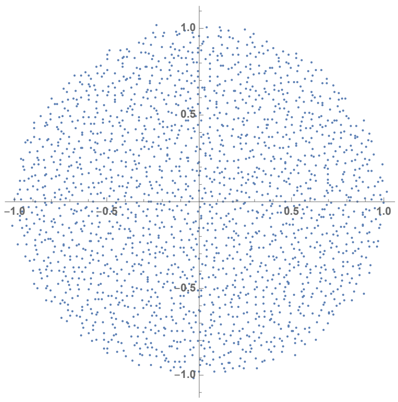
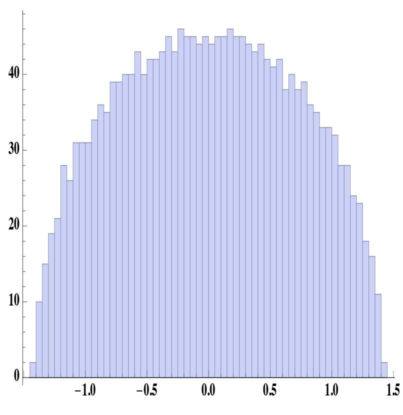
RANDOM MATRICES AND FREE PROBABILITY

PROJECT PROPOSED BY ANTOINE DAHLQVIST

When you sample an $N \times N$ array A of random numbers, say with centred independent entries, that N is quite big, say $N = 2000$, and look at the eigenvalues, once divided by $\sqrt{N}$ you usually get something like this [1]

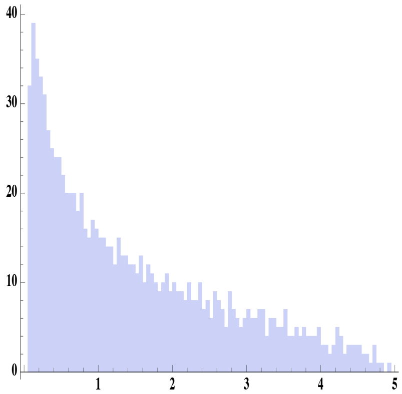


Assume now that your matrix is symmetric, say with independent entries above the diagonal, such that diagonal entries have a variance twice bigger than the one away from the diagonal (as what happens if you symmetrise the previous matrix considering $S = A + A^*$), if you divide again by $\sqrt{N}$ and plot the histogram of the eigenvalues you shall see



[1] If the entries have variance 1.

If now you consider an array A of size $N \times M$, that you form the $N \times N$ matrix[2] AA^* and look at the histogram of the eigenvalues, for $N = 1000$ and $M = 1500$ you shall see after scaling by N



These three samples are manifestations of universality results, alike the central limit theorem in probability, which allow to predict the shapes that appear in the above pictures. In contrast to the classical probability setting, the random variables we are looking at here, random matrices, typically don't commute and are typically free from any algebraic relation.

Project:

(1) A first aim of the project will be to understand that the last two laws can be understood as a central limit theorem and a law of rare events in a framework called *free probability.* Free probability is now a well-established part of mathematics, first introduced by Voiculescu around 1983, where a lot of theorems of probability find their mirror image.

(2) A second less-known aspect is that free probability allows to understand not only eigenvalues of random matrices but also eigenvectors. A second goal of this project will consist in applying this idea to different examples, including perturbations of Wishart matrices.

References

[1] Free probability for probabilists, Philippe Biane, 1998
[2] Free Probability and Random Matrices, James Mingo, Roland Speicher, Springer 2017
[3] Lectures on the combinatorics of Free probability, A. Nica R. Speicher, London Mathematical Society, Lecture Note Series 335.

Related Modules: Probability and Statistics, Probability models, Random processes.

[2]This is called a Wishart matrix. It appears naturally when you want to form the empirical covariance matrix of N quantities that you observe M times.