MSc Dissertation topics:

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1. Geometric motions

2. Stochastic Differential Equations

Geometric motions and applications I

See Gurtin, 1993 for details.

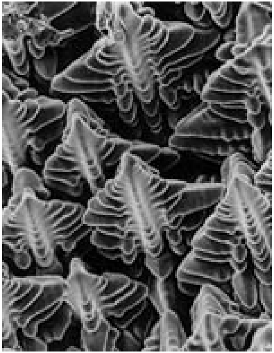
- Curves, surfaces, and more generally manifolds, tradionally viewed as static objects lying in a surrounding space.
- Geometric motions is the branch of maths that makes manifolds move.
- Geometric motions $\Leftrightarrow$ special classes of partial differential equations (PDE's) with loads of differential geometry.
- In this project(s) we view them instead as moving within the surrounding space: immersed manifolds.
- Differential Geometry (at the heart of Geometric Motions) is a mature mathematical theory.
- Geometric Motions is (surprisingly) quite young: picked-up in the late seventies of the past century.
- Recent work of G. Perelman (solving the Poincaré conjecture) has made geometric motions even more trendy.

Applications

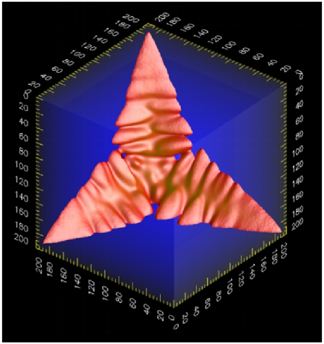
Geometric motions play in applications which range from phase transition to cyrstal growth (see piccie) and from fluid dynamics, image processing and cell mobility.

Dendrites

A real-life dendrite lab picture:



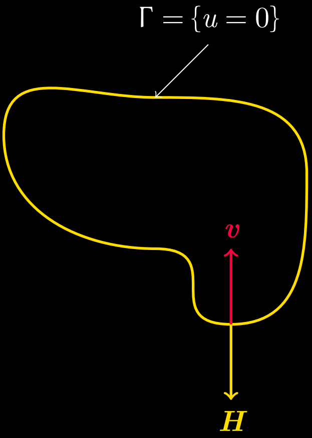
A phase-field simulation:



Mean Curvature Flow

- Manifold (curve in picture) Γ.
- Velocity (speed) at each (space-time) point of Γ is $\boldsymbol{v}(\boldsymbol{x}, t)$
- Curvature (average of principal curvatures) $\boldsymbol{H}(\boldsymbol{x}, t)$.
- Mean curvature flow obeys

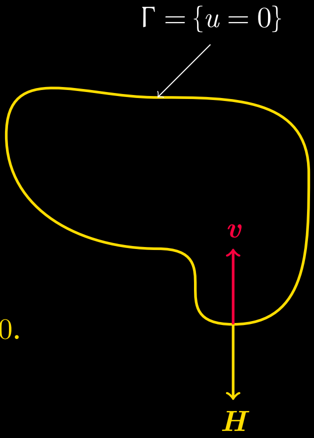
$$\boldsymbol{v}(\boldsymbol{x}, t) = -\boldsymbol{H}(\boldsymbol{x}, t).$$



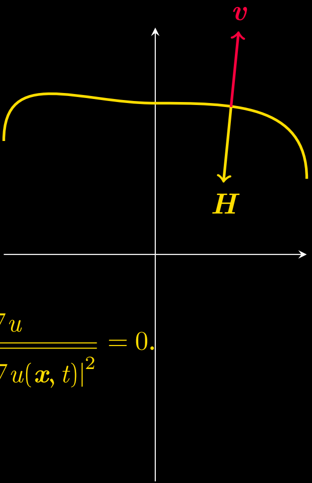
Level set method

- $\Gamma(t) = \{\boldsymbol{x} \in \mathbb{R}^{d+1} : \phi(\boldsymbol{x}, t) = 0\}$
- ϕ is a function defined for all $\boldsymbol{x} \in \mathbb{R}^{d+1}$.
- ϕ satisfies partial differential equation

$$\frac{\partial_t \phi(\boldsymbol{x}, t)}{|\nabla \phi(\boldsymbol{x}, t)|} - \operatorname{div} \frac{\nabla \phi(\boldsymbol{x}, t)}{|\nabla \phi(\boldsymbol{x}, t)|} = 0.$$



Nonparametric/Graph method



- Only sections of the curve can be represented.
- Loads of “standard” analysis can be used.
- Γ =graph of function u.
- u satisfies

$$\frac{\partial_t u(\boldsymbol{x}, t)}{\sqrt{1 + |\nabla u(\boldsymbol{x}, t)|^2}} - \operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u(\boldsymbol{x}, t)|^2}} = 0.$$

Computations

One way of peforming this extension would be to implement computer code simulating geometric motions and analysing the algorithms.

- Finite element method.
- Matlab.
- C/C++.

Requirements

- Love for geometry and analysis (e.g., Bethuel et al., 1999; Spivak, 1979; Struwe, 1996
- Interest for "physically useful" mathematics.
- Technically, you need some PDE and (if you want to compute as well) Numerical Analysis.

Recommended courses

- Numerical Solution of Partial Differential Equations,
- Differential Geometry,
- Measure and Integration,
- Numerical Linear Algebra,
- Advanced Numerical Analysis,
- Introduction to Mathematical Biology,
- Topology courses.

Differential equations (ordinary and stochastic)

(Deterministic) ordinary differential equation (ODE) consists in finding a function $\boldsymbol{u} : [0, T] \subset \mathbb{R} \to \mathbb{R}^n$ such that

$$\frac{d}{dt}\boldsymbol{u}(t) = \boldsymbol{f}(\boldsymbol{u}(t), t) \quad \forall\, t \in [0, T], \tag{ODE}$$

given a function $\boldsymbol{f}\mathbb{R}^n\mathbb{R}^n$ and an initial value

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{IC}$$

A stochastic (ordinary) differential equation (SDE) consists in finding a stochastic process $\boldsymbol{X} : [0, T] \times \Omega \to \mathbb{R}^n$ such that

$$\frac{d}{dt}\boldsymbol{X}(t, \omega) = \boldsymbol{f}(\boldsymbol{X}(t, \omega), t, \omega) \quad \forall\, t \in [0, T], P\text{-a.e. } \omega \in \Omega. \tag{SDE}$$

and

$$\boldsymbol{X}(0) = \boldsymbol{X}_0. \tag{1}$$

References: Higham and Kloeden, 2006; Kloeden and Platen, 1999.

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$$\frac{d}{dt}\boldsymbol{X}(t,\omega) = \boldsymbol{f}(\boldsymbol{X}(t,\omega), t, \omega) \quad \forall\, t \in [0, T], P\text{-a.e. } \omega \in \Omega. \tag{2}$$

- What does $\frac{d}{dt}\boldsymbol{X}(t)$ really mean? (Stochastic processes are generally continuous but not differentiable!)
- How does the "outcome parameter" ω enter the picture.
- Stochastic processes model uncertainty, e.g., white noise: e.g.,

$$\boldsymbol{f}(\boldsymbol{x}, t, \omega)\, dt = \boldsymbol{\alpha}(\boldsymbol{x})\, dt + \boldsymbol{\sigma}(\boldsymbol{x})\, dW(t,\omega). \tag{3}$$

Where W is a Wiener process (not differentiable!). What does it mean?

Answers

How to give mathematical meaning to a stochastic differntial equation (SDE)

The SDE

$$\frac{d}{dt}\boldsymbol{X}(t,\omega) = \boldsymbol{f}(\boldsymbol{X}(t,\omega), t, \omega) \quad \forall\, t \in [0, T], P\text{-a.e. } \omega \in \Omega. \tag{4}$$

can be rewritten in integral form

$$\boldsymbol{X}(t) = \boldsymbol{X}_0 + \int_0^t \boldsymbol{f}(\boldsymbol{X}(s), s)\, ds. \tag{5}$$

For example if

$$\boldsymbol{f}(\boldsymbol{x}, t, \omega)\, dt = \boldsymbol{\alpha}(\boldsymbol{x})\, dt + \boldsymbol{\sigma}(\boldsymbol{x})\, dW(t,\omega) \tag{6}$$

we write the SDE in the integral form

$$\boldsymbol{X}(t) = \boldsymbol{X}_0 + \int_0^t \boldsymbol{\alpha}(\boldsymbol{X}(s))\, ds + \int_0^t \boldsymbol{\sigma}(\boldsymbol{X}(s))\, dW(s). \tag{7}$$

provided we give meaning to the last “integral”.

$$\boldsymbol{X}(t) = \boldsymbol{X}_0 + \int_0^t \boldsymbol{\alpha}(\boldsymbol{X}(s))\, ds + \int_0^t \boldsymbol{\sigma}(\boldsymbol{X}(s))\, dW(s). \tag{8}$$

provided we give meaning to the last “integral”.
This is solved, in theory, by introducing Itô’s calculus, e.g., see Li, 2004.
Beautiful theorey implies Project.
But what how do we “compute” solutions? implies Project.

Why do we care?

About computing stochastic differential equations

Uncertainty arises in many applications:

structural uncertainty risk analysis (e.g., due to weather pattern), climate change, meteorology, stock market,

data uncertainty partial or incomplete measurements (e.g., geology),

Bayesian uncertainty reliability of data collection (e.g., medicine) or the difficulty to find "averages" (e.g., what is the average size of the human heart?)

inherent "noise" in systems astrophysics, neurology, etc.

Other fields connected to SDE's

- Feynman–Kac path integrals in particle physics,
- Harmonic functions and potential theory,
- Brownian motion and diffusion equations (Einstein's paper and Lagevin's equations),
- Discrete stochastic processes,
- Multiscale problems (involving SDE's at some scales), e.g., models of non-Newtonian fluids,
- Statistical mechanics,
- Theory of computability (Computer science).

Recommended courses

Indicative, as usual

- Probability Models,
- Random Processses,
- Advanced Numerical Analysis,
- Mathematical Models in Industry and Finance,
- Partial differential equations,
- Numerical Linear Algebra.

Refrences I

[1] F. Bethuel et al. Calculus of variations and geometric evolution problems. Vol. 1713. Lecture Notes in Mathematics. Lectures given at the 2nd C.I.M.E. Session held in Cetraro, June 15–22, 1996, Edited by S. Hildebrandt and M. Struwe, Fondazione C.I.M.E.. [C.I.M.E. Foundation]. Berlin: Springer-Verlag, 1999, pp. vi+294. ISBN: 3-540-65977-3.

[2] Morton E. Gurtin. Thermomechanics of evolving phase boundaries in the plane. Oxford Mathematical Monographs. New York: The Clarendon Press Oxford University Press, 1993, pp. xi+148. ISBN: 0-19-853694-1.

[3] Desmond Higham and Peter Kloeden. "An introduction to the numerical simulation of stochastic differential equations". Lecture Notes for a Compact Course for Students of the Bavarian Graduate School in Computational Engineering. 2006.

[4] Peter E. Kloeden and Eckhard Platen. Numerical solution of stochastic differential equations. Corrected 3rd Printing. Vol. 23. Applications of Mathematics (New York). Berlin: Springer-Verlag, 1999, pp. xxxvi+636. ISBN: 3-540-54062-8.

[5] Zenghu Li. Stochastic differential equations with applications in finance. 2004. URL: http://math.bnu.edu.cn/~lizh.

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