Double phase functionals: Regularity, Irregularity, and Calderón-Zygmund estimates

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the talk

- Part 1: Non-autonomous functionals
- Part 2: Irregularity
- Part 3: Main results
- Part 4: Similarities and heuristics
- Part 5: Proofs sketches

Part 1: Non-autonomous functionals

Classical facts

consider variational problems of the type

$$W^{1,1} \ni v \mapsto \int_\Omega f(x, Dv)\, dx \qquad \Omega \subset \mathbb{R}^n$$

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for $p > 1$, and the problem is well settled in $W^{1,p}$
a model example is

$$v \mapsto \int_\Omega c(x)|Dv|^p\, dx$$

Cassical regularity facts (in a model case)

consider the model case

$$v \mapsto \int_\Omega c(x)|Dv|^p \, dx$$

then

- If $c(x)$ is measurable then $u \in C^{0,\gamma}$ for some $\gamma > 0$
- If $c(x)$ is continuous then $u \in C^{0,\gamma}$ for every $\gamma < 1$
- If $c(x)$ is Hölder then $Du \in C^{0,\gamma}$ for some $\gamma > 0$

Non-standard growth conditions

consider now variational problems of the type

$$W^{1,1} \ni v \mapsto \int_\Omega f(x, Dv)\, dx \qquad \Omega \subset \mathbb{R}^n$$

with

$$|z|^p \lesssim f(x,z) \lesssim |z|^q + 1 \qquad \text{and } q > p > 1$$

Pioneers: Marcellini and Zhikov in the 80s
A few

- Marcellini Ann. IHP 86
- Zhikov Izv. Akad. Nauk SSSR 86
- Marcellini-Fonseca JGA 97
- Fonseca-Malý Ann. IHP 97
- Kristensten Proc. Edin. 97 + Calc. Var. 98
- Bouchitté-Fonseca-Malý Ann. Proc. Edin. 98

Lower semicontinuity results

Theorem (Fonseca & Malý Ann. IHP 97)

Let

$$W^{1,1} \ni v \mapsto \int_\Omega f(Dv)\,dx \qquad \Omega \subset \mathbb{R}^n$$

be a quasiconvex functional satisfying assumptions

$$0 \leq f(z) \lesssim |z|^q + 1 \, .$$

Then

$$\int_\Omega f(Dv)\,dx \leq \liminf_k \int_\Omega f(Dv_k)\,dx$$

whenever $\{v_k\} \subset W^{1,q}$ *is such that* $v_k \rightharpoonup v$ *in* $W^{1,p}$ *provided*

$$\frac{q}{p} < \frac{n}{n-1}$$

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$$\frac{q}{p} < \frac{n}{n-1} \quad \left(> \frac{n+1}{n} \text{ by Marcellini Ann. IHP 86} \right)$$

Regularity - A basic condition

$$W^{1,1} \ni v \mapsto \int_\Omega f(Dv)\,dx \qquad \Omega \subset \mathbb{R}^n$$

with

$$|z|^p \lesssim f(z) \lesssim |z|^q + 1 \qquad \text{and } q > p > 1$$

then

$$\frac{q}{p} < 1 + o(n)$$

is a sufficient (Marcellini) and necessary (Giaquinta and Marcellini) condition for regularity

Non-standard growth conditions

Bounded minimisers give better bounds

$$q < p + 1$$

the first example of this result I know is from a paper of Uraltseva & Urdaletova (1984).

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Several results have been obtained in this direction (in particular I mention a recent result of Carozza & Kristensen & Passarelli (ann. IHP 2011), where the bound is $q < p + 2$)

Several people on non-uniformly elliptic operators

- Leon Simon
- Uraltseva & Urdaletova
- Zhikov
- Marcellini
- Hong
- Lieberman
- Fusco-Sbordone
- many, many, many others (including me)

Non-autonomous functionals

I am interested in non-autonomous functionals of the type

$$v \mapsto \int_\Omega f(x, Dv)\, dx$$

new phenomena appear in this situation, and the presence of x *is not any longer a perturbation*

Three functionals of Zhikov

Zhikov introduced, between the 80s and the 90s, the following functionals:

$$v \mapsto \int_\Omega |Dv|^2 w(x)\, dx \qquad w(x) \geq 0$$

$$v \mapsto \int_\Omega |Dv|^{p(x)}\, dx \qquad p(x) \geq 1$$

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx \qquad a(x) \geq 0$$

motivations: modelling of strongly anisotropic materials, Elasticity, Homogenization, Lavrentiev phenomenon etc

Part 3: Irregularity (based on pioneering work of Zhikov)

Irregularity I

Theorem (Esposito-Leonetti-Min. JDE 2004)

For every choice of $n \geq 2$, $\Omega \subset \mathbb{R}^n$ *and of*

$$\varepsilon > 0 \qquad \textit{and} \qquad \alpha \in (0,1)$$

there exists a non-negative function $a(\cdot) \in C^{0,\alpha}$, *a boundary datum* $u_0 \in W^{1,\infty}(B)$ *and exponents* p, q *satisfying*

$$n - \varepsilon < p < n < n + \alpha < q < n + \alpha + \varepsilon$$

such that the solution to the Dirichlet problem

$$\begin{cases} u \mapsto \min\limits_{w} \displaystyle\int_B (|Dv|^p + a(x)|Dv|^q)\,dx \\ \qquad w \in u_0 + W_0^{1,p}(B) \end{cases}$$

does not belong *to* $W^{1,q}_{\mathrm{loc}}(B)$

The example goes via Lavrentiev phenomenon

$$
\begin{aligned}
&\inf_{w \in u_0 + W^{1,p}_0(B)} \int_B (|Dv|^p + a(x)|Dv|^q)\, dx \\
&\qquad < \inf_{w \in u_0 + W^{1,p}_0(B) \cap W^{1,q}_{\mathrm{loc}}(B)} \int_B (|Dv|^p + a(x)|Dv|^q)\, dx
\end{aligned}
$$

Irregularity II

Theorem (Fonseca-Malý-Min. ARMA 2004)

For every choice of $n \geq 2$, $\Omega \subset \mathbb{R}^n$ *and of* $\varepsilon > 0$, $\alpha > 0$, *there exists a non-negative function* $a(\cdot) \in C^{[\alpha]+\{\alpha\}}$, *a boundary datum* $u_0 \in W^{1,\infty}(B)$ *and exponents* p, q *satisfying*

$$n - \varepsilon < p < n < n + \alpha < q < n + \alpha + \varepsilon$$

such that the solution to the Dirichlet problem

$$\begin{cases} u \mapsto \min\limits_{w} \displaystyle\int_B (|Dv|^p + a(x)|Dv|^q)\, dx \\ \qquad\qquad w \in u_0 + W^{1,p}_0(B) \end{cases}$$

has a singular set of essential discontinuity points of Hausdorff dimension larger than $n - p - \varepsilon$

First (old) regularity

Theorem (Esposito-Leonetti-Min. JDE 2004)

Let $u \in W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$ *be a local minimiser of the functional*

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx$$

and assume that

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \qquad \text{and} \qquad \frac{q}{p} < 1 + \frac{\alpha}{n}$$

then

$$u \in W^{1,q}_{\mathrm{loc}}(\Omega)$$

Part 3: Main results, with Maria Colombo (ETH)

Three papers:

- Regularity for double phase variational problems - ARMA 15
- Bounded minimisers of double phase variational integrals - ARMA 15
- Calderón-Zygmund estimates and non-uniformly elliptic operators - JFA 15

Classical Regularity

Theorem (Uraltseva Zap. LOMI 68 - Uhlenbeck Acta Math. 77)

Let $u \in W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$, be a local minimiser of the functional

$$v \mapsto \int_\Omega |Dv|^p \, dx \,, \qquad p > 1 \,.$$

Then

Du is Hölder continuous

Theorem (Colombo-Min. ARMA 15)

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Theorem (Colombo-Min. ARMA 15)

Let $u \in W^{1,p}(\Omega)$ *be a* **bounded** *local minimiser of the functional*

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx$$

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then

Du is Hölder continuous

Notice the the delicate borderline case $q = p + \alpha$ is achieved

Theorem (Iwaniec Studia Math. 83 - DiBenedetto & Manfredi Amer. J. Math. 93)

Let $u \in W^{1,p}$ be a distributional solution to

$$\operatorname{div}\,(|Du|^{p-2}Du) = \operatorname{div}\,(|F|^{p-2}F) \qquad \textit{in } \mathbb{R}^n$$

Then it holds that

$$F \in L^q \Longrightarrow Du \in L^q \qquad p \leq q < \infty$$

Theorem 3

Theorem (Colombo-Min. JFA 15)

Let $u \in W^{1,p}(\Omega)$ *be a distributional solution to*

$$\operatorname{div}\left(|Du|^{p-2}Du + a(x)|Du|^{q-2}Du\right) = \operatorname{div}\left(|F|^{p-2}F + a(x)|F|^{q-2}F\right)$$

and assume that

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \qquad \textit{and} \qquad \frac{q}{p} \leq 1 + \frac{\alpha}{n}$$

then

$$(|F|^p + a(x)|F|^q) \in L^{\gamma}_{\mathrm{loc}} \Longrightarrow (|Du|^p + a(x)|Du|^q) \in L^{\gamma}_{\mathrm{loc}}$$

for every $\gamma \geq 1$

Theorem (Colombo-Min. JFA 15)

*Let $u \in W^{1,p}(\Omega)$ be a **bounded** minimiser of the functional*

$$v \mapsto \int [|Dv|^p + a(x)|Dv|^q - (|F|^{p-2} + a(x)|F|^{q-2})\langle F, Dv\rangle]$$

and assume that

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \qquad \textit{and} \qquad q \leq p + \alpha$$

and

$$\sup_{B_\varrho} \varrho^{p_0} \fint_{B_\varrho} [|F|^p + a(x)|F|^q]\, dx < \infty \qquad \textit{for some} \quad p_0 < p$$

then

$$(|F|^p + a(x)|F|^q) \in L^\gamma_{\mathrm{loc}} \Longrightarrow (|Du|^p + a(x)|Du|^q) \in L^\gamma_{\mathrm{loc}}$$

The general viewpoint

is to consider functionals as

$$v \mapsto \int_\Omega f(x, v, Dv)\, dx$$

where

$$H(x, |z|) \lesssim f(x, u, z) \lesssim H(x, |z|) + 1$$

with

$$H(x, |z|) = |z|^p + a(x)|z|^q$$

being a replacement of

$$|z|^p$$

Heuristic explanations - dependence on α of the bound

The Euler equation of the functional is

$$\operatorname{div} a(x, Du) = \operatorname{div} \left(|Du|^{p-2}Du + (q/p)a(x)|Du|^{q-2}Du\right) = 0$$

then

$$\begin{aligned}\frac{\text{highest eigenvalue of } \partial_z a(x, Du)}{\text{lowest eigenvalue of } \partial_z a(x, Du)} &\approx 1 + a(x)|Du|^{q-p} \\ &\approx 1 + R^{\alpha}|Du|^{q-p}\end{aligned}$$

Heuristic explanation - the bound $q \le p + \alpha$

Consider the usual p-capacity for $p < n$

$$\mathrm{cap}_p(B_r) = \inf \left\{ \int_{\mathbb{R}^n} |Dv|^p \, dx \, : \, f \in W^{1,p}, f \ge 1 \text{ on } B_r \right\}$$

we have

$$\mathrm{cap}_p(B_r) \approx r^{n-p}$$

then consider the weighted capacity

$$\mathrm{cap}_{q,\alpha}(B_r) = \inf \left\{ \int_{\mathbb{R}^n} |x|^\alpha |Dv|^q \, dx \, : \, f \in C_0^\infty(\mathbb{R}^n), f \ge 1 \text{ on } B_r \right\}$$

we then have (the ball is centered at the origin)

$$\mathrm{cap}_{q,\alpha}(B_r) \approx r^{n-q+\alpha}$$

We then ask for

$$\mathrm{cap}_{q,\alpha}(B_r) \lesssim \mathrm{cap}_p(B_r)$$

that is

$$r^{n-q+\alpha} \leq r^{n-p}$$

for r small enough, so that

$$q \leq p + \alpha$$

A parallel with Muckenhoupt weights

A maximal theorem holds

$$\int_\Omega [H(x, |M(f)|)]^t \, dx \lesssim \int_\Omega [H(x, |f|)]^t \, dx$$

where Mf is the usual (localised) Hardy-Littlewood maximal operator, together with a Sobolev-Poincaré type inequality

$$\left(\fint_{B_R} \left[H\left(x, \left| \frac{f - (f)_{B_R}}{R} \right| \right) \right]^d dx \right)^{1/d} \leq c \fint_{B_R} [H(x, |Df|)] dx$$

for $d > 1$

A parallel with Muckenhoupt weights

A non-negative function $w \in L^t$ is said to be of class A_t if

$$\sup_{B_R} \left(-\!\!\!\!\!\!\int_{B_R} |w| \, dx \right) \left(-\!\!\!\!\!\!\int_{B_R} |w|^{1/(1-t)} \, dx \right)^{1/(p-1)} < \infty$$

then it follows

$$\int_\Omega |M(f)|^p w(x) \, dx \lesssim \int_\Omega |f|^w (x) \, dx$$

holds for $t > 1$ and

$$\left(-\!\!\!\!\!\!\int_{B_R} \left[H\left(x, \left|\frac{f-(f)_{B_R}}{R}\right|\right)\right]^d dx \right)^{1/d} \leq c -\!\!\!\!\!\!\int_{B_R} H(x, |Df|) dx$$

holds for $d > 1$

Questions

- Study more general conditions for which such abstract results hold in connection to regularity theorems, for instance
- Define the quantity

$$\begin{aligned}&\mathrm{cap}_H(B_r)\\ &\qquad = \inf\left\{\int_{\mathbb{R}^n} H(x, Dv)\,dx \,:\, f \in C_0^\infty(\mathbb{R}^n), f \geq 1 \text{ on } B_r\right\}\end{aligned}$$

  and prove it is a capacity in the usual sense when $q \leq p + \alpha$; also consider the condition $q/p < 1 + \alpha/n$
- Consider removability of singularities problems using this capacity, and in connection obstacle problems

A parallel with Muckenhoupt weights

Minima of functionals of the type

$$v \to \int f(x, v, Dv)\, dx$$

with

$$f(x, v, z) \approx |z|^p w(x) \equiv H(x, |z|)$$

are locally Hölder continuous provided
Fabes-König-Serapioni (Comm. PDE 1982) - Modica (Ann. Mat. Pura Appl. 1985)

Questions

- Study more general conditions for which such abstract results hold in connection to regularity theorems, for instance
- Define the quantity

$$\begin{aligned} &\mathrm{cap}_H(B_r) \\ &\quad = \inf\left\{ \int_{\mathbb{R}^n} H(x, Dv)\, dx \,:\, f \in C_0^\infty(\mathbb{R}^n), f \geq 1 \text{ on } B_r \right\} \end{aligned}$$

  and prove it is a capacity in the usual sense when $q \leq p + \alpha$; also consider the condition $q/p < 1 + \alpha/n$
- Consider removability of singularities problems using this capacity, and in connection obstacle problems
- Consider weights with respect to this new norm

Part 5: Proofs sketches

The proofs: Separation of phases and universal threshold

There exists a universal threshold $M \equiv M(n, p, q, \alpha)$ such that if on the ball B_R

$$a_i(R) := \inf_{x \in B_R} a(x) \leq M[a]_{0,\alpha} R^\alpha$$

Then our functional is essentially equivalent to

$$v \mapsto \int_{B_R} |Dv|^p \, dx$$

there exists a universal threshold $M \equiv M(n, p, q, \alpha)$ such that if on the ball B_R

$$a_i(R) := \inf_{x \in B_R} a(x) > M[a]_{0,\alpha} R^\alpha$$

then our functional is essentially equivalent to

$$v \mapsto \int_{B_R} (|Dv|^p + a_i(R)|Dv|^q)\, dx$$

Implementation of this is very delicate and goes though a delicate analysis involving an exit time argument

Tool 1: Reverse Hölder inequality

Lemma

Let $u \in W^{1,p}(\Omega)$ be a local minimiser of the functional

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx$$

and let B_R be a ball such that

$$\inf_{x \in B_R} a(x) \leq M[a]_\alpha R^\alpha \qquad \text{and} \qquad \frac{q}{p} < 1 + \frac{\alpha}{n}$$

hold. then there exists a positive constant $c \equiv c(M)$ such that

$$\left(\fint_{B_{R/2}} |Du|^{2q-p}\, dx \right)^{1/(2q-p)} \leq c \left(\fint_{B_R} |Du|^p\, dx \right)^{1/p}$$

Tool 2: Caccioppoli type inequality

Lemma

Let $u \in W^{1,p}(\Omega)$ *be a* **bounded** *local minimiser of the functional*

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx$$

and let B_R *be a ball such that*

$$\inf_{x \in B_R} a(x) \leq M[a]_\alpha R^\alpha \qquad \textit{and} \qquad q \leq p + \alpha$$

hold. then there exists a positive constant $c \equiv c(M)$ *such that*

$$\fint_{B_{R/2}} |Du|^p\, dx \leq c \fint_{B_R} \left| \frac{u - (u)_{B_R}}{R} \right|^p dx$$

Theorem on bounded minimisers

Theorem (Colombo-Min. ARMA 15)

Let $u \in W^{1,p}(\Omega)$ *be a* **bounded** *local minimiser of the functional*

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx$$

and assume that

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \qquad \textit{and} \qquad q \leq p + \alpha$$

then

Du is Hölder continuous

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and assume that

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then

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A parabolic theorem is on its way

Proof goes in ten different Steps

- Step 1: Low Hölder continuity (to treat the borderline case $q = p + \alpha$)
- Step 2: p-harmonic approximation to handle the p-phase
- Step 3: Decay estimate on all scales in the (p, q)-phase
- Step 4: Exit time argument implies $u \in C^{0,\gamma}$ for every $\gamma < 1$
- Step 5: Previous Step implies that Du is in every Morrey space
- Step 6: Morrey space regularity of the gradient implies absence of Lavrentiev phenomenon
- Step 7: Gradient fractional Sobolev regularity
- Step 8: Upgraded Caccioppoli inequality via interpolation inequalities in fractional Sobolev spaces
- Step 9: Higher integrability of the gradient implies a better p-harmonic approximation in the p-phase
- Step 10: Hölder gradient continuity via weighted separation of phases

The excess functional

I will consider for simplicity the case $p \geq 2$

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$$E(u; x_0, R) := \left(\fint_{B_R(x_0)} |u - (u)_{B_R(x_0)}|^p \, dx \right)^{1/p}$$

You want to prove that

$$E(u; x_0, \tau^k R) \leq \tau^{k\gamma} E(u; x_0, R)$$

and this implies that

$$u \in C^{0,\gamma}$$

Step 1: Preliminary microscopic Hölder continuity

u is locally Hölder continuous with some potentially microscopic exponent $\gamma_0 \in (0, 1)$. This essentially serve to catch the borderline case $q = p + \alpha$.

Step 2: p-phase

Assume

$$\inf_{x \in B_R} a(x) \leq M[a]_{0,\alpha} R^{\alpha}$$

holds for some number $M \geq 1$. then for every $\gamma \in (0,1)$ there exists a positive radius $R_* \equiv R_*(M,\gamma)$ and $\tau \equiv \tau(M,\gamma) \in (0,1/4)$ such that the decay estimate

$$E(u; x_0, \tau R) \leq \tau^{\gamma} E(u; x_0, R)$$

holds whenever $0 < R \leq R_*$

→ Caccioppoli inequality in the p-phase becomes

$$\int_{B_{R/2}} |Du|^p \, dx \leq c \int_{B_R} \left| \frac{u-(u)_{B_R}}{R} \right|^p dx = \left(\frac{E(u;x_0,R)}{R} \right)^p ,$$

→ then define

$$v(x) := \frac{u(x_0+Rx)}{E(u;x_0,R)} , \qquad x \in B_1$$

so that

$$\fint_{B_{1/2}} |Dv|^p \, dx \leq c$$

Step 2: p-phase

$\to$ moreover, v solves, for every $\varphi \in C_0^\infty(B_1)$

$$\int_{B_1} \langle |Dv|^{p-2}Dv + (q/p)\tilde{a}(x)R^{p-q}[E(u;x_0,R)]^{q-p}|Dv|^{q-2}Dv, D\varphi\rangle \, dx = 0$$

this means that

$$\begin{aligned}
&\left| \fint_{B_1} \langle |Dv|^{p-2}Dv, D\varphi\rangle \, dx \right| \\
&\qquad \le cMR^{p+\alpha-q}[E(u;x_0,R)]^{q-p}\|D\varphi\|_{L^\infty(B_{1/2})} \fint_{B_{1/2}} |Dv|^{q-1}\, dx \\
&\qquad \le cR^{p+\alpha-q+\gamma_0(q-p)}\|D\varphi\|_{L^\infty(B_{1/2})} \left(\fint_{B_{1/2}} |Dv|^p \, dx \right)^{\frac{q-1}{p}} \\
&\qquad \le C_* R_*^{p+\alpha-q+\gamma_0(q-p)}\|D\varphi\|_{L^\infty(B_{1/2})}
\end{aligned}$$

Step 2: p-phase

$\to$ we conclude that

$$\left| \fint_{B_1} \langle |Dv|^{p-2} Dv, D\varphi \rangle \, dx \right| \leq \varepsilon \|D\varphi\|_{L^\infty(B_{1/2})}$$

by taking R_* suitably small

Step 2: p-phase

$\rightarrow$ apply the p-harmonic approximation lemma

Theorem (Duzaar - Min. Calc. Var. 04)

Given $\varepsilon > 0$ and $L > 0$, there exists $\delta \in (0,1]$ such that whenever $v \in W^{1,p}(B_{1/2})$ satisfies

$$\fint_{B_{1/2}} |Dv|^p \, dx \leq L$$

and

$$\fint_{B_{1/2}} \langle |Dv|^{p-2} Dv, D\varphi \rangle \, dx \leq \delta \|D\varphi\|_{L^\infty(B_{1/2})}$$

holds for all $\varphi \in C^1_0(B_{1/2})$. there exists a p-harmonic map $h \in W^{1,p}(B_{1/2})$, that is $\operatorname{div}(|Dh|^{p-2}Dh) = 0$, such that

$$\fint_{B_{1/2}} |v - h|^p \, dx \leq \varepsilon^p$$

Step 2: p-phase

$\rightarrow$ we conclude that

$$\left| \fint_{B_1} \langle |Dv|^{p-2} Dv, D\varphi \rangle \, dx \right| \leq \varepsilon \|D\varphi\|_{L^\infty(B_{1/2})}$$

by taking R_* suitably small

$\rightarrow$ find a p-harmonic map h such that

$$\fint_{B_{1/2}} |v - h|^p \, dx \leq \varepsilon^p$$

$\rightarrow$ for harmonic maps you know that you have a good excess decay, and therefore, since v and h are close, then also v has the same property; scaling back, the same property holds for u

Step 3: (p,q)-phase

assume

$$\inf_{x \in B_R} a(x) > M[a]_{0,\alpha} R^{\alpha}$$

holds for some number $M \geq 1$. Fix $\gamma \in (0,1)$; there exist positive constants $M_1 \geq 4$ and $\tau \in (0,1/4)$, with depending on γ, such that if $M \geq M_1$, then the decay estimate

$$\begin{aligned} & E(u; x_0, \tau^k R) \\ & \quad \lesssim \tau^{k\gamma} R \left[\fint_{B_{2R}} \left(\left| \frac{u-(u)_{B_{2R}}}{R} \right|^p + a(x) \left| \frac{u-(u)_{B_{2R}}}{R} \right|^q \right) dx \right]^{1/p} \end{aligned}$$

holds for every integer $k \geq 0$

Step 4: Separation of phases via exit time

- $\to$ choose $\gamma \in (0,1)$
- $\to$ Find $M \geq 1$ and τ_2 from Step 2
- $\to$ Use this M in Step 1 and find R_* and τ_1 from Step 1
- $\to$ consider the sequence of balls

$$....B_{R_{k+1}} \subset B_{R_k} ... \subset B_{R_1} \subset B_R \, , \qquad\qquad R_k = \tau_1^k R_0$$

and the condition

$$\inf_{x \in B_{R_k}} a(x) \leq M R_k^\alpha \tag{1}$$

the exit time index is

$$m := \min \left\{ k \in \mathbb{N} \cup \{\infty\} \ : \ (1) \, \text{fails} \right\} \, .$$

Step 4: Separation of phases via exit time

$\rightarrow$ keep on using Step 1 as long as the exit time is not reached, this yields

$$E(u;x_0,\tau_1^k R_0) \le \tau_1^{k\gamma} E(u;x_0,R_0) \qquad \text{for every } k \in \{0,\ldots,m\}\ .$$

$\rightarrow$ after the exit time you can use Step 2 to get

$$\begin{aligned} E(u;x_0,\tau_2^k\tau_1^m R_0) &\lesssim \tau_2^{k\gamma} E(u;x_0,2\tau_1^m R_0) \\ &+\tau_2^{k\gamma}\tau_1^m R_0 \left(\fint_{B_{2\tau_1^m R_0}} a(x) \left| \frac{u-(u)_{B_{2\tau_1^m R_0}}}{\tau_1^m R_0} \right|^q dx \right)^{1/p} \end{aligned}$$

$\rightarrow$ match the two inequalities using the exit time condition and ones again the bound $q \le p+\alpha$

this tells that

$$\int_{B_R} |Du|^p \, dx \lesssim R^{n-\theta} \qquad \forall\ \theta > 0$$

Step 6: Absence of Lavrentiev phenomenon

there exists a sequence of smooth functions $\{u_n\}$ such that

$$\int_B (|Du_n|^p + a(x)|Du_n|^q)\, dx \to \int_B (|Du|^p + a(x)|Du|^q)\, dx$$

for every ball $B \subset \Omega$

Step 7: Fractional differentiability

We get suitable uniform estimates in

$$Du \in W^{\beta/p,p} \qquad \text{for every } \beta < \alpha$$

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we recall that this means

$$\int_{\Omega'}\int_{\Omega'} \frac{|Du(x) - Du(y)|^p}{|x-y|^{n+\beta}} < \infty$$

for every $\Omega' \Subset \Omega$

Step 7: Fractional differentiability

the proof goes via approximation

$$\begin{cases} v_n \mapsto \min_w \int_B (|Dv|^p + [a(x)+\sigma_n]|Dv|^q)\,dx \\ \qquad\qquad w \in u_n + W_0^{1,q}(B) \end{cases}$$

where $0 < \sigma_n \to 0$

$$\int_B (|Du_n|^p + a(x)|Du_n|^q)\,dx \to \int_B (|Du|^p + a(x)|Du|^q)\,dx$$

and

$$u_n \in C^\infty(B)$$

the proof goes via approximation

$$\begin{cases} v_n \mapsto \min_{w} \int_B (|Dv|^p + [a(x) + \sigma_n]|Dv|^q)\, dx \\ \qquad\qquad w \in u_n + W_0^{1,q}(B) \end{cases}$$

where $0 < \sigma_n \to 0$

$$\int_B (|Du_n|^p + a(x)|Du_n|^q)\, dx \to \int_B (|Du|^p + a(x)|Du|^q)\, dx$$

and

$$u_n \in C^\infty(B)$$

this implies $v_n \to u$

Step 8: Upgraded Caccioppoli inequality

the following improved Caccioppoli type inequality holds:

$$
\begin{aligned}
&\fint_{B_{R/2}} |Du|^{2q-p}\, dx \\
&\lesssim \frac{1}{R^{\alpha/2}} \left[\fint_{B_{2R}} \left(\left| \frac{u-(u)_{B_R}}{R} \right|^p + a(x) \left| \frac{u-(u)_{B_{2R}}}{R} \right|^q \right) dx + 1 \right]^b
\end{aligned}
$$

we use the fractional interpolation inequality

$$\|f\|_{W^{\tilde{s},t}} \leq c\|f\|^{\theta}_{W^{s_1,p_1}}\|f\|^{1-\theta}_{W^{s_2,p_2}}$$

with

$$\tilde{s} = \theta s_1 + (1-\theta)s_2 \qquad \frac{1}{t} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$$

we apply as

$$\|Dv_n\|_{L^t} \leq c[v_n]_{s,p_1}^{\theta} \|Dv_n\|_{W^{\beta/p,p}}^{1-\theta}$$

with exponents

$$1 = \theta s + (1-\theta)\left(1 + \frac{\beta}{p}\right) \qquad \frac{1}{t} = \frac{\theta}{p_1} + \frac{1-\theta}{p}$$

and

$$[v_n]_{s,p_1} := \left(\int\int \frac{|v_n(x) - v_n(y)|^{p_1}}{|x-y|^{n+sp_1}}\, dx\, dy\right)^{1/p_1}$$

and take s close to 1 as you please and p_1 as large as you like

Step 9: Improved estimate in the p-phase

if for some $M \geq 1$

$$a_i(R) = \inf_{x \in B_R} a(x) \leq M[a]_{0,\alpha} R^\alpha$$

then solve

$$\begin{cases} v \mapsto \min_w \int_{B_R} |Dv|^p \, dx \\ \quad w \in u + W_0^{1,p}(B_R) \end{cases}$$

and find

$$\fint_{B_R} |Du - Dv|^p \, dx \leq M^2 R^\alpha$$

Step 9: Improved estimate in the p-phase

if for some $M \geq 1$

$$a_i(R) = \inf_{x \in B_R} a(x) \leq M[a]_{0,\alpha} R^{\alpha}$$

then solve

$$\begin{cases} v_R \mapsto \min\limits_{w} \displaystyle\int_{B_R} \left(|Dv|^p + a_i(R)|Dv|^q\right) dx \\ \qquad\qquad w \in u + W_0^{1,p}(B_R) \end{cases}$$

and get

$$\fint_{B_R} |Du - Dv|^p \, dx \lesssim \frac{1}{M} \fint_{B_{2R}} \left(\left| \frac{u - (u)_{B_R}}{R} \right|^p + a(x) \left| \frac{u - (u)_{B_{2R}}}{R} \right|^q \right) dx$$

Step 10: Final gradient continuity

$\to$ take B_R and $M > 0$ and consider the functionals

$$v \mapsto \int_{B_R} \left(|Dv|^p + a_i(R)|Dv|^q\right) \, dx$$

where

$$a_i(R) := \begin{cases} 0 & \text{if } \quad \inf_{x \in B_R} a(x) \leq M[a]_{0,\alpha} R^\alpha \\ \\ \inf_{x \in B_R} a(x) & \text{if } \inf_{x \in B_R} a(x) > M[a]_{0,\alpha} R^\alpha \end{cases}$$

$\to$ solve

$$\begin{cases} v_R \mapsto \min_w \int_{B_R} \left(|Dv|^p + a_i(R)|Dv|^q\right) \, dx \\ \qquad\qquad w \in u + W_0^{1,p}(B_R) \end{cases}$$

Thank you, with a work of Serena Nono