Nonlocal geometric flows

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In collaboration with
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A classical geometric evolution

Motion by mean curvature: $t \mapsto E_t \subset \mathbb{R}^d$

$$V = H_{\partial E_t} \qquad \text{su } \partial E_t \tag{MCM}$$

V is the normal velocity of ∂E_t

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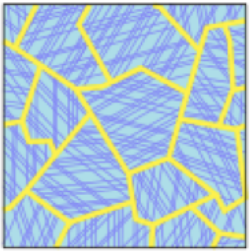
Proposed by Mullins (1956) to describe the evolution of crystalline grains:



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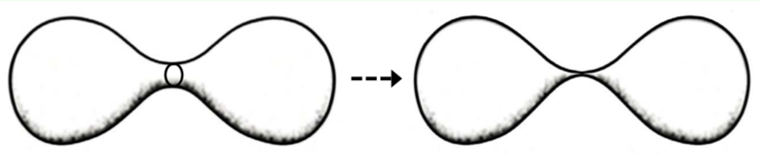

Figure : An example of pinching singularity (Grayson '89).

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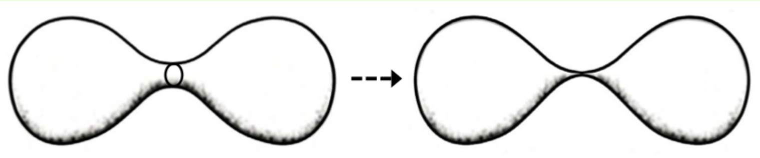

Figure : An example of pinching singularity (Grayson '89).

Question: how to define a global-in-time solution?

The level set approach

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- Global existence and uniqueness for (EqIL) has been established in Evans & Spruck (1991) and Chen-Giga-Goto (1991) within the formalism of viscosity solutions.

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- One defines a discrete-in-time evolution by iteration:

$$E_{n+1} \in \operatorname{argmin} \left(\operatorname{Per}(F) + \frac{1}{\tau} \int_{F \Delta E_{n-1}} d(x, \partial E_{n-1})\, dx \right)$$

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- Chambolle, A.: An algorithm for mean curvature motion. *Interfaces Free Bound.* **6** (2004), 195-218.

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J is submodular $\iff$ $\widetilde{J}$ is convex (Chambolle, Giacomini, Lussardi 2010).

Generalized curvatures

Definition

We say that $\kappa(\cdot, E)$ *is the curvature of* ∂E *associated with* J *if per any smooth family of diffeomorphisms* $(\Phi_\varepsilon)_\varepsilon$, *with* $\Phi_0 = Id$, *one has*

$$\frac{d}{d\varepsilon} J\big(\Phi_\varepsilon(E)\big)_{|_{\varepsilon=0}} = \int_{\partial E} \kappa(x, E)\, X(x) \cdot \nu^E(x) d\mathcal{H}^{N-1}(x).$$

where $X := \frac{\partial \Phi_\varepsilon}{\partial \varepsilon}_{|_{\varepsilon=0}}$ *and* ν^E *is the outer normal to* E.

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- Non degeneracy: $\inf_{\rho>0} \min_{x \in \partial B_\rho} \kappa(x, B_\rho) > -\infty$.

Consequences of convexity

Lemma (Subgradient inequalities)

Let $E \in C^2$ and $x \in \partial E$. Then

$$J(E) - J(E \setminus W_n) \leq |W_n \cap E|(\kappa(x, E) + o(1))$$

and

$$J(E \cup W_n) - J(E) \geq |W_n \setminus E|(\kappa(x, E) + o(1))$$

if $W_n \to \{x\}$ in the Hausdorff sense.

Consequences of convexity

Lemma (Monotonicity)

Let $E, F \in C^2$ with $E \subseteq F$ and let $x \in \partial F \cap \partial E$. Then

$$\kappa(x, F) \leq \kappa(x, E).$$

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- The strong formulation of (GMC) becomes meaningless when singularities appear
- Need for a weak formulation:
  - Representation via super-level sets
  - Viscosity formulation of the corresponding equation

Level set formulation

Representing $E(0) := \{u_0 \geq 0\}$, one is led to the Cauchy problem:

$$\begin{cases} u_t(x,t) + |Du(x,t)|\kappa(x, \{y:\ u(y,t) \geq u(x,t)\}) = 0 \\ u(0,\cdot) = u_0. \end{cases}$$

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- Weak formulation: The curvature κ is defined only on regular sets. We consider viscosity solutions.
- Functional setting: Evolution of sets with compact boundary. Therefore, u and the test functions are constant outside a compact set.

Definition of viscosity solution

Definition

A continuous function $u : \mathbb{R}^d \times [0, T] \to \mathbb{R}$ *is a viscosity subsolution if* $u(0, \cdot) \leq u_0$, *and for any test function* φ *of class* C^2 *s.t.* $u - \varphi$ *has a maximum at* $z := (x, t)$, *one has*

$$\varphi_t(z) + |D\varphi(z)| \, \kappa \left(x, \{y : \varphi(y, t) \geq \varphi(z)\}\right) \leq 0,$$

if the level set $\{\varphi(\cdot, t) = \varphi(z)\}$ *is not critical, and* $\varphi_t(z) \leq 0$ *if* $D\varphi(z) = 0$ *(and* φ *"flat enough" at* z*).*

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- Extension of κ to non-regular sets by semicontinuity.
- Perron's Method extends to this setting

The minimizing movements scheme

For any fixed time step $h > 0$, let $T_h E$ be the minimal solution to

$$\min_{F \subset \mathbb{R}^d} \left\{ J(F) \; + \; \frac{1}{h} \int_{F \triangle E} \mathrm{dist}(x, \partial E) \, dx \right\}$$

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Lemma (Discrete Comparison Principle)

$E \subseteq E' \Longrightarrow T_h E \subseteq T_h E'.$

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- $u \in BUC(\mathbb{R}^d)$, $s > s' \Longrightarrow T_h\{u \geq s\} \subseteq T_h\{u \geq s'\}$.

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- Equicontinuity in space: follows from discrete comparison principle + translation invariance
- Equicontinuity in time: follows from discrete comparison principle + estimate on how fast balls shrink.

The main consistency result

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Theorem (Chambolle-M.-Ponsiglione, ARMA 2015)

The limiting function u is a viscosity solution of

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- Non degeneracy: $\inf_{\rho>0} \min_{x\in\partial B_\rho} \kappa(x, B_\rho) > -\infty$.

Uniqueness

We consider two different hypotheses:

(FO): (*First order curvatures*) Let $\Sigma \in \mathfrak{C}^{1,1}$ and $x \in \partial\Sigma$. Then

$$\sup\left\{\kappa(x,F)\ :\ F \in C^2, F \supseteq \Sigma, x \in \partial F\right\} = \inf\left\{\kappa(x,F)\ :\ F \in C^2, \mathring{F} \subseteq \Sigma, x \in \partial F\right\}.$$

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(UC) (*Uniform C^2-continuity*) Given $r > 0$, there exists ω_r such that:

For every $E \in C^2$, $x \in \partial E$ satisfying a ball condition of radius r at x and for every diffeomorphisms $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ of class C^2

$$|\kappa(x,E) - \kappa(\Phi(x),\Phi(E))| \leq \omega_r(\|\Phi - Id\|_{C^2}).$$

Uniqueness

Theorem (Chambolle-M.-Ponsiglione, ARMA 2015)

Assume that (FO) *or* (UC) *hold. Then*

$$\begin{cases} u_t(x,t) + |Du(x,t)|\kappa(x, \{y : u(y,t) \geq u(x,t)\}) = 0 \\ u(0,\cdot) = u_0. \end{cases}$$

admits a unique viscosity solution.

Examples and applications: the fractional mean curvature flow

- For $\alpha \in (0,1)$ consider the fractional perimeter

$$J^\alpha(E) \ := \int\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{N+\alpha}}\,dxdy = [\chi_E]^2_{H^{\frac{\alpha}{2}}}\,.$$

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Minimizing movements and fractional mean curvature flow

- κ^α is a first order curvature.

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Corollary

The minimizing movements algorithm converges to the unique viscosity solution of

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- Our approximation is new and alternative to Caffarelli&Souganidis ARMA (2010).

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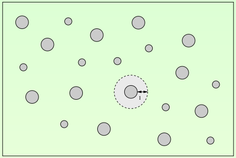
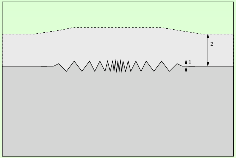


"relaxation" of boundary oscillations



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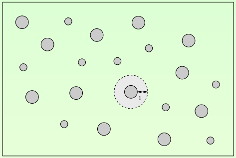

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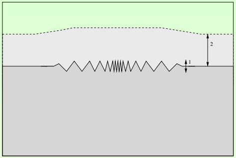

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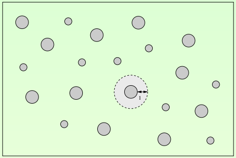

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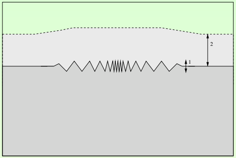

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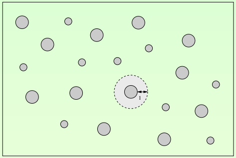

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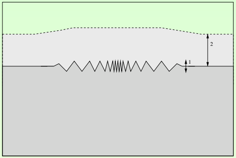

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- the associated curvature satisfies (UC)

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The case $p = 2$ is similar to the Hele-Show type flow studied in

P. Cardaliaguet and O. Ley. Some flows in shape optimization. Arch. Ration. Mech. Anal., **183** (1):21–58, 2007.

The anisotropic mean curvature flow

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- If ϕ is smooth, then (AMC) falls within the previous theory

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If ϕ is non-smooth or crystalline, then the anisotropic mean curvature is not well-defined.

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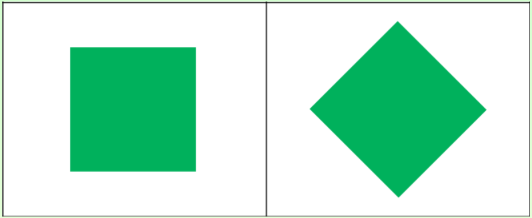
If ϕ is non-smooth or crystalline, then the anisotropic mean curvature is not well-defined. Thus the previous theory does not apply to the crystalline mean curvature flow!

The crystalline case



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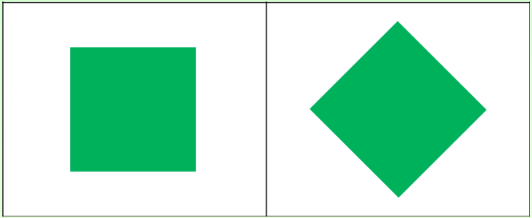

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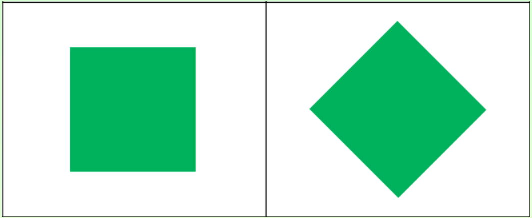

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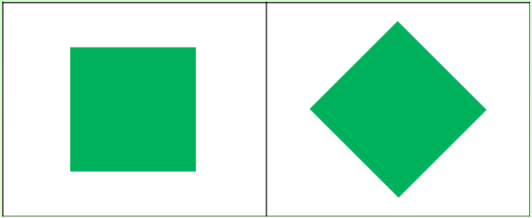

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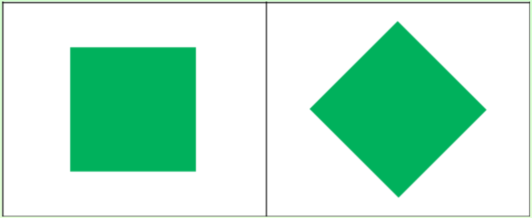

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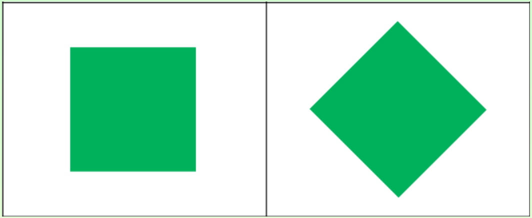

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Latest developments

New result (Chambolle-M.-Ponsiglione 2015)

Let ϕ be any crystalline anisotropy. Then, the crystalline mean curvature equation

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- Our result holds only for the "natural" mobility $m = \phi$.

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- Since the curvatures of the s-level sets of d are non-increasing in s, we have

$$\partial_t d \geq \mathrm{div}(\nabla\phi(\nabla d)) \qquad \text{in } \{d > 0\}. \tag{1}$$

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Let $t \mapsto E(t)$ be a smooth flow and assume ϕ to be smooth.

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$$\partial_t d = \mathrm{div}(\nabla\phi(\nabla d)) \qquad \text{on } \partial E(t) = \partial\{d(\cdot, t) = 0\}.$$

- Since the curvatures of the s-level sets of d are non-increasing in s, we have

$$\partial_t d \geq \mathrm{div}(\nabla\phi(\nabla d)) \qquad \text{in } \{d > 0\}. \tag{1}$$

- Analogously, setting $d^c(\cdot, t) := \mathrm{dist}(\cdot, E^c(t))$, we have

$$\partial_t d^c \geq \mathrm{div}(\nabla\phi(\nabla d^c)) \qquad \text{in } \{d^c > 0\}. \tag{2}$$

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Open Problem: the general mobility case (future investigations).

THE END

THANK YOU!!