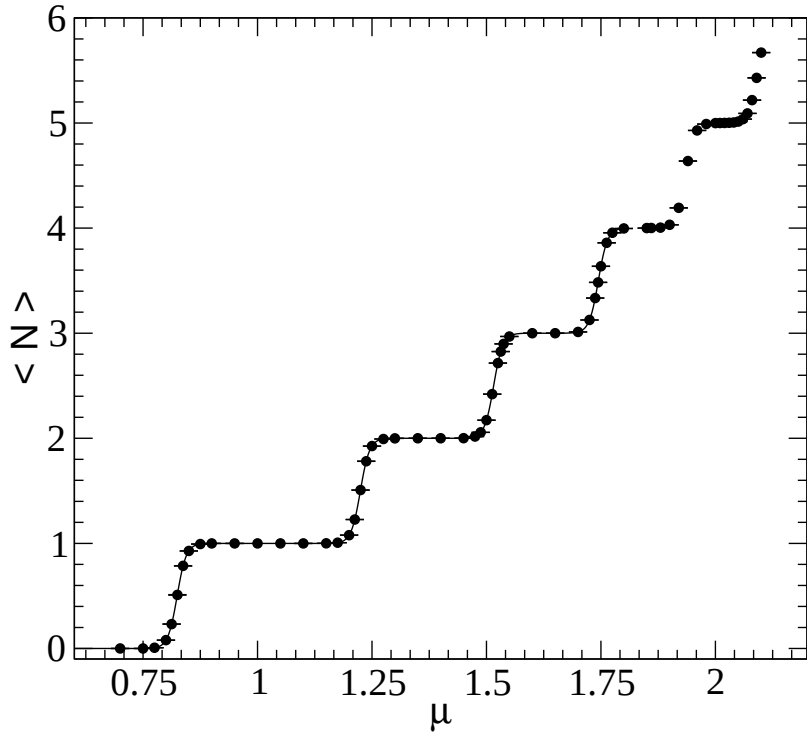
$$\mathscr{N} = \frac{1}{\beta}\left(\frac{\partial \ln Z}{\partial \mu}\right) = \frac{-1}{\beta Z}\int [\mathrm{d}\theta]\, e^{-S}\left(\frac{\partial S}{\partial \mu}\right)$$

$$\mathscr{N} \xrightarrow[\beta\to\infty]{} \frac{2N_f}{Z}\int [\mathrm{d}\theta]\, e^{-S}\sum_{l=1}^{\infty}\sum_{i=1}^{N} l(l+1)\left[\frac{e^{\beta\mu}}{e^{\beta\mu} + e^{-i\theta_i + \beta(l+1/2)/R}}\right]$$

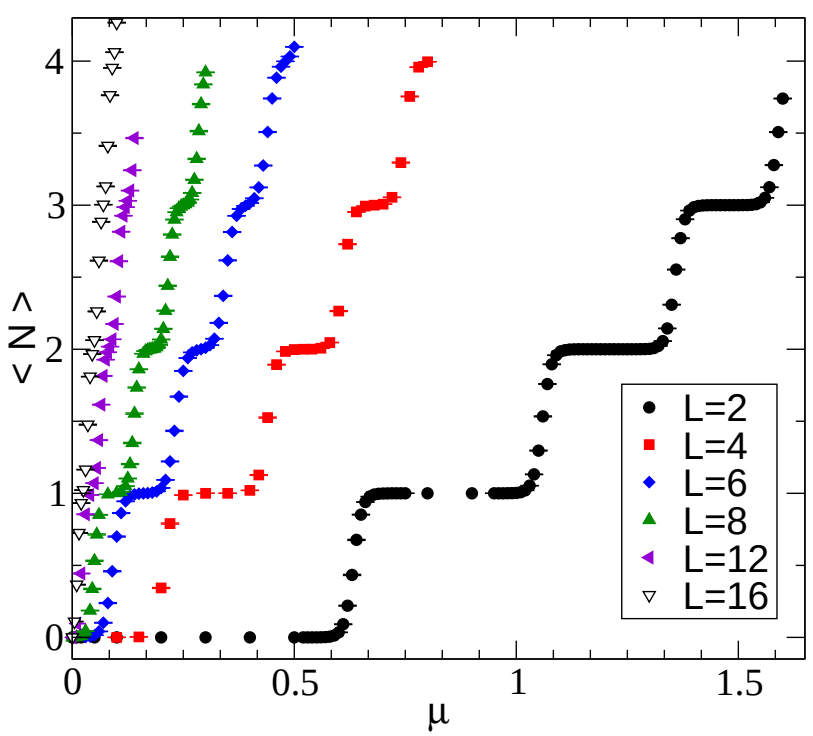
Each level L starts at $(\mu R)_0 = L + 1/2$ and has:

$$\text{height}: \quad h_L = NN_f \sum_{l}^{L} 2l(l+1); \qquad \text{width}: \quad w = \Delta(\mu R) = 1$$

Classical non-linear $O(2)$ sigma-model results from lattice (Banerjee and Chandrasekharan, arXiv:1001.3648)

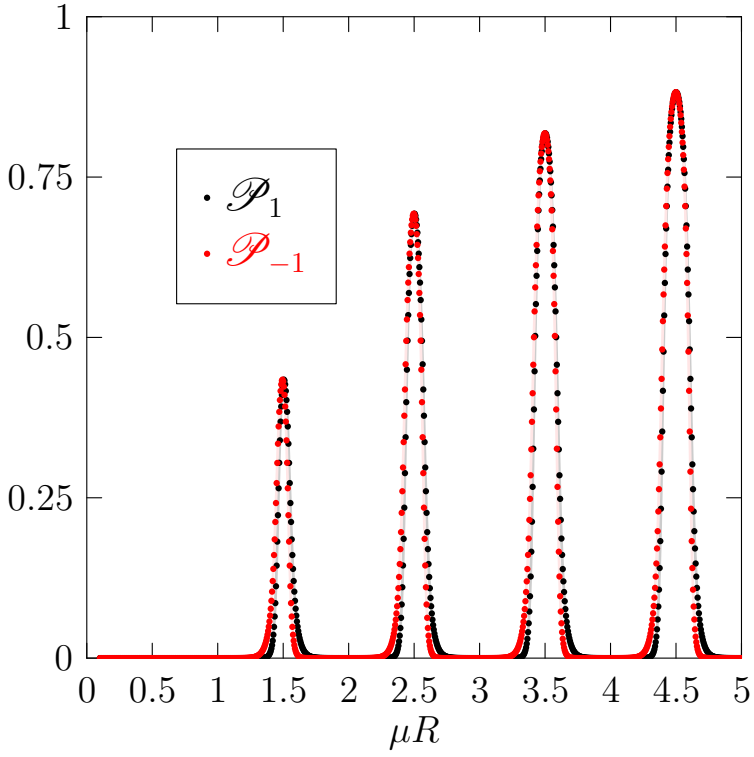


Average charge number



Levels appear to go away with increasing $L_s = L$, but this may not be true if L_t/L_s is kept fixed at a large value.

Polyakov lines: $\mathscr{P}_1 = \langle \mathrm{Tr} P \rangle$ and $\mathscr{P}_{-1} = \langle \mathrm{Tr} P^\dagger \rangle$ for $m = 0$



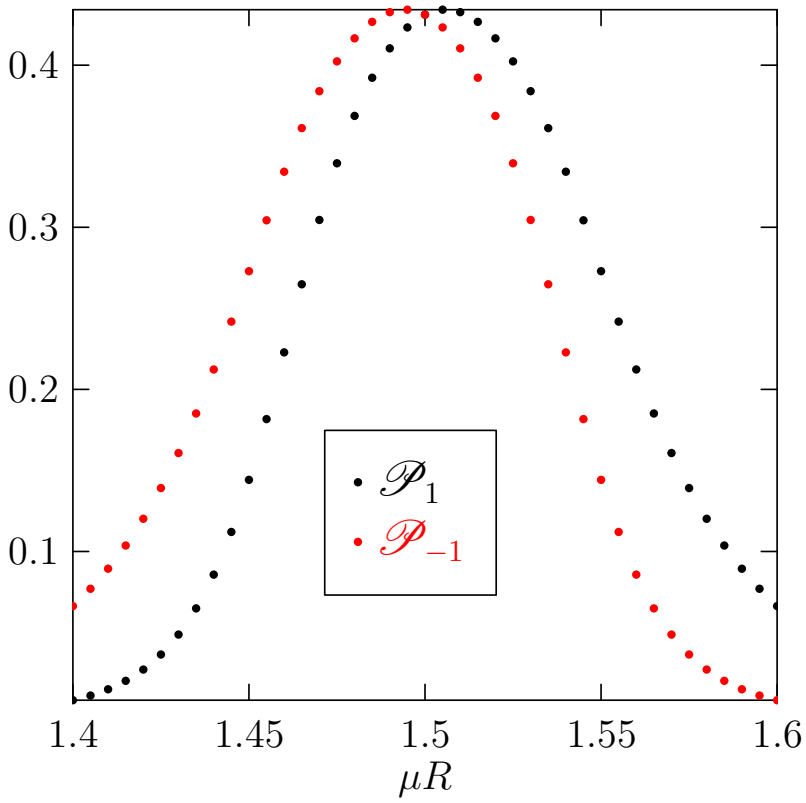
$$\mathscr{P}_1 \equiv \frac{\int [\mathrm{d}\theta] e^{-S} \sum_{i=1}^{N} e^{i\theta_i}}{Z},$$

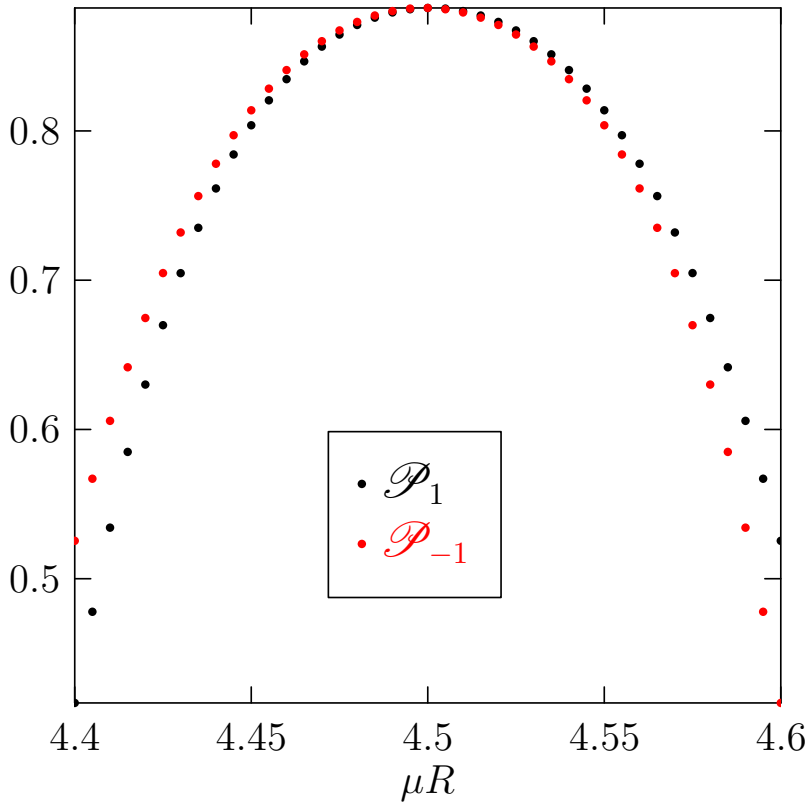
$$\mathscr{P}_{-1} \equiv \frac{\int [\mathrm{d}\theta] e^{-S} \sum_{i=1}^{N} e^{-i\theta_i}}{Z}.$$

Because of the bias in the temporal direction introduced by a non-zero chemical potential $\mathscr{P}_1 \neq \mathscr{P}_{-1}$

- There is a spike in $\mathscr{P}_1$ and $\mathscr{P}_{-1}$ corresponding to each jump in the average fermion number, $\mathscr{N}$.
- Deconfinement occurs in between the levels, as they are being filled.
- Given our notation, $\mathscr{P}_{-1}$ always preceeds $\mathscr{P}_1$ at each transition.

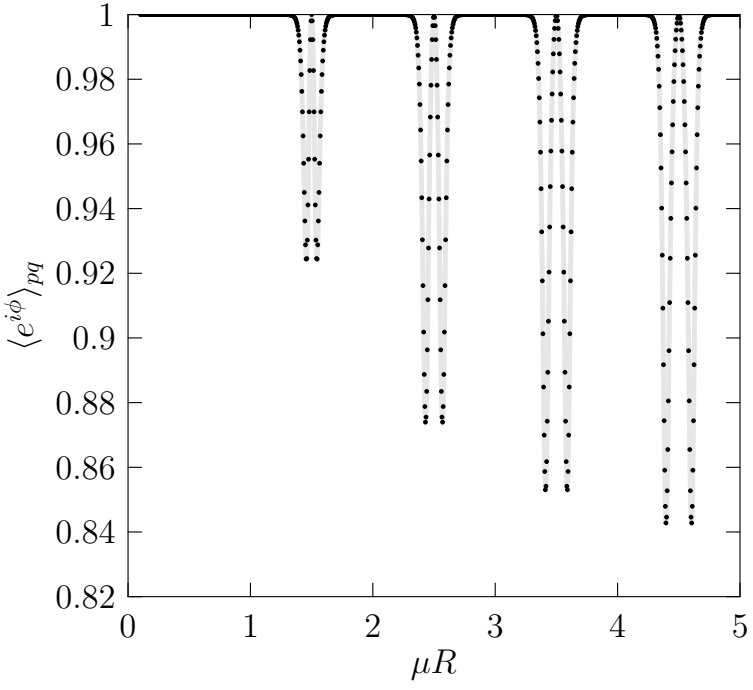
Polyakov lines: $\mathscr{P}_1$ and $\mathscr{P}_{-1}$ with increasing μ





- As μ increases the peaks of $\mathscr{P}_1$ and $\mathscr{P}_{-1}$ get wider indicating that the regions of deconfinement become larger with increasing μ.

Average phase $\langle e^{i\phi} \rangle_{pq}$ for $m = 0$



$$\langle e^{i\phi} \rangle_{pq} \equiv \frac{Z}{\int [\mathrm{d}\theta] e^{Re[-S]}},$$

where the denominator is the "phase quenched" (real action) partition function:

$$Z_{pq} = \int [\mathrm{d}\theta] \left| e^{-S} \right| = \int [\mathrm{d}\theta] e^{Re[-S]}$$

- The average phase shows where the sign problem is severe.
- $\langle e^{i\phi} \rangle_{pq}$ is smallest (largest) when $|\mathscr{P}_1 - \mathscr{P}_{-1}|$ is largest (smallest).

Continuum results (large mR)

Since all of our observables are a function only of β/R, mR, or μR, then we can obtain a continuum limit by taking:

- β/R small (high T perturbation theory),
- μR large (high density perturbation theory),
- mR large (heavy quarks).

We take mR large. Then, in the vicinity of $\mu = m$:

$$z_f(n\beta/R, mR) = 2\sum_{l=0}^{\infty} l(l+1)e^{-n\beta\sqrt{(l+1/2)^2R^{-2}+m^2}}$$

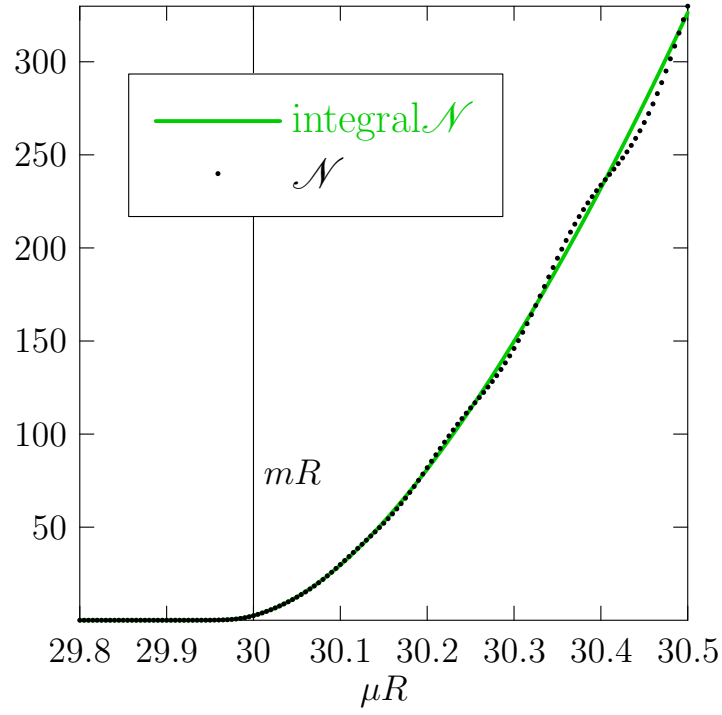
$$= 2\int_0^{\infty} \mathrm{d}y \left(y^2 - \frac{1}{4}\right) e^{-\frac{n\beta}{R}\sqrt{y^2+m^2R^2}}$$

$$+ 4\int_{mR}^{\infty} dy \frac{y^2+\frac{1}{4}}{e^{2\pi y}+1} \sin\left(\frac{n\beta}{R}\sqrt{y^2-m^2R^2}\right)$$

$$\xrightarrow[mR\to\infty]{} 2\int_0^{\infty} \mathrm{d}y \left(y^2 - \frac{1}{4}\right) e^{-n(\beta/R)\sqrt{y^2+m^2R^2}}$$

$\mathcal{N}$ for $m \to \infty$

- For non-zero quark mass the expectation value $\mathcal{N}$ exhibits "Silver Blaze" behavior: Bulk observables are zero until onset.
- Onset occurs at the mass of the lightest particle $\mu \simeq m$.



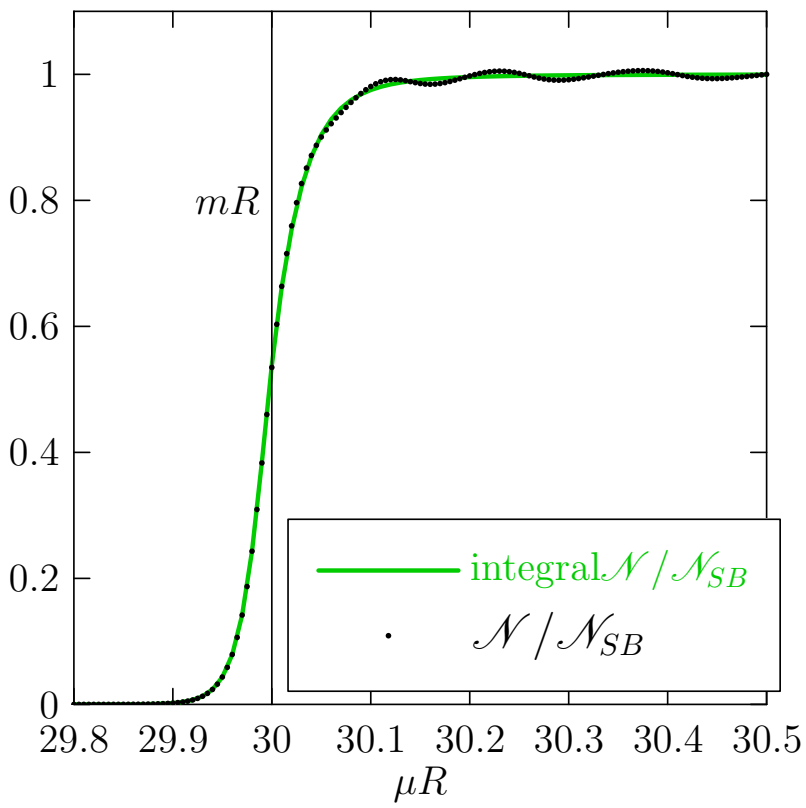
$$\mathcal{N} \xrightarrow[\beta\to\infty]{} \frac{2N_f}{Z}\int [\mathrm{d}\theta]\, e^{-S}\int_0^\infty \mathrm{d}y(y^2-1/4)\sum_{i=1}^{N}\left[\frac{e^{\beta\mu}}{e^{\beta\mu}+e^{-i\theta_i+(\beta/R)\sqrt{y^2+m^2R^2}}}\right]$$

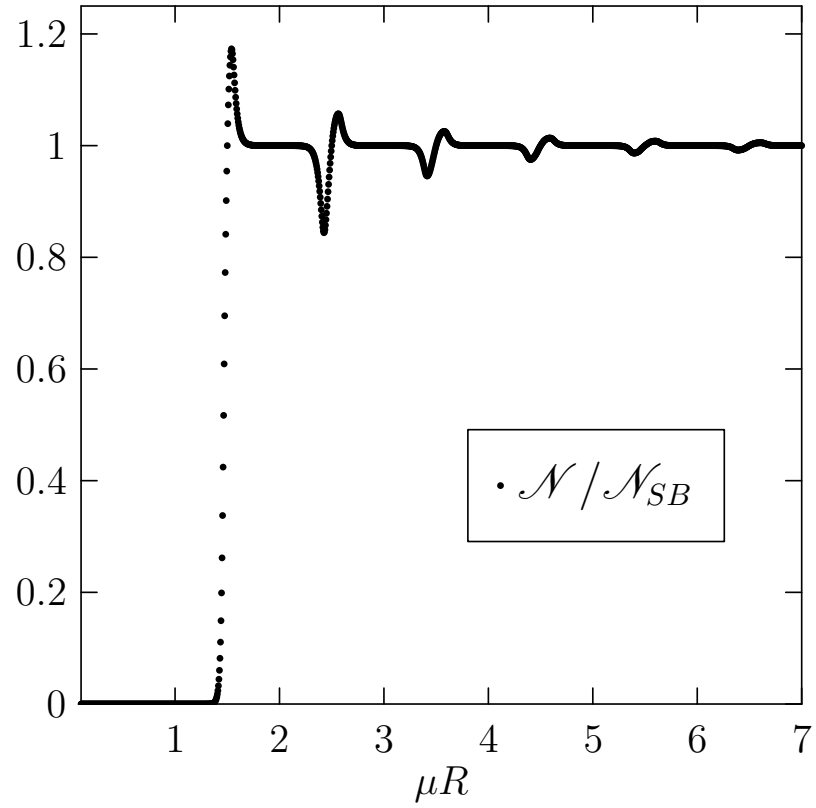
Each level L has:

$$\text{height}: \quad h_L = NN_f\sum_{l=1}^{L} 2l(l+1) \to h_y = NN_f\int_0^\infty \mathrm{d}y\, 2(y^2-1/4)$$

$$\text{width}: \quad \Delta(\mu R) \to \left(\sqrt{(y+\mathrm{d}y)^2+m^2R^2}-\sqrt{y^2+m^2R^2}\right) \to 0$$

$\mathcal{N}$, $\mathcal{P}$, E approach the Stefan-Boltzmann limit

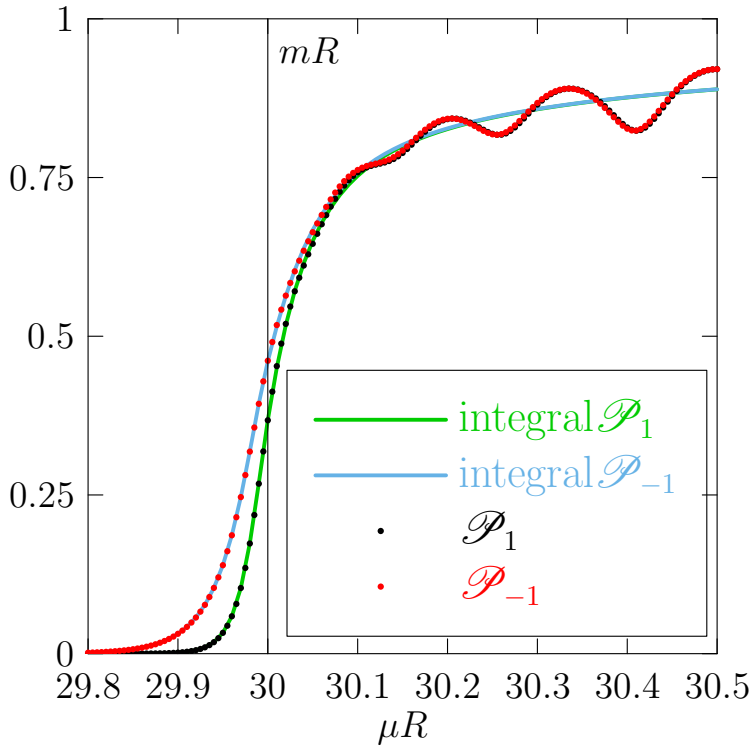




- The Stefan-Boltzmann limit is the zero interaction free fermion limit. On $S^1 \times S^3$ we obtain it from the one-loop result taking all the $\theta_i = 0$, corresponding to the "deconfined" phase, e. g.

$$\mathcal{N}_{SB} \xrightarrow[\beta\to\infty]{} 2NN_f \int_0^\infty \mathrm{d}y(y^2 - 1/4) \left[\frac{e^{\beta\mu}}{e^{\beta\mu} + e^{(\beta/R)\sqrt{y^2+m^2R^2}}} \right]$$

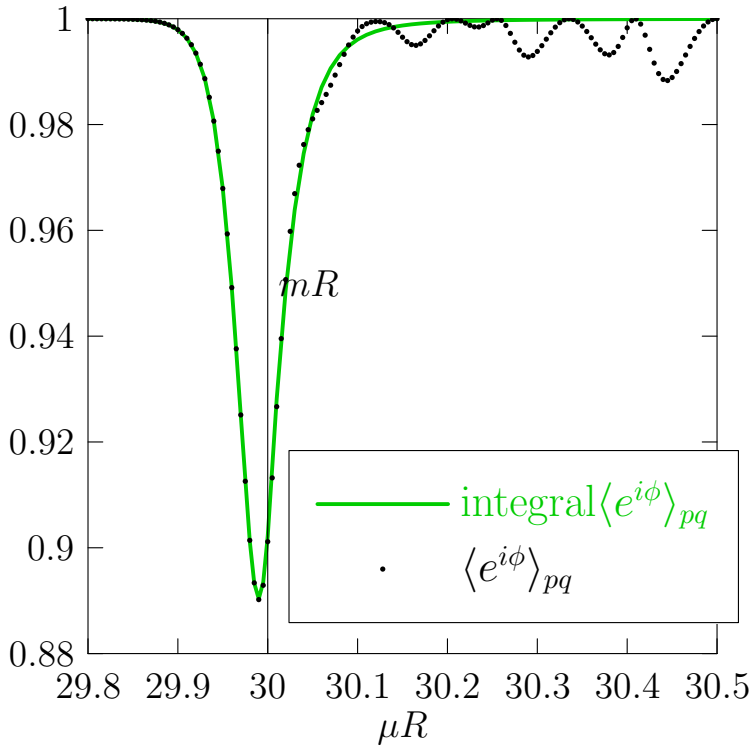
Polyakov liness: $\mathscr{P}_1$ and $\mathscr{P}_{-1}$ for $m \to \infty$



$\mathscr{P}_1$ and $\mathscr{P}_{-1}$ as a function of chemical potential for large quark mass near onset at $\mu R = mR = 30$. $N = 3$, $N_f = 1$ $\beta/R = 30$ (low T).

- At low but non-zero temperatures the confinement-deconfinement oscillations can be delayed by taking $mR \to \infty$.
- The transition in μR occurs around onset at mR and becomes sharper with increasing mR.
- The integral approximation to z_f (curves) breaks down shortly after the onset transition and the oscillations return. The larger we take mR, the farther in μR we can go before breakdown.

Average phase $\langle e^{i\phi} \rangle_{pq}$



- In the limit of large mR, spike in the average phase as a function of μR marks the onset transition. This is followed by a brief respite from large phase fluctuations.
- Again we find that $\langle e^{i\phi} \rangle_{pq}$ is smallest (largest) when $|\mathscr{P}_1 - \mathscr{P}_{-1}|$ is largest (smallest).

Large N theory at low T

In the large N limit the saddle point method is valid and it is possible to solve for several observables analytically. Considering a single level transition and performing the sum over n the action reduces to

$$S(\theta_i) = -\frac{1}{2}\sum_{i,j=1}^{N} \log \sin^2\left(\frac{\theta_i - \theta_j}{2}\right) + N\sum_{i=1}^{N} V(\theta_i)$$

$$V(\theta) = i\mathcal{N}\theta - \sigma \log\left(1 + \xi e^{i\theta}\right)$$

- $\mathcal{N}$ is a Lagrange multiplier necessary to satisfy the $\det P = 1$ constraint: $\sum_{i=1}^{N} \theta_i = 0$.
- $\sigma \equiv \sigma_l \equiv 2l(l+1)\frac{N}{N_f}$
- $\xi \equiv \exp\left(\beta(\mu - \varepsilon)\right)$
- $\varepsilon \equiv \varepsilon_l \equiv \sqrt{m^2 + (l+1/2)^2 R^{-2}}$

Equation of Motion

The saddle point solution is found by solving the equation of motion $\partial S/\partial\theta_i = 0$. This becomes:

$$i\mathcal{N} - \frac{i\sigma\xi e^{i\theta_i}}{1+\xi e^{i\theta_i}} = \frac{1}{N}\sum_{j(\neq i)} \cot\left(\frac{\theta_i - \theta_j}{2}\right)$$

Define the eigenvalues of the Polyakov line: $z_i = e^{i\theta_i}$. Then the equation of motion is

$$\mathcal{N} - \frac{\sigma\xi z_i}{1+\xi z_i} = \frac{1}{N}\sum_{j(\neq i)} \frac{z_i + z_j}{z_i - z_j}$$

From the definition of the Polyakov line:

$$\mathscr{P}_n = \langle \mathrm{Tr} P^n \rangle = \frac{1}{N}\sum_{i=1}^{N} e^{in\theta_i}$$

it is clear that $\mathscr{P}_{-n} \neq \mathscr{P}_n^*$ ($\langle \mathrm{Tr} P^{-n} \rangle \neq \langle \mathrm{Tr} P^n \rangle^*$) for the saddle point solution.

Fermion number

Adding EOMs for all the θ_is we find that the Lagrange multiplier is

$$\mathcal{N} = \frac{1}{N}\sum_i \frac{\sigma\xi z_i}{1+\xi z_i} = \frac{T}{N^2}\frac{\partial \log Z}{\partial \mu}$$

which is the effective fermion number, $\mathcal{N} = \mathscr{N}/N^2$, valid in the large N limit.

Limits:

$$\begin{aligned} &\text{As } \xi \to 0 && \mathcal{N} \to 0, \\ &\text{As } \xi \to \infty && \mathcal{N} \to \sigma. \end{aligned}$$

This is in agreement with the $N = 3$ results for a single level transition.

Small ξ confined phase

Hypothesis: As ξ increases from 0 the eigenvalues are continuously distributed along a closed contour $\mathcal{C}$ in the z-plane.

It is useful to consider a map between the theory on the unit circle and the theory in the complex z-plane of the Polyakov line eigenvalues. To this end

$$\frac{1}{N}\sum_i \longrightarrow \int_{-\pi}^{\pi}\frac{ds}{2\pi} = \oint_{\mathcal{C}}\frac{dz}{2\pi i}\varrho(z)\ ,$$

The contour is given by the inverse map $z(s)$, which can be obtained by solving the differential equation

$$i\frac{ds}{dz} = \varrho(z)$$

subject to the initial condition $z = e^{is}$ when $\xi = 0$.

Constraints

The distribution must satisfy the normalization condition

$$\oint_{\mathcal{C}} \frac{dz}{2\pi i} \varrho(z) = 1$$

and the $\det P = 1$ constraint

$$\int_{\mathcal{C}} \frac{dz}{2\pi i} \varrho(z) \log z = 0$$

EOM for eigenvalues on a closed contour

Using

$$\frac{1}{N}\sum_i \longrightarrow \int_{-\pi}^{\pi}\frac{ds}{2\pi} = \oint_{\mathcal{C}}\frac{dz}{2\pi i}\varrho(z)$$

we convert the EOM to an integral form

$$zV'(z) = \mathfrak{P}\oint_{\mathcal{C}}\frac{dz'}{2\pi i}\,\varrho(z')\frac{z+z'}{z-z'}\ , \qquad zV'(z) = \mathcal{N} - \frac{\sigma\xi z}{1+\xi z}\ .$$

where $\mathfrak{P}$ indicates principal value and the integral over the closed contour allows for evaluation of the right-hand side using Cauchy's theorem.

Distribution $\varrho(z)$ for eigenvalues on a closed contour $\mathcal{C}$

We start from the distribution $\varrho(z)$ with $z \equiv re^{i\phi}$ in the form of delta-functions:

$$\begin{aligned}\frac{1}{N}\sum_{j=1}^{N} &= \int \mathrm{d}r\mathrm{d}\phi \frac{1}{N}\sum_{j=1}^{N}\delta(r-r_j)\delta(\phi-\phi_j) \\ &= \oint \frac{\mathrm{d}z}{iz}\frac{1}{N}\sum_{j=1}^{N}\delta(\phi-\phi_j - i\log(r/r_j))\end{aligned}$$

so

$$\varrho(z) = \frac{2\pi}{zN}\sum_{j=1}^{N}\delta(\phi-\phi_j - i\log(r/r_j))$$

then we can solve the EOM assuming all poles in $\mathcal{C}$ to obtain the form

$$\varrho(z) = \frac{c_1}{c_2 z} - \frac{\mathcal{N}}{c_2 z} + \frac{\sigma\xi/c_2}{1+\xi z}$$

Distribution $\varrho(z)$ in the small ξ confined phase

The small ξ confined phase with the pole $-\xi^{-1}$ outside. The EOM and the normalization condition give

$$\varrho(z) = \frac{1}{z} + \frac{\sigma\xi}{1+\xi z} \ .$$

with $\mathcal{N} = 0$ as expected.

Solving the differential equation $i\frac{ds}{dz} = \varrho(z)$ leads to

$$e^{is} = z(1+\xi z)^{\sigma}$$

which we invert to get the contour $z(s)$. The Polyakov lines are

$$\mathscr{P}_1 = \int_{\mathcal{C}} \frac{dz}{2\pi i}\varrho(z)z = 0 \ , \qquad \mathscr{P}_{-1} = \int_{\mathcal{C}} \frac{dz}{2\pi i}\varrho(z)\frac{1}{z} = \sigma\xi \ .$$

where $\mathscr{P}_{-1} \neq \mathscr{P}_1^*$ as advertised.

Extent of the small ξ confined phase

As ξ is increased the condition that the pole $-\xi^{-1}$ lies outside $\mathcal{C}$ must break down. Indeed, as ξ is increased there comes a point when $\varrho(z)$ vanishes, $z = -\frac{1}{\xi(1+\sigma)}$. This happens when

$$\xi = \xi_1 = \frac{\sigma^\sigma}{(1+\sigma)^{1+\sigma}}$$

and a gap opens up on the negative z-axis signaling a phase transition as in the matrix model of Gross and Witten [Phys. Rev. D **21** (1980) 446].

In terms of μ and ε the line of transitions in the $(\mu,\ T)$ plane is

$$\mu = \varepsilon - T\big[(1+\sigma)\log(1+\sigma) - \sigma\log\sigma\big]$$

valid in the low T limit.

Large ξ confined phase

The vanishing of the potential term in the action in the large ξ limit requires that the contour closes here too. The analysis is similar to that of the smal ξ confined phase and we find that

$$\varrho(z) = \frac{1+\sigma+\xi z}{z(1+\xi z)} \ ,$$

from the requirement that $-\xi^{-1}$ lies inside $\mathcal{C}$. This gives $\mathcal{N} = \sigma$ as expected and the level is occupied.

The Polyakov line expectation values are

$$\mathscr{P}_1 = \frac{\sigma}{\xi} \ , \qquad \mathscr{P}_{-1} = 0 \ .$$

where comparing with the small ξ confined phase $\mathscr{P}_{\pm 1}$ swaps over along with the replacement $\xi \to \xi^{-1}$.

Extent of the large ξ confined phase

As ξ is decreased the condition that the pole $-\xi^{-1}$ lies inside $\mathcal{C}$ must break down. Indeed, as ξ is decreased there comes a point when $\varrho(z)$ vanishes, $z = -\frac{1+\sigma}{\xi}$. This happens when

$$\xi = \xi_2 = \frac{(1+\sigma)^{1+\sigma}}{\sigma^\sigma} \ .$$

and a gap opens up again on the negative z-axis.

In terms of μ and ε the line of transitions in the $(\mu,\ T)$ plane is

$$\mu = \varepsilon + T\big[(1+\sigma)\log(1+\sigma) - \sigma\log\sigma\big]$$

valid in the low T limit.

The deconfined (open) phase: $\xi_1 \leq \xi \leq \xi_2$

In the deconfined phase the distribution has a gap and the eigenvalues lie on an arc $\mathcal{C}$ in the complex z-plane defined by a square root branch cut with endpoints $\tilde{z}$ and $\tilde{z}^*$. To solve for the case where the contour is open it is necessary to use the resolvent / spectral curve method.
In anaolgy with the Gross-Witten model the resolvent is

$$\omega(z) = -\frac{1}{N}\sum_j \frac{z+z_j}{z-z_j} = -\int_{\mathcal{C}} \frac{dz'}{2\pi i}\varrho(z')\frac{z+z'}{z-z'}$$

where it is clear that

$$\lim_{|z|\to 0} \omega(z) = 1 \ , \qquad \lim_{|z|\to\infty} \omega(z) = -1 \ .$$

We take the resolvent to be everywhere continuous except over the branch cut. Then from the Plemelj formulae the EOM is

$$zV'(z) = -\frac{1}{2}\left[\omega(z+\epsilon) + \omega(z-\epsilon)\right] \ , \qquad z \in \mathcal{C}$$

The distribution of the eigenvalues in the deconfined phase

The spectral density of eigenvalues is obtained from

$$z\varrho(z) = \frac{1}{2}\left[\omega(z+\epsilon) - \omega(z-\epsilon)\right] \ , \qquad z \in \mathcal{C} \ .$$

which implies that we can solve for various observables using an average of the form

$$\int_{\mathcal{C}} \frac{dz}{2\pi i}\varrho(z)F(z) = \oint_{\tilde{\mathcal{C}}} \frac{dz}{4\pi i z}\omega(z)F(z)$$

Following the technique of Wadia [EFI-79/44-Chicago] we solve the EOM for the resolvent and density of eigenvalues

$$\omega(z) = -zV'(z) + f(z)\sqrt{(z-\tilde{z})(z-\tilde{z}^*)} \ , \quad z\varrho(z) = f(z)\sqrt{(z-\tilde{z})(z-\tilde{z}^*)} \ .$$

where

$$f(z) = \frac{\sigma}{(1+\xi z)\left|\frac{1}{\xi}+\tilde{z}\right|} ,$$

$$\tilde{z} = \frac{-1}{\xi\left(1+\sigma-\mathcal{N}\right)^2}\left[\mathcal{N}^2 + 1 + \sigma - \mathcal{N}\sigma + 2i\sqrt{\mathcal{N}\left(\sigma-\mathcal{N}\right)\left(1+\sigma\right)}\right] .$$

Fermion number and Polyakov lines

We impose the $SU(N)$ condition

$$\oint_{\tilde{\mathcal{C}}} \frac{dz}{4\pi i z}\, \omega(z) \log z = 0 \ .$$

to obtain the effective fermion number $\mathcal{N}$ from

$$\xi = \frac{(\sigma - \mathcal{N})^{\sigma-\mathcal{N}} (1+\mathcal{N})^{1+\mathcal{N}}}{\mathcal{N}^{\mathcal{N}} (1+\sigma-\mathcal{N})^{1+\sigma-\mathcal{N}}} \ .$$

The Polyakov lines are obtained from an expansion of the resolvent

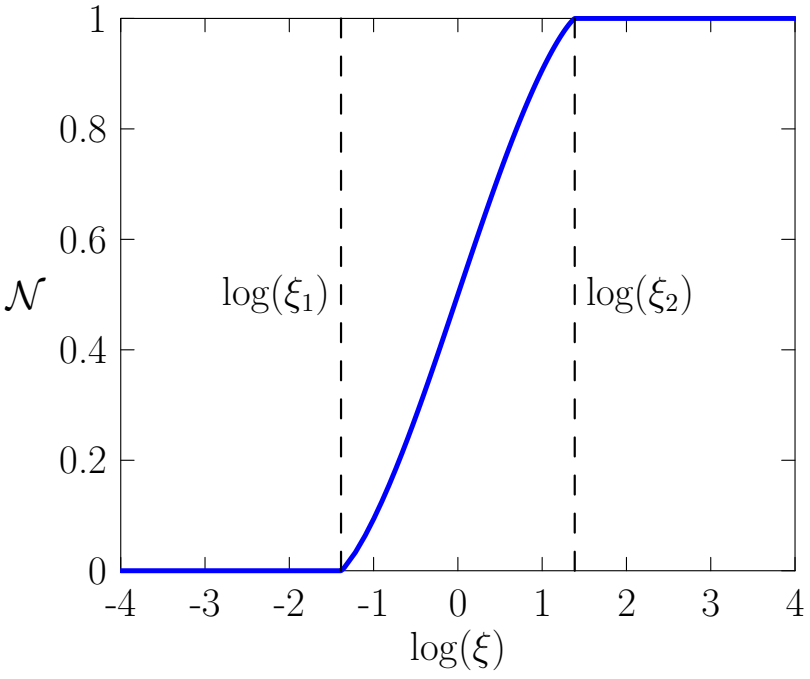
$$\omega(z) = -1 - 2\sum_{n=1}^{\infty} \frac{1}{z^n} \mathscr{P}_n$$

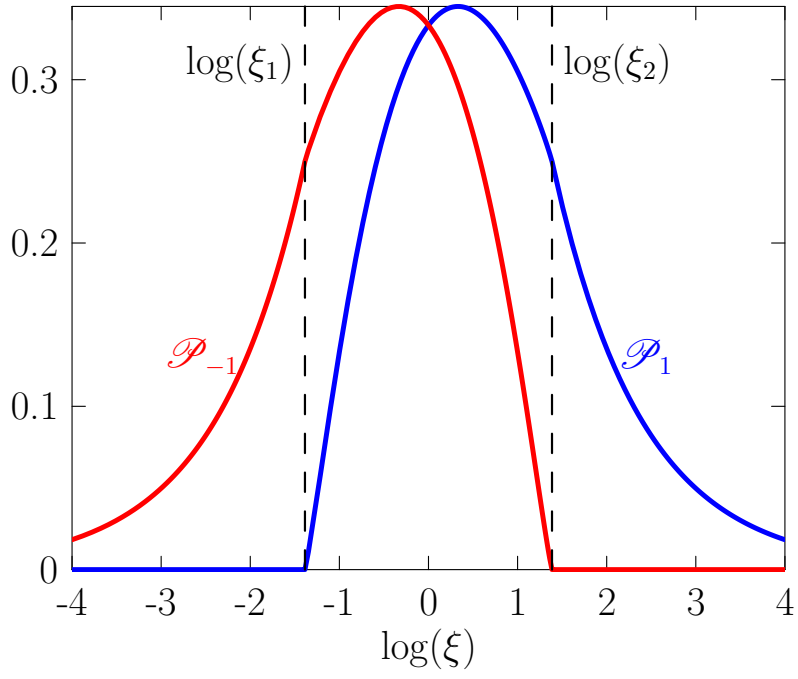
$$\omega(z) = 1 + 2\sum_{n=1}^{\infty} z^n \mathscr{P}_{-n}$$

For a single winding

$$\mathscr{P}_1 = \frac{\mathcal{N}}{\sigma+1-\mathcal{N}} \frac{1}{\xi} \ , \qquad \mathscr{P}_{-1} = \frac{\sigma-\mathcal{N}}{1+\mathcal{N}} \xi$$

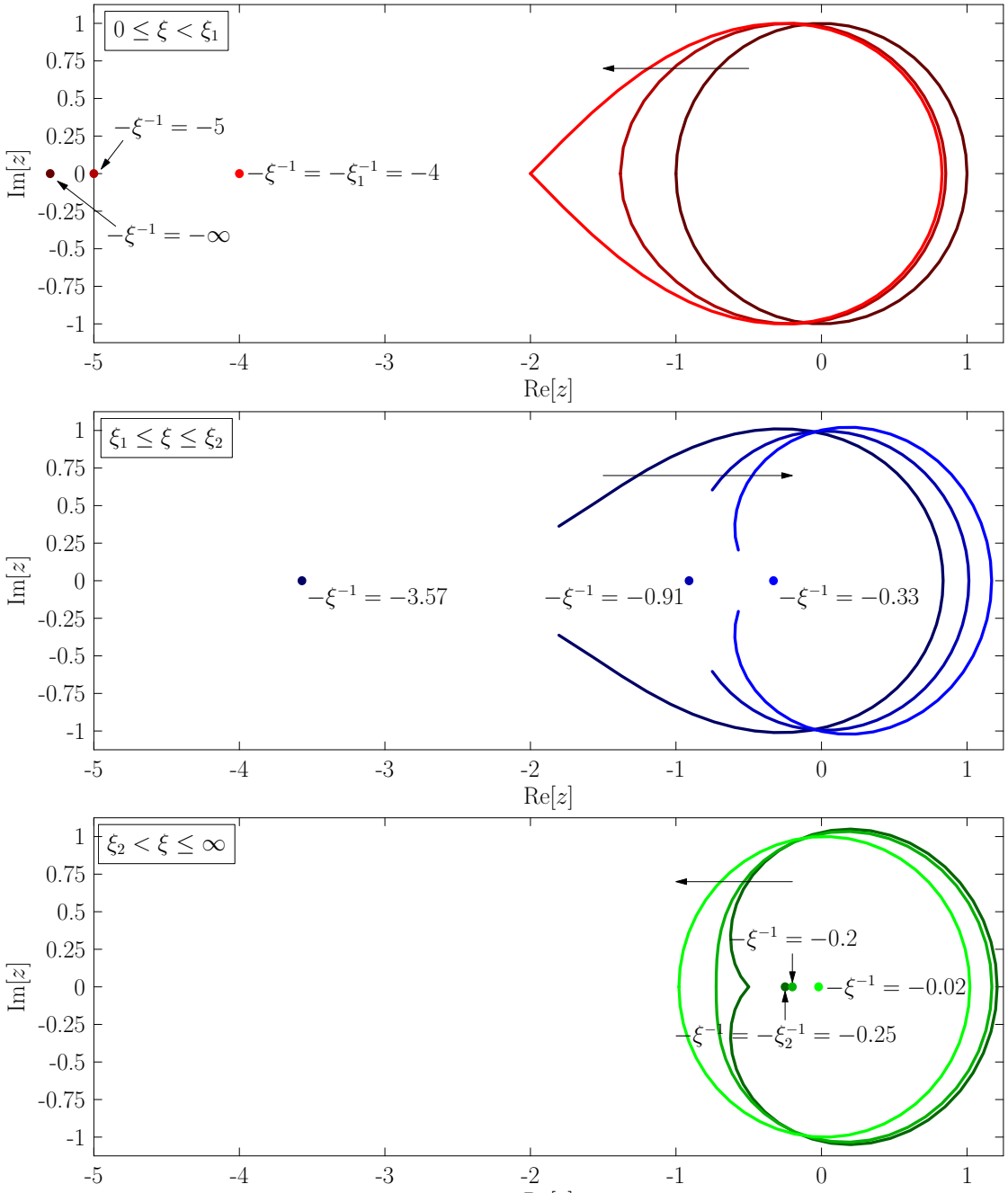
Large N theory at low T





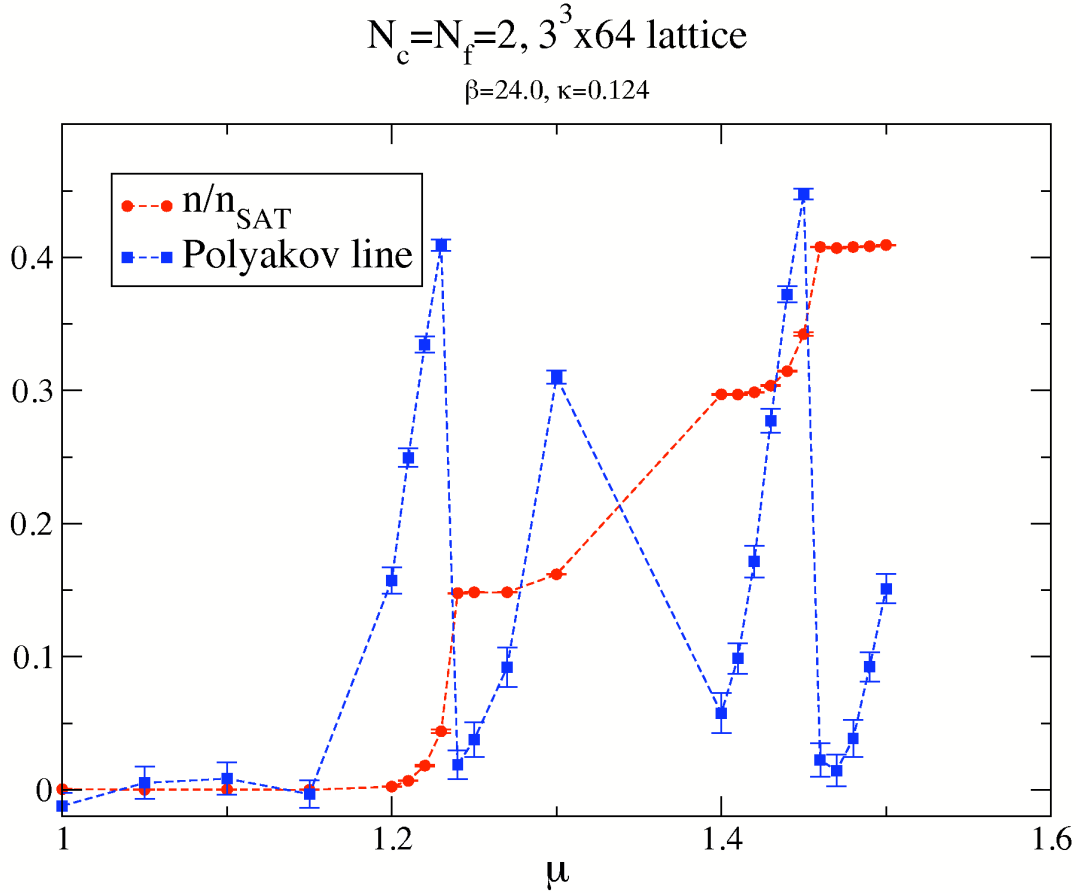
- The discontinuities in the effective fermion number and the Polyakov lines mark the third-order Gross-Witten transitions.

Distribution in the deconfined phase



- The contour $\mathcal{C}$, which gives the distribution of the eigenvalues of the Polyakov line, showing the transition from the small ξ closed phase (in red), the open phase (in blue) and the large ξ closed phase (green).

Preliminary lattice results from 2-color QCD



Simulation results for $N = 2$ QCD confirm the level structure of the fermion number and the associated spikes in the Polyakov line at each level transition. The curious smooth $\longrightarrow$ sharp feature of the observables at the transitions needs study to determine if it is a result of larger coupling, or perhaps resulting from working on the 4-torus.

Conclusions

- QCD at finite chemical potential on $S^1 \times S^3$ has a complex action which results the stationary solution lying in the configuration space of complexified gauge fields.
- Expectation values for observables can be obtained at finite N by numerically integrating over the gauge fields.
- Observables and the distribution of the gauge field eigenvalues can be calculated analytically in the large N limit using the saddle point method of Gross and Witten generalized to deal with a complex action.
- For small mR, the fermion number as a function of the chemical potential suggests a level-structure where the level transitions correspond to spikes in the Polyakov line.
- For large mR, a continuum limit is obtained and the observables exhibit the "Silver blaze" feature, remaining zero until onset is reached at $\mu = m$. The confinement-deconfinement transitions return for sufficiently large μ.

Outlook

- Add more flavors and look for color-superconducting phases through calculation of observables like $\psi\psi$, $(\bar{\psi}\psi)^2$
- Make a connection with Complex Langevin which is a non-perturbative technique
- Consider higher-loop corrections and go beyond the Gaussian approximation to obtain effects from increased coupling strength
- Formulate a related theory from the gravity side (eg. $\mathcal{N} = 4$ SYM + fundamental flavor branes and chemical potential)