Calculation of Contraction Metrics

Stability of an equilibrium in a dynamical system given by an ODE of the form $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, can be studied by a Lyapunov function. Such a function measures in some way the distance of a point to the equilibrium. An alternative method is to consider two solutions with adjacent starting points and study whether the distance between them decreases. If the distance decreases, then one can show that the long-time evolution is the same, and they both converge to an equilibrium point.

The distance can be measured with respect to the Euclidean, or with respect to a general metric, described by a matrix-valued function $M\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$, where $M(x)$ is a symmetric, positive definite matrix. Then $\langle v, w\rangle_x = v^T M(x) w$ defines a point-dependent scalar product for $v, w \in \mathbb{R}^n$. One can derive a partial differential equation for this function of the form

$$M'(x) + Df(x)^T M(x) + M(x) Df(x) = -I \qquad (1)$$

where $M'(x) = \nabla M(x) \cdot f(x)$ denotes the orbital derivative.

We will use a method to solve the matrix-valued partial differential equation (1) using meshfree collocation.

The new element of the project is to apply the same computational method to examples which do not have an asymptotically stable equilibrium, but a saddle point, or an unstable equilibrium, for example. The conjecture is that a metric, satisfying (1), can still be computed, but it will not be positive definite any more. Instead, it will provide information about the stable and unstable manifolds.

The project will include programming in MATLAB.

References:

Paul Glendinning: Stability, instability and chaos: an introduction to the theory of nonlinear differential equations. Cambridge University Press, 1994.

Gregory E. Fasshauer. Meshfree approximation methods with MATLAB. World Scientific 2007.

P. Giesl and H. Wendland: Kernel-based Discretisation for Solving Matrix-Valued PDEs. SIAM J. Numer. Anal. 56 No. 6 (2018), 3386-3406.

P. Giesl and H. Wendland: Construction of a Contraction Metric by Meshless Collocation. accepted at Discrete Contin. Dyn. Syst. Ser. B
P. Giesl: Construction of Finsler-Lyapunov functions with meshless collocation. ZAMM - Journal of Applied Mathematics and Mechanics Z. Angew. Math. Mech. 99 No. 4 (2019), e201800141.

Keywords: differential equations, stability, contraction metric, approximation, MATLAB, meshfree collocation.
Recommended modules: Dynamical systems, Introduction to Mathematical Biology.