The Morse–Sard theorem, generalized Luzin property and level sets of Sobolev functions

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The theorems of A.P. Morse (1939) and A. Sard (1942)

- $f\colon \mathbb{R}^n \to \mathbb{R}^m$ differentiable map
- The **critical set** $C_f = \{x \in \mathbb{R}^n : \operatorname{rank} Df(x) < \max(m,n)\}$

$n \leq m$: $\mathcal{H}^m(f(C_f)) = 0$

$n > m$: **If** $f \in \mathrm{C}^k$ **with** $k \geq n - m + 1$, **then** $\mathcal{H}^m(f(C_f)) = 0$

H. Whitney (1935): False if only $k \leq n - m$.

Global statements

- N^n second countable smooth manifold, M^m smooth manifold
- $f\colon N^n \to M^m$ differentiable map
- The **critical set** $C_f = \{x \in N^n : \mathrm{rank} Df(x) < \max(m,n)\}$

$n \leq m$: $\mathcal{H}^m(f(C_f)) = 0$

$n < m$: **If** $f \in \mathrm{C}^k$ **with** $k \geq n - m + 1$**, then** $\mathcal{H}^m(f(C_f)) = 0$

The theorem of A.Ya. Dubovitskiĭ (1957)

- Dimensions $n > m$
- $f\colon \mathbb{R}^n \to \mathbb{R}^m$ C^k map
- Critical set $C_f = \{x \in \mathbb{R}^n :\ \mathrm{rank} Df(x) < m\}$
- $\ell = \max(n - m + 1 - k, 0)$

Then $\mathcal{H}^\ell(C_f \cap f^{-1}\{y\}) = 0$ **for** $\mathcal{L}^m$ **a.e.** $y \in \mathbb{R}^m$

The theorems of A.Ya. Dubovitskiĭ (1966) and H. Federer (1966)

- $f\colon \mathbb{R}^n \to \mathbb{R}^m$ C^k map
- The d–**critical set** $C_f^d = \{x \in \mathbb{R}^n : \operatorname{rank} Df(x) < d\}$ where $d \in \{1, \dots, \min(m,n)\}$

Then $\mathcal{H}^\beta(f(C_f^d)) = 0$ **when** $\beta \geq d - 1 + \frac{n-d+1}{k}$

$\Rightarrow$ **If** $n > m = d$**, then** $\mathcal{H}^\beta(f(C_f^m)) = 0$ **when** $\beta \geq m - 1 + \frac{n-m+1}{k}$**.**

Result is sharp on C^k **scale**

Improvements to Hölder spaces

- $f\colon \mathbb{R}^n \to \mathbb{R}^m$ $\mathrm{C}^{k,\alpha}_{\mathrm{loc}}$ map
- The d–**critical set** $C^d_f = \{x \in N^n : \operatorname{rank} Df(x) < d\}$, where $d \in \{1, \ldots, \min(m,n)\}$

Then $\mathcal{H}^\beta(f(C^d_f)) = 0$ **when** $\beta \geq d - 1 + \frac{n-d+1}{k+\alpha}$

Result is sharp on $\mathrm{C}^{k,\alpha}_{\mathrm{loc}}$ **scale**

Y. Yomdin (1983), A. Norton (1986), S.M. Bates (1993), C.G.T. De A. Moreira (2001), ...

An example (Dubovitskiĭ, Federer, Moreira)

- If $f \colon \mathbb{R}^2 \to \mathbb{R}$ is $\mathrm{C}^{1,1}_{\mathrm{loc}}$, then $\mathcal{H}^1(f(C_f)) = 0$.
- False when f is only $\mathrm{C}^{1,\alpha}_{\mathrm{loc}}$ for $\alpha < 1$.

Central Cantor sets: C_λ for $0 < \lambda < 1/2$

- remove from $[0,1]$ the central open interval $I_{1,1}$ of proportion $1 - 2\lambda$
- remove from two remaining intervals the central open intervals $I_{2,1}$ and $I_{2,2}$ of proportion $1 - 2\lambda$, and repeat ...
- $$C_\lambda = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{2^{i-1}} [0,1] \setminus I_{i,j}$$

An example

- Let $\phi\colon \mathbb{R} \to [0,1]$ be C^∞ and

$$\phi(t) = \begin{cases} 0 & t \le 0 \\ 1 & t \ge 1. \end{cases}$$

- Let $\frac{1}{4} < \lambda < \frac{1}{2}$.
- Let $I_{ij}^{\lambda} = (a,b) \subset [0,1]$ and $I_{ij}^{\frac{1}{4}} = (c,d) \subset [0,1]$ be corresponding removed intervals in construction of C_λ and $C_{\frac{1}{4}}$, respectively.
- Put

$$g(t) = c + (d-c)\phi\left(\frac{t-a}{b-a}\right), \quad t \in (a,b),$$

and $g(t) = 0$ for $t \le 0$, $g(t) = 1$ for $t \ge 1$.
- Extend g by continuity to C_λ; hereby $g\colon \mathbb{R} \to \mathbb{R}$ is continuous and $g(C_\lambda) = C_{\frac{1}{4}}$.

An example

- $g \in \mathrm{C}^{1,\alpha}$ for $\alpha = \ln 4/\ln(1/\lambda) - 1$
- $\ln 4/\ln(1/\lambda) - 1 \nearrow 1$ as $\lambda \nearrow 1/2$
- $g'(t) = 0$ for $t \in C_\lambda$ (since $\phi'(0) = \phi'(1) = 0$)
- Put $f(x,y) = g(x) + 2g(y)$, $(x,y) \in \mathbb{R}^2$.

$\Rightarrow f$ is $\mathrm{C}^{1,\alpha}$, $C_f \supset C_\lambda \times C_\lambda$ so

$$\begin{aligned} f(C_f) &\supseteq f(C_\lambda \times C_\lambda) = g(C_\lambda) + 2g(C_\lambda) \\ &= C_{\frac{1}{4}} + 2C_{\frac{1}{4}} = [0,3] \end{aligned}$$

Other constructions, incl. H. Whitney (1935), R. Kaufman (1979), Y. Yomdin (1983), E.L. Grinberg (1985), T. Körner (1988), G. Comte (1996), M. Csörnyei (2002), J. Kolar (2003), Z.-Y. Wen & L.-F. Xi (2009) ...

Summary

- Given dimensions $m < n$.
- Maps $f\colon \mathbb{R}^n \to \mathbb{R}^m$ of class $\mathrm{C}^{n-m,\alpha}_{\mathrm{loc}}$ enjoys Morse–Sard property iff $\alpha = 1$.
- $\mathrm{C}^{n-m,1}_{\mathrm{loc}} = \mathrm{W}^{n-m+1,\infty}_{\mathrm{loc}}$ (precise representatives)

Always further to go... A. Norton (1994)

Maps $f\colon \mathbb{R}^n \to \mathbb{R}^m$ of class C^{n-m} have the Morse–Sard property provided $D^{n-m}f$ is locally **Zygmund continuous**:

$$|D^{n-m}f(x+h) + D^{n-m}f(x-h) - 2D^{n-m}f(x)| \leq c|h|.$$

Extension to Sobolev functions

- L. De Pascale (2001): If $f \in \mathrm{W}^{k,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ and $k \geq n - m + 1$, $p > n$, then $\mathcal{H}^m(f(C_f)) = 0$.
- Note that $\mathrm{W}^{k,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m) \subset \mathrm{C}^1(\mathbb{R}^n, \mathbb{R}^m)$ when $k \geq n - m + 1$, $p > n$.

E.M. Landis (1951), B. Bojarski, P. Hajlasz and P. Strzelecki (2005), D. Pavlica and L. Zajíček (2006), A. Figalli (2008), D. Bucur, A. Giacomini and P. Trebeschi (2008), R. Van der Putten (2012), G. Alberti, S. Bianchini and G. Crippa (2013), P. Hajlasz and S. Zimmerman (2014), G. Alberti, M. Csörnyei, E. D'Aniello and B. Kirchheim (2012), S. Hencl and P. Honzík (2015) ...

The critical set when the function is not differentiable

The precise representative: of $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ is

$$f(x) := \limsup_{r \searrow 0} \frac{1}{\mathcal{H}^n(B_r(x))} \int_{B_r(x)} f(y)\,\mathrm{d}y \qquad (1)$$

Then $f \colon \mathbb{R}^n \to [-\infty, \infty]$ Borel and (1) is a limit in $\mathbb{R}$ for $\mathcal{H}^n$ almost all x.

Sobolev functions: $f \in \mathrm{W}^{k,p}_{\mathrm{loc}}(\mathbb{R}^n)$. Then

- f is continuous when $p > 1$ & $kp > n$ or when $p = 1$ & $k = n$
- (1) exists as a limit in $\mathbb{R}$ at $\mathcal{H}^t$ almost all x for each $t > n - kp$ when $p > 1$ & $kp < n$ (and for $\mathcal{H}^{n-k}$ almost all x when $p = 1$ & $n > k$)

The critical set when the function is not differentiable

- $f \in \mathrm{W}^{k,\frac{n}{k}}(\mathbb{R}^n, \mathbb{R}^m)$ with $k \in \{1, \ldots, n\}$
- $E = \{x \in \mathbb{R}^n : x$ is **not** an L^n Lebesgue point of $Df\}$
- $d \in \{1, \ldots, \min(m,n)\}$

The d–critical set: $C_f^d := \{x \in \mathbb{R}^n \setminus E : \mathrm{rank} Df(x) < d\}$

The critical set when the function is not differentiable

Let $f \in \mathrm{W}^{n,1}(\mathbb{R}^n)$. Then

- $f \in \mathrm{C}_0(\mathbb{R}^n)$
- $\mathcal{H}^1(E) = 0$
- f is Fréchet differentiable at $\mathcal{H}^1$ almost all $x \in \mathbb{R}^n$ with differential

$$Df(x) = \lim_{r \searrow 0} (Df)_{x,r}.$$

The critical set: $C_f := \{x \in \mathbb{R}^n \setminus E : \, Df(x) = 0\}$

The generalized Luzin property

Theorem I (Bourgain, Korobkov & K, 2010, 2012)
Let $f \in \mathrm{W}^{n,1}(\mathbb{R}^n)$. Then

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall A \subset \mathbb{R}^n : \quad \mathcal{H}^1_\infty(A) < \delta \Rightarrow \mathcal{H}^1(f(A)) < \varepsilon.$$

Remarks:

- $\mathcal{H}^1(f(E)) = 0$
- Theorem I also holds for $f \in \mathrm{B}V^n(\mathbb{R}^n)$.

Morse–Sard for Sobolev functions

Theorem II (Bourgain, Korobkov & K, 2010, 2012)
Let $f \in \mathrm{W}^{n,1}(\mathbb{R}^n)$. Denote

$$E = \{x \in \mathbb{R}^n : x \text{ not } \mathrm{L}^n \text{ Lebesgue point for } Df\}$$

and

$$C_f := \{x \in \mathbb{R}^n \setminus E : Df(x) = 0\}.$$

Then $\mathcal{H}^1(f(C_f)) = 0$.

Remark: Theorem II also holds for $f \in \mathrm{B}V^n(\mathbb{R}^n)$. When $n = 2$ we used another definition of the critical set.

Typical level sets for Sobolev functions

Theorem III (Bourgain, Korobkov & K, 2010, 2012)
Let $f \in \mathrm{W}^{n,1}(\mathbb{R}^n)$. For $\mathcal{H}^1$ almost all $y \in f(\mathbb{R}^n)$ the level set $f^{-1}\{y\}$ is a compact $(n-1)$–dimensional C^1 submanifold in $\mathbb{R}^n$.

Remark: f need not be C^1, but is only differentiable $\mathcal{H}^1$ almost everywhere.

Morse–Sard in vector–valued case

Theorem IV (Korobkov & K, 2013)
Assume

- $f \in \mathrm{W}^{k,\frac{n}{k}}(\mathbb{R}^n, \mathbb{R}^m)$ for a $k \in \{1, \dots, n\}$
- $d \in \{1, \dots, \min(m, n)\}$

Then $\mathcal{H}^\beta\big(f(C_f^d)\big) = 0$ when $\beta \geq d - 1 + \frac{n+1-d}{k}$

In particular when $n > m = d$: $\mathcal{H}^\beta\big(f(C_f^m)\big) = 0$ when $\beta \geq m - 1 + \frac{n+1-m}{k}$

Remark: $\mathrm{W}^{k,\frac{n}{k}}$ functions on $\mathbb{R}^n$ can be discontinuous

Vector–valued case: Sobolev–Lorentz functions

Lorentz space: $f \in \mathrm{L}^{p,1}(\mathbb{R}^n)$ if

$$\|f\|_{p,1} = \int_0^\infty \mathcal{H}^n\Big(\{x \in \mathbb{R}^n : |f(x)| > t\}\Big)^{\frac{1}{p}} \,\mathrm{d}t < \infty.$$

- $f \in \mathrm{W}^{k,p,1}(\mathbb{R}^n)$ if $f \in \mathrm{W}^{k,p}(\mathbb{R}^n)$ and $|D^k f| \in \mathrm{L}^{p,1}$.

Proposition:
Let $f \in \mathrm{W}^{k,p,1}(\mathbb{R}^n)$ where $k \in \{2, \dots, n-1\}$ and $p = \frac{n}{k}$. Then
• $f \in \mathrm{C}_0(\mathbb{R}^n)$
• There exists a Borel set $A \subset \mathbb{R}^n$ such that $\mathcal{H}^t(A) = 0$ for $t > \frac{n}{k}$, f is (Fréchet–)differentiable at each $x \in \mathbb{R}^n \setminus A$ with differential $Df(x)$ and x is an L^n–Lebesgue point of the weak derivative Df.

Remark: $k = 1$, $t = p = n$ due to Stein (1970),
$k = n$, $t = p = 1$ due to Dorronsoro (1989)

Luzin N type property for Sobolev–Lorentz maps

Theorem V (Korobkov & K, 2013) Assume

- $f \in \mathrm{W}^{k,\frac{n}{k},1}(\mathbb{R}^n, \mathbb{R}^m)$ for a $k \in \{2, \dots, n\}$
- $t > \frac{n}{k}$

Then $\forall \varepsilon > 0 \, \exists \delta > 0$, $\mathcal{H}^t_\infty(S) < \delta \Rightarrow \mathcal{H}^t_\infty(f(S)) < \varepsilon$

In particular: $\mathcal{H}^t_\infty\big(f(E)\big) = 0$ when $t > \frac{n}{k}$

Typical level sets for Sobolev-Lorentz maps

Theorem VI (Korobkov & K, 2013)
Let $2 \leq m \leq n$, $k = n - m + 1$, and $f \in \mathrm{W}^{k,\frac{n}{k},1}(\mathbb{R}^n, \mathbb{R}^m)$.
Then for $\mathcal{H}^m$ almost all $y \in f(\mathbb{R}^n)$ the level set $f^{-1}\{y\}$ is a compact C^1 $(n-m)$–dimensional submanifold of $\mathbb{R}^n$

Properties of Sobolev–Lorentz maps

Theorem VII: (Korobkov & K, 2015)
Assume $f \in \mathrm{W}^{k,\frac{n}{k},1}(\mathbb{R}^n,\mathbb{R}^m)$ for a $k \in \{2,\dots,n-1\}$. Then:

(i)
$$\forall \varepsilon > 0 \quad \exists \delta > 0, \quad \mathcal{H}^{\frac{n}{k}}_{\infty}(S) < \delta \Rightarrow \mathcal{H}^{\frac{n}{k}}_{\infty}(f(S)) < \varepsilon$$

(ii) f is j times differentiable (in the classical Fréchet–Peano sense) with j–th differential $D^j f$ at $\mathcal{H}^{j\frac{n}{k}}$ almost all points for each $j \in \{1,\dots,k\}$

Comments on proof for Theorem I

Generalized Luzin property: Let $f \in \mathrm{W}^{n,1}(\mathbb{R}^n)$.
If $A \subset \mathbb{R}^n$ and $\mathcal{H}^1_\infty(A) = 0$, then $\mathcal{H}^1(f(A)) = 0$.

- Trivial when f locally Lipschitz.
- When $f \in \mathrm{W}^{n,1}(\mathbb{R}^n)$, then $f \in \mathrm{C}_0(\mathbb{R}^n)$ and f is differentiable $\mathcal{H}^1$ almost everywhere but not locally Lipschitz.

Comments on proof for Theorem I

Lemma: Let $f \in \mathrm{W}^{n,1}(\mathbb{R}^n)$. For a cube $Q \subset \mathbb{R}^n$,

$$\operatorname{diam} f(Q) \leq c \left(\frac{1}{\ell(Q)^{n-1}} \int_Q |Df| \, \mathrm{d}x + \int_Q |D^n f| \, \mathrm{d}x \right)$$

First attempt: For $\delta > 0$ take cubes Q_j so $A \subset \bigcup_j Q_j$ and $\sum_j \ell(Q_j) < \delta$.

By Lemma

$$\begin{aligned} \mathcal{H}^1(f(A)) &\leq \sum_j \operatorname{diam} f(Q_j) \\ &\leq c \sum_j \left(\frac{1}{\ell(Q_j)^{n-1}} \int_{Q_j} |Df| \, \mathrm{d}x + \int_{Q_j} |D^n f| \, \mathrm{d}x \right) \end{aligned}$$

Comments on proof for Theorem I

Refine covering to gain control of first term

Lemma (BKK 2012) Given nonoverlapping dyadic cubes $\{Q_j\}$ there exists nonoverlapping dyadic cubes $\{I_k\}$ satisfying

- $\bigcup_j Q_j \subseteq \bigcup_k I_k$ and $\sum_k \ell(I_k) \leq \sum_j \ell(Q_j)$
- For any dyadic cube $Q \subset \mathbb{R}^n$,

$$\ell(Q) \geq \sum_{k\colon I_k \subseteq Q} \ell(I_k).$$

Idea of proof: Take $\{I_k\}$ to be the maximal cubes in family

$$\mathfrak{F} := \left\{ I \text{ dyadic } : \sum_{j\colon Q_j \subseteq I} \ell(Q_j) \geq \ell(I) \right\}$$

Comments on proof for Theorem I

Measure

$$\mu := \Big(\sum_k \ell(I_k)^{1-n} \mathbf{1}_{I_k} \Big) \mathcal{L}^n$$

satisfies

$$(*) \qquad \mu(Q) \leq 2^{n+2} \ell(Q) \qquad \forall \text{ cubes } Q$$

- **Adams–Maz'ya embedding/trace theorem:** for $b \in \mathrm{W}^{n,1}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |Db| \, \mathrm{d}\mu \leq c(n) \int_{\mathbb{R}^n} |D^n b| \, \mathrm{d}x.$$

- **From definition:** $\forall \varepsilon > 0 \; \exists g \in \mathrm{C}^\infty_c$, $b \in \mathrm{W}^{n,1}$ such that

$$f = g + b \quad \text{and} \quad \|g\|_{\mathrm{W}^{n,\infty}} < k_\varepsilon, \quad \|b\|_{\mathrm{W}^{n,1}} < \varepsilon.$$

Comments on proof of Theorem VII

- For nonnegative Borel measure μ on $\mathbb{R}^n$ and $\beta \in (0, n)$:

$$[\mu]_\beta := \sup_Q \frac{\mu(Q)}{\mathcal{L}^n(Q)^{\frac{\beta}{n}}}$$

- Riesz potential of order $\alpha \in (0, n)$ of f:

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, \mathrm{d}y$$

Adams–Maz'ya trace/embedding theorem: Let $\alpha > 0$, $1 < p < \infty$ and $\alpha p < n$. Then $\forall q > p \quad \exists c < \infty$ so

$$\int_{\mathbb{R}^n} |I_\alpha f|^q \, \mathrm{d}\mu \leq c[\mu]_\beta \|f\|^q_{\mathrm{L}^p}$$

for $\beta = (n - \alpha p)\frac{q}{p}$.

Remark: False for $q = p$

Trace/embedding theorem for Sobolev–Lorentz maps

Theorem VIII: (Korobkov & K, 2015)
For $\alpha > 0$, $1 < p < \infty$, $\alpha p < n$:

$$\int_{\mathbb{R}^n} |I_\alpha f|^p \, \mathrm{d}\mu \leq c[\mu]_{n-\alpha p} \|f\|_{\mathrm{L}^{p,1}}^p$$

Comments on proof for Theorem III

Auxiliary result: **Luzin–type theorem** (BKK 2012)

Let $f \in \mathrm{W}^{n,1}(\mathbb{R}^n)$. Then $\forall \varepsilon > 0$ $\exists$ open $U \subset \mathbb{R}^n$ with $\mathcal{H}^1_\infty(U) < \varepsilon$ and $g \in \mathrm{C}^1(\mathbb{R}^n)$ so

$$f = g \text{ and } Df = Dg \text{ on } \mathbb{R}^n \setminus U.$$

Similar to result by B. Bojarski, P. Hajlasz and P. Strzelecki (2005) for $\mathrm{W}^{k,p}$ valid for $p > 1$.

Comments on proof for Theorem III

Fix $\varepsilon > 0$. By Theorems I & II, $\mathcal{H}^1(f(E \cup C_f) = 0$ so $\exists$ open set $V_1 \supset f(E \cup C_f)$ with $\mathcal{H}^1(V_1) < \varepsilon/2$. For given $\delta > 0$ by Luzin $\exists$ U, g so $\mathcal{H}^1_\infty(U) < \delta$ and

$$f = g \text{ and } Df = Dg \text{ on } \mathbb{R}^n \setminus U.$$

Theorem I allows to take δ so $\mathcal{H}^1(f(U)) < \varepsilon/2$. Take open set $V_2 \supset f(U)$ with $\mathcal{H}^1(V_2) < \varepsilon/2$. Put $V = V_1 \cup V_2$ and record:

$$f(E \cup C_f) \subset V, \quad f|_{f^{-1}(\mathbb{R}\setminus V)} = g|_{f^{-1}(\mathbb{R}\setminus V)}$$

$$Df|_{f^{-1}(\mathbb{R}\setminus V)} = Dg|_{f^{-1}(\mathbb{R}\setminus V)}, \quad \mathcal{H}^1(V) < \varepsilon.$$

Comments on proof for Theorem III

Fix $y \in f(\mathbb{R}^n) \setminus V$, $y \neq 0$.
Then

(i) $f^{-1}\{y\}$ compact ($\Leftarrow f \in \mathrm{C}_0(\mathbb{R}^n)$)

(ii) $f^{-1}\{y\} \subseteq g^{-1}\{y\}$

(iii) $Df = Dg \neq 0$ on $f^{-1}\{y\}$

(iv) f diff. at each $x \in f^{-1}\{y\}$ with differential $Df(x)$ and Df has Lebesgue point at x

Claim: $\forall x_0 \in f^{-1}\{y\} \, \exists r > 0$ so
$f^{-1}\{y\} \cap B_r(x_0) = g^{-1}\{y\} \cap B_r(x_0)$
($\Rightarrow$ conclusion)

Comments on proof for Theorem III

Suppose not.
Then $\exists x_0 \in f^{-1}\{y\}$ and $x_j \in g^{-1}\{y\} \setminus f^{-1}\{y\}$ with $x_j \to x_0$.
Put

$$I_x = \left\{ x + \frac{Dg(x_0)}{|Dg(x_0)|} t : \; |t| < r \right\}.$$

Since $g|_{I_x}$ strictly monotone:

$$I_x \cap g^{-1}\{y\} = \{x\} \quad \forall x \in g^{-1}\{y\} \cap B_r(x_0)$$

when $r > 0$ sufficiently small. By (ii) we get for large j:

$$I_{x_j} \cap f^{-1}\{y\} = \emptyset$$

So either $f > y$ or $f < y$ on I_{x_j}. WLOG $f > y$ on I_{x_j}, so by cont. $f \geq y = f(x_0)$ on I_{x_0}. But f diff. at x_0 so $Df(x_0) = 0$ contradicting (iii) and (iv).

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