Asymptotic analysis of integrals and applications

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Consider the sequence $a_n = n!$, where $n = 1, 2, 3, \ldots$ That is, we have $a_1 = 1$, $a_2 = 2 \cdot 1 = 2$, $a_3 = 3 \cdot 2 \cdot 1 = 6$, $a_4 = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ and so on. These numbers seem to grow without bound. But how fast?

The statement

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1 \tag{1.1}$$

gives us information on how fast $n!$ grows. This formula is called *Stirling's approximation* and is one of the most famous examples of what is called *asymptotic approximation*. It tells us, in the most reasonable and explicit terms one can hope for, what is the behaviour of $n!$ when n becomes very large.

A starting point for the proof of (1.1) is to remember the integral

$$n! = \int_0^\infty x^n e^{-x}\, dx = n^{n+1} \int_0^\infty e^{-n\phi(u)}\, du \tag{1.2}$$

where we changed variable $x = un$ and defined $\phi(u) = u - \log(u)$. This already seems to capture the factor n^n in (1.1) but the other terms like $\sqrt{2\pi}$ remain mysterious at this stage.

The aim of this project will be to develop the skills to perform an asymptotic analysis of integrals, including for example ones of the type (1.2). This is a very practical and fundamental skill not often taught at the undergraduate level, despite its appearance in many areas of research in probability, statistics and mathematical physics.

A particular goal could be to apply the methods learnt to some currently active research topics, such as those in random matrix theory. Depending on the progress, the proof of some fundamental results on the eigenvalue distribution of large random matrices could form part of the project.

Relevant modules (in order): Complex Analysis (G5110), Probability and Statistics (G5098), Perturbation th & Calc of variations (840G1).

Key words: asymptotic analysis, Laplace method, Watson's lemma, steepest descent method, stationary phase approximation.

References

[1] Carl M. Bender and Steven A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory: v. 1* Springer, no. 1, 1999.