Random matrix theory and the Riemann zeta function

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Given a complex number $s = x + iy$, the Riemann zeta function is defined by the following series

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\ &= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots\end{aligned}$$

By standard convergence tests, this sum is finite provided $\mathrm{Re}(s) = x > 1$. It is known that the function $\zeta(s)$ and its zeros encode information about the prime numbers: $2, 3, 5, 7, 11, \dots$. The possible location of zeros of $\zeta(s)$ (*Riemann hypothesis*) has been described as one of the greatest unsolved problems of pure mathematics.

On the other hand, random matrix theory is concerned with matrices, say of size $n \times n$

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \cdots & M_{nn} \end{pmatrix}$$

where the entries M_{ij} are random and taken from a given distribution (for example, normally distributed and independent)[1]. For large n, it has been observed since the 1960s that on a statistical level, the *eigenvalues* of M seem to have a close relationship with *zeros* of $\zeta(s)$.

The aim of this project will be to study some of the interactions between these two subjects. Goals:

- To understand the basics of the Riemann zeta function and the connection to prime numbers.
- Basic principles of random matrix theory: definitions and key notions. Basic calculations.

Relevant modules (in order): Complex Analysis (G5110), fluency in Linear Algebra (G5088) and Probability and Statistics (G5098).

Key words: Random matrix theory, Riemann zeta function, Riemann hypothesis, analytic number theory, complex analysis, normal distribution.

[1] It is usually also required that M is Hermitian: $M_{ij} = \overline{M_{ji}}$.

References

[1] J.P. Keating and N.C. Snaith, *Random matrix theory and* $\zeta(1/2+it)$. Communications in Mathematical Physics, no. 1, 57-89, 2000.

[2] M. L. Mehta, *Random Matrices* Pure and Applied Mathematics Series 142, (3rd ed), Singapore: Elsevier, 2004.