Dimension of attractors in dynamical systems (only MMath and MSc)

We consider a dynamical system given by an Ordinary Differential Equation of the form $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. The long-term behaviour is determined by the (global) attractor, which is a compact, invariant set in $\mathbb{R}^n$ which attracts all bounded sets, and in particular all solutions. The attractor can consist of just one point or a periodic solution, but can also be a more complicated set such as the famous Lorenz attractor. The dimension of the attractor is an important quantity and we seek to compute it numerically in this project.
The (Hausdorff) dimension $\dim_H A$ of the attractor can bounded in the following way: Denote the eigenvalues of the matrix $\frac{1}{2}(Df(x)^T + Df(x))$ by $\lambda_1(x) \geq \ldots \geq \lambda_n(x)$; here, $Df(x)$ denotes the Jacobian of f. If there exists an integer $d \in \{0, \ldots, n\}$ and $s \in [0, 1]$ such that

$$\lambda_1(x) + \ldots + \lambda_d(x) + s\lambda_{d+1}(x) \quad < \quad 0 \tag{1}$$

holds for all $x \in A$, then $\dim_H A < d + s$.
This estimate can be improved by means of a Lyapunov function: let $v \in C^1(\mathbb{R}^n, \mathbb{R})$ be a function such that (1) is replaced by

$$\lambda_1(x) + \ldots + \lambda_d(x) + s\lambda_{d+1}(x) + v'(x) \quad < \quad 0 \tag{2}$$

where $v'(x) = \nabla v(x) \cdot f(x)$ denotes the orbital derivative, then the same conclusion holds.
The goal of the project is to numerically find a function v satisfying (2). We will use mesh-free collocation, in particular with Radial Basis Functions, which has been already employed to find Lyapunov functions.
The project will consist of understanding the theory, developing a numerical method and programming it to apply it to examples in MATLAB.

Keywords: differential equations, attractors, Hausdorff dimension, MATLAB, mesh-free collocation.
Recommended modules: Dynamical systems, Introduction to Mathematical Biology.

References:

- L. Barreira & K. Gelfert, 2011 Dimension estimates in smooth dynamics: a survey of recent results *Ergod. Th. & Dynam. Sys.* **31**, 641–671.
- P. Giesl, 2007 Construction of global Lyapunov functions using radial basis functions. Lecture Notes in Math. **1904**, Springer, Berlin.