

Equilibrium Investment in High-Frequency Trading Technology: A Real Options Approach

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Abstract

This paper derives an optimal timing strategy for a regular slow trader considering investing in a high-frequency trading technology. The market is fragmented, and slow traders compete with fast traders for trade execution. Assuming all traders adhere to the optimal strategy derived for a single trader, I then determine the equilibrium level of fast trading in the market across all traders, as well as the socially optimal level. I show that there is a unique level of market fragmentation such that the equilibrium level of fast trading and the socially optimal level coincide. Moreover, the real options approach to investment yields an equilibrium level which is less socially optimal than the equilibrium level obtained via the classical net present value approach.

Keywords: High frequency trading, Fragmented markets, Real options.

JEL Classification Numbers: C61, G10, G20.

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1 Introduction

This paper uses a real options approach to determine and analyse the optimal time for financial market investors (referred to as “traders” hereafter) to invest in a high frequency trading (HFT) technology. Assuming this optimal strategy is adhered to by all traders in the market, I then determine and compare the equilibrium level of fast trading in the market and the socially optimal level.

HFT is a type of algorithmic trading that uses sophisticated computer algorithms to implement vast amounts of trades in extremely small time intervals. The analysis is motivated by the fact that over the last decade, the state of financial markets has changed considerably. In the first instance, markets have become highly fragmented. There are now more than 50 trading venues for U.S. equities - 13 registered exchanges and 44 so called Alternative Trading Systems (see Biais et al. [2] and O’Hara and Ye [16]). Hence, traders must search across many markets for quotes and doing so can be costly as it may delay full execution of their orders.

In response to the increase in market fragmentation, so called HFT technologies have been developed to reduce the associated costs borne by traders. For example, traders can buy colocation rights (the placement of their computers next to the exchange’s servers) which gives them fast access to the exchange’s data feed, they can invest in smart routers which can instantaneously compare quotes across all trading venues and then allocate their orders accordingly, or they can invest in high-speed connections to the exchanges via fiber optic cables or microwave signals. Proprietary trading desks, hedge funds, and so called pure-play HFT outlets are investing large sums of money into such technologies in an effort to outpace the competition. Indeed, according to Hoffman [11], recent estimates suggest that HFTs are now responsible for more than 50% of trading in U.S. equities.

In a recent paper, O’Hara [15] details the many ways in which market microstructure has changed over the past decade and calls for a new approach to research in this area which “reflects the new realities of the high frequency world”. Nevertheless, there has been a growth in the literature on HFT in recent years (see a survey by Foucault [6]).

Much of the literature is empirical and, on the whole, the consensus has been that HFT improves liquidity through lower bid-ask spreads (Hendershott et al. [10] and Hasbrouck and Saar [8]); is highly profitable (Menkveld [12] and Baron et al. [1]); and facilitates price discovery (Hendershott and Riordan [9] and Brogaard et al. [3]). The theoretical literature has also been growing in recent years. For example, Hoffman [11] presents a stylised model of HFT in a limit order market where agents differ in their trading speed; Pagnotta and Philippon [17] propose a model in which trading venues invest in speed and compete for traders who choose where and how much to trade; and Rosu [18] develop a model in which traders receive a stream of private signals and differ in their information processing speeds.

The theoretical model most closely related to the one in this paper, however, is that of Biais et al. [2] who develop a model of HFT in a Glosten and Milgrom [7] type framework. In particular, I incorporate some features of their static model into a dynamic framework such that a slow trader has the option to adopt the HFT technology now or at some future date. To the best of my knowledge, this is the first paper to examine the adoption strategy of HFT technologies in a dynamic setting. Moreover, while optimal investment timing has been well-developed in the real options literature through its application to many different types of problems (see, for example, Dixit and Pindyck [4]; Farzin et al. [5]; Thijssen et al. [19]; Murto [13]); and Nishihara and Shibata [14]), it is the first application of the approach in a HFT environment.

The optimal time for traders to invest in the HFT, as a function of the level of HFT activity already accounted for in the market as well as the degree of market fragmentation, is determined analytically. Assuming that this optimal strategy is adhered to by all traders in the market, I determine the equilibrium level of fast trading in the market, as well as the socially optimal level. I also determine these same levels by assuming that the classical net present value (NPV) approach to investment, which prescribes investment as soon as the discounted expected future payoff exceeds the sunk investment cost, is adhered to. This allows for the determination of the implications of the value of waiting for equilibrium investment and social optimality.

The novel results generated by my model are as follows. (i) In both the real options and the NPV cases, there is always a unique equilibrium level of fast trading, and the equilibrium level under the real options approach is never higher than under the NPV approach. Hence, the value of waiting reduces the market equilibrium level of investment in HFT technology. (ii) In both cases, the equilibrium level of fast trading increases in the degree of market fragmentation. (iii) There is a unique level of market fragmentation such that the socially optimal level of fast trading and the equilibrium level coincide. This is true in both the options case and the NPV case. (iv) The real options approach to investment yields an equilibrium level of HFT which is less socially optimal than the equilibrium level obtained by the NPV approach. Hence, adopting an approach to evaluating the investment opportunity which does not account for the option to wait induces social optimality.

The remainder of this paper is organised as follows. The set-up of the model is described in the next section. In Section 3 the solution to the optimal stopping problem is given, as well as a brief discussion on the implications it generates. Section 4 analyses and compares the market equilibrium and market socially optimal levels of fast trading as well as discussing some implications for policy, and Section 5 concludes. All proofs are placed in the appendix.

2 The Model

Consider a risk-neutral market trader contemplating investment in a HFT technology. Time is continuous, the horizon is infinite, and indexed by $t \in [0, \infty)$. The trader discounts the future at the risk-free rate $r > 0$. Investing in the technology incurs a sunk cost $I > 0$. Before investing, the trader is a regular (slow) trader who trades in fragmented markets where fast high frequency traders (HFTs) also trade. Hence, he is exposed to the impact such traders have on the likelihood of his orders getting executed at favourable prices. Once he invests, however, he becomes one of the HFTs.

2.1 The Trading Environment

The market is fragmented where slow traders compete with HFTs and, similar to Biais et al. [2], I capture this by assuming there is a size-one continuum of trading venues distributed on a circle and indexed clockwise from 0 to 1. When markets are fragmented, traders must search for quotes across markets which can lead to delayed execution of orders. At every instant a fraction $\lambda < 1$ of the trading venues are “liquid”. In the context of this model, a trading venue is liquid if the trader’s order, when sent to the exchange venue, is fully executed.

At any instant, a trader can only send an order to one trading venue. His choice of venue is random and uniformly drawn from the unit circle. HFTs have extremely fast connection speeds to the market and can observe all venues instantaneously. Thus, they find a liquid one with certainty upon arrival to the market and send their order to one of the liquid venues immediately. The traders are indifferent between liquid venues. However, a slow trader must search for liquid trading venues and finding one can take time. Thus, at each instant they execute their trade with probability λ . Otherwise, with complementary probability, they must continue to search for a liquid venue. Note that λ can also be interpreted as a measure of the degree of market fragmentation because the more fragmented the market, the more traders must search for quotes across markets and the lower is the probability of immediate execution; i.e., the lower is λ .

I also assume that there is a size-one continuum of traders in the market and at each instant a fraction $\alpha < 1$ of them are HFTs.

2.2 Valuations

The trading activities of a slow trader yields a flow of returns X^S in perpetuity, and the activities of a fast trader yields a flow of returns X^F , such that both processes depend on a stochastic process $(X_t)_{t \geq 0}$ which is a geometric Brownian motion of the form:

$$dX = \mu X dt + \sigma X dW, \tag{1}$$

for constants $\mu < r$ and $\sigma > 0$, which represent the drift and volatility of the process respectively, and $W = (W_t)_{t \geq 0}$ is a standard Brownian motion.

I describe the different return process specifications for slow and fast traders later in the section, but will determine a process for both types that depends positively on the stochastic component X . Hereafter, I refer to X simply as the “return process”.¹

2.2.1 The Search Process

As discussed, traders without access to the HFT technology cannot observe all trading venues instantaneously and, thus, sometimes send orders to venues which may not be liquid. They can only send orders to one trading venue in any instant. If they send an order to some venue at some instant t , it will be executed with probability λ . If it is not executed, I assume that the trader continues to search for a liquid venue for *that order until it is executed*.² To describe the process, we let the dependence on time be explicit so that X_t denotes the value of X at time t . The process works as follows:

- At $t = 0$: the trader submits an order which delivers a return X_0 with probability (w.p.) λ , otherwise gets 0.
- At $t = 1$ the trader will do *one* of the following:
 1. If his order is executed at $t = 0$, he submits another order at $t = 1$ which will yield, if filled, X_1 . It gets filled with probability λ .

¹Assuming the returns process follows a geometric Brownian motion is analytically very tractable, but it implies that the level of returns must always be positive which, in reality, will not always be the case. Returns will be positive if the trader is short the stock and buys it back at a lower price to which he sold, and if he is long the stock and sells at a higher price. The opposite can, of course, occur, but as I describe later in the paper, the point of the model is that slow traders are searching for venues to execute their chosen orders given that HFTs in the market compete with them. It is plausible to assume that they will only ever try to get orders filled that will yield them a positive return. Moreover, since HFTs can observe all venues instantaneously, it is also plausible that their returns will always be positive. Hence, assuming a geometric Brownian motion for the return process is not inappropriate.

²Note that Biais et al. [2] assume that if the order is not liquidated, the trader will abandon his search for a liquid venue for that order with a certain probability and, once he abandons, he does not trade again. However, their model is static. In a dynamic setting, if we assumed that he could abandon his search for execution of a specific order, we cannot simply assume he ceases trading altogether, but would need to assume he begins searching for liquidation of a different order instead. This yields a very cumbersome search process for which the abandoning aspect is likely to have, at best, a negligible impact on the result.

2. If, however, his $t = 0$ order is not filled, at $t = 1$ he is searching for a liquid venue for his $t = 0$ order and gets it filled w.p. λ . If filled, it yields the return X_0 .

Thus,

$$\begin{aligned} E^0[X_1^S] &= \lambda^2 E^0[X_1] + (1 - \lambda)\lambda X_0 \\ &= \lambda X_0 (\lambda e^\mu + (1 - \lambda)) \end{aligned} \tag{2}$$

since X_1 and X_0 are governed by (1).

- At $t = 2$ the trader will do *one* of the following:
 1. If his order at both $t = 0$ and $t = 1$ is executed, he submits a new order at $t = 2$ which gets filled with probability λ . If filled, the return is X_2 .
 2. If his order at $t = 0$ is filled, but his order at $t = 1$ is not, at $t = 2$ he is still trying to get the $t = 1$ order filled and does so w.p. λ . If it gets executed, he gets X_1 .
 3. If he does not get his order at $t = 0$ filled, but gets that order filled at $t = 1$, he submits an order at $t = 2$ which, if filled, gives him the X_1 return.³
 4. If he does not get his order filled at $t = 0$ or at $t = 1$, he is still searching for a liquid venue to fill his $t = 0$ order. With probability λ he gets X_0 .

Thus

$$\begin{aligned} E^0[X_2^S] &= \lambda^3 E^0[X_2] + \lambda(1 - \lambda)\lambda E^0[X_1] + (1 - \lambda)\lambda^2 E^0[X_1] + (1 - \lambda)^2 \lambda X_0 \\ &= \lambda X_0 [\lambda^2 e^{2\mu} + 2\lambda(1 - \lambda)e^\mu + (1 - \lambda)^2]. \end{aligned} \tag{3}$$

- The process continues in this way as long as the trader is slow.

³Since he missed the opportunity at $t = 1$ to obtain X_1 owing to his search for liquidity of the $t = 0$ order, we assume that he tries to obtain X_1 at $t = 2$. If we assume that, for example, he fills his $t = 0$ order at $t = 1$ and that he does not attempt to get X_1 at all, but just tries for X_2 at the next instant $t = 2$, the general expression for X_t^S becomes too cumbersome to handle. Thus, we impose the restriction that if he misses the opportunity to have an order filled at a particular instant, he tries to get that order filled at the next possible instant, rather than abandoning that order all together and trying for the most up to date order available.

Therefore, for some arbitrary instant $t \geq 0$, the search process described implies that

$$\begin{aligned} E^0[X_t^S] &= \lambda(1 - \lambda)^t X_0 \sum_{i=0}^t \binom{t}{i} \left(\frac{\lambda e^{\mu}}{1 - \lambda} \right)^i \\ &= \lambda X_0 ((1 - \lambda) + \lambda e^{\mu})^t \end{aligned} \tag{4}$$

by the Binomial theorem.

For a fast trader, he gets his order filled with certainty whenever he sends it to the market because he can observe which venues are liquid. Thus, he does not need to search for liquidity and at every instant t that he trades, he gets a return of X_t . Thus, $X_t^F \equiv X_t$ and

$$E^0[X_t^F] = X_0 e^{\mu t}, \tag{5}$$

which is, in fact, Eq. (4) for $\lambda = 1$. Moreover, since $\lambda < 1$, $E^0[X_t^F] > E^0[X_t^S]$ for all t . This says that fast traders have higher returns in expectation than slow traders because their execution delay costs are zero. This is intuitive and necessary because if this condition did not hold, investment in the HFT technology would never be optimal.

Much of the literature in this area documents that HFTs generate adverse selection costs for slow traders because of their accelerated access to value-relevant information for the asset (see, for example, empirical studies by Baron et al. [1] and Brogaard et al. [3]). Moreover, as documented in Biais et al. [2], anecdotal evidence suggests that the profitability of HFTs has declined in recent years which may be due to the fact that the level of high frequency trading in the market place has increased. I capture these stylized facts by assuming that $X_0 := (1 + \alpha)^{-1}x$, for some $x > 0$. (Recall from subsection 2.1 that $\alpha < 1$ denotes the mass of HFTs in the market at each instant.) Therefore, since the returns process follows a geometric Brownian motion, the current (discounted) expected return from trading for *each type* of trader, for any future time, decreases in the mass of HFTs in the market.

3 The Optimal Stopping Problem

If the trader decides to adopt the technology at time τ , the current ($t = 0$) value of the investment, denoted by $V(x)$ is given by

$$\begin{aligned} V(x) &= E^0 \left[\int_0^\tau e^{-rt} X_t^S dt + \int_\tau^\infty e^{-rt} X_t^F dt - e^{-r\tau} I \right] \\ &= E^0 \left[\int_0^\infty e^{-rt} X_t^S dt + e^{-r\tau} E^\tau \left[\int_0^\infty e^{-rt} (X_t^F - X_t^S) dt - I \right] \right], \end{aligned} \quad (6)$$

where $E^{t'}$ denotes the expectation operator applied at time t' .

The problem is to find a value function $V^*(x)$ and a stopping time $\tau^* > 0$ such that the following optimal stopping problem is solved:

$$V^*(x) = \sup_{\tau \in \mathcal{T}} E^0 [e^{-r\tau} F(X_\tau^S, X_\tau^F)], \quad (7)$$

for \mathcal{T} the set of stopping times, and

$$F(X_\tau^S, X_\tau^F) = E^\tau \left[\int_0^\infty e^{-rt} (X_t^F - X_t^S) dt \right] - I \quad (8)$$

is the trader's payoff from adopting the technology.

Letting $\delta := r - \mu > 0$, and using Eqs. (4) and (5) and the strong Markov property of diffusions,

$$\begin{aligned} F(X_\tau^S, X_\tau^F) &= X_\tau \left(\frac{1}{\delta} - \lambda \int_0^\infty e^{-rt} ((1 - \lambda) + \lambda e^\mu)^t dt \right) - I \\ &= \Omega(r, \mu, \lambda) X_\tau - I \\ &\equiv F(X_\tau), \end{aligned} \quad (9)$$

where the scaling term

$$\Omega(r, \mu, \lambda) = \frac{1}{\delta} - \frac{\lambda}{(r - \ln((1 - \lambda) + \lambda e^\mu))} > 0 \quad (10)$$

represents the relative advantage (in terms of returns) from being fast. This is induced

by the elimination of delayed execution arising from the need to search for quotes in fragmented markets. For completeness, I prove that $\Omega(r, \mu, \lambda) > 0$ in Appendix A. This is useful since its positivity is necessary to derive other comparative static results later in the paper.

Theorem 1. *Let $\beta_1 > 1$ be the positive root of the quadratic equation*

$$\mathcal{Q}(\beta) = \frac{1}{2}\sigma^2\beta(\beta - 1) + (r - \delta)\beta - r = 0. \quad (11)$$

Investment takes place at the first passage time $\tau^ = \inf\{t : X_t \geq X^*\}$, for some constant*

$$X^* = \frac{\beta_1}{\beta_1 - 1} (\Omega(r, \mu, \lambda))^{-1} I. \quad (12)$$

Moreover, the optimal stopping problem (7) is solved by

$$V^*(x) = \begin{cases} \frac{\lambda(1+\alpha)^{-1}x}{(r - \ln((1-\lambda) + \lambda e^\mu))} + \left(\frac{(1+\alpha)^{-1}x}{X^*}\right)^{\beta_1} F(X^*) & \text{if } x < (1 + \alpha)X^* \\ \frac{(1+\alpha)^{-1}x}{\delta} - I & \text{if } x \geq (1 + \alpha)X^*, \end{cases} \quad (13)$$

where $F(X)$ is given by Eq. (9).

Proof. See Appendix B. ■

3.1 Model Implications

In this subsection I discuss the economics underlying the optimal investment strategy.

Proposition 1. *An increase in the level of fast trading, α , makes it optimal for a slow trader to wait longer before investing.*

The mass of HFTs in the market does not directly affect the return threshold below which it is not optimal to become fast. However, an increase in the level of α reduces the expected present value of trading returns for both fast institutions and slow institutions, but it reduces the expected returns of fast institutions more since the expected present

value from investing decreases in α ; i.e.,

$$\frac{\partial F((1 + \alpha)^{-1}x)}{\partial \alpha} = -(1 + \alpha)^{-2}x\Omega < 0, \quad (14)$$

where $F(X)$ is given by Eq. (9).

This leads to the proposition that higher the level of fast trading in the market, the longer it is optimal to wait before adopting because the relative advantage from being fast decreases in α more than the value of waiting. Another way of seeing this is that an increase in α reduces the expected present value of returns of a slow trader and this increases the distance between the threshold X^* and the current level of returns $x(1+\alpha)^{-1}$.

The reason for the effect on the relative advantage of being fast is owing to the fact that once the search cost is removed, the relative importance of α on the expected future return increases. The expected return of slow traders are already suppressed because of the search cost and this appears to have a dampening effect of an increase in α on their expected return. Hence, HFTs lose more than slow traders when α goes up because removing the search cost exacerbates the relative importance of α .

Proposition 2. *An increase in the degree of market fragmentation (a decrease in λ) makes it optimal for the trader to invest earlier.*

This impacts the optimal threshold via its effect on the relative advantage of being fast; in particular, through its effect on $\Omega(r, \mu, \lambda)$. The relative advantage of being fast increases as the market becomes more fragmented and thus, it is optimal to invest sooner if the market level of fragmentation increases. This is because it becomes harder for the slow institution to find quotes quickly and owing to the search process, the more fragmented the market, the more they lag behind the HFTs in terms of trading returns, i.e.; $E^0[X^F] - E^0[X^S]$ increases as markets become more fragmented.

The effects of the remaining parameters, which are not particular to the HFT environment, but standard in such real option models, is stated in the following proposition.

Proposition 3. *The relative advantage from being fast, $\Omega(r, \mu, \lambda)$, decreases in r , increases in μ , and is invariant to σ .*

Proof. See Appendix C. ■

Since $\Omega(r, \mu, \lambda)$ is invariant to the volatility of the return process, σ affects the optimal investment threshold via its impact on the value of the option to invest (i.e., the value of waiting) only. This is measured by $\frac{\partial}{\partial \beta_1} \left(\frac{\beta_1}{\beta_1 - 1} \right) \frac{\partial \beta_1}{\partial \sigma} > 0$. Hence, the more volatile and uncertain is the return process, the longer it is optimal for the trader to wait before adopting because of the positive effect of σ on the value of the option to wait. This is a standard result in all real option models of this type.

On the other hand, r and μ impact the optimal time to invest via the option effect and via their impacts on the relative value of being fast, in other words, the net present value effect. In particular, for $\zeta \in \{r, \mu\}$

$$\frac{\partial X^*}{\partial \zeta} = -\frac{1}{(\beta_1 - 1)^2} \frac{\partial \beta_1}{\partial \zeta} \frac{I}{\Omega} + \left(-\frac{\beta_1}{\beta_1 - 1} \frac{\partial \Omega}{\partial \zeta} \frac{I}{\Omega^2} \right), \quad (15)$$

where the first term on the right hand side measures the option effect and the second term the present value effect. Note that $\partial \beta_1 / \partial r > 0$ and $\partial \beta_1 / \partial \mu < 0$, implying that the option effect of r and μ on X^* is, respectively, negative and positive. However, the present value effect of r and μ on X^* are respectively positive and negative since, from Proposition 3, $\partial \Omega / \partial r < 0$ and $\partial \Omega / \partial \mu > 0$. Therefore, these two competing forces make knife-edge comparative static results of the effects of r and μ on X^* difficult to obtain. However, Figs. 1 and 2 below show that for reasonable parameter values, X^* increases in r and decreases in μ . Therefore, for both parameters, their effect on the optimal time to invest is driven by their effects on the relative value of being fast. A high discount rate implies a low present value of the relative advantage of being fast which discourages immediate investment in the HFT. On the other hand, a high μ implies a high relative advantage of being fast in present value terms, which means that early adoption is optimal.

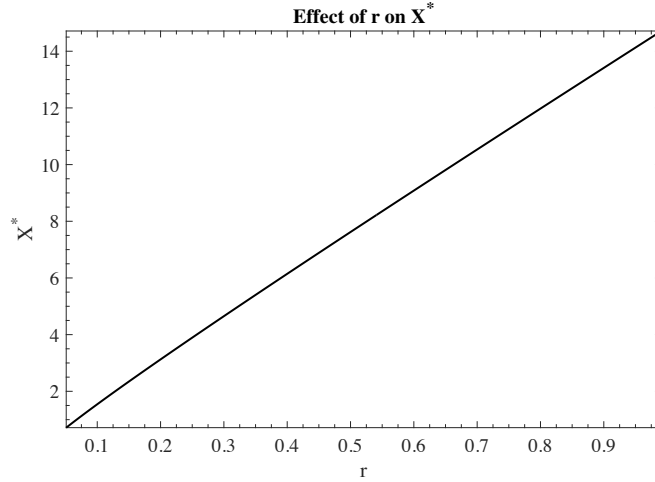


Figure 1: The following parameter values are used: $\mu = 0.05$, $\sigma = 0.2$, $\lambda = 0.25$, and $I = 10$.

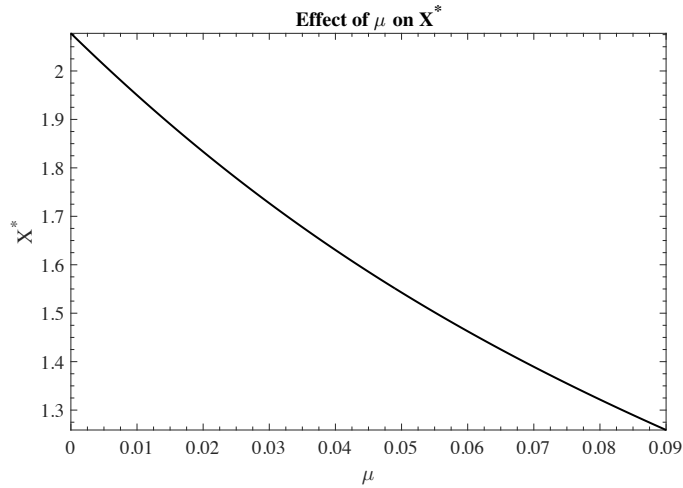


Figure 2: The following parameter values are used: $r = 0.1$, $\sigma = 0.2$, $\lambda = 0.25$, and $I = 10$.

4 Levels of Fast Trading

In this section the market equilibrium level of fast trading is determined by assuming that all traders adhere to the optimal investment strategy derived in Section 3. I analyse how this level compares with the market socially optimal level of fast trading, and also with the equilibrium level of fast trading by assuming all traders evaluate the investment as a “now or never” decision; i.e., such that the value of waiting is zero. This is referred to as the net present value (NPV) approach to investment. We can then determine how the value of waiting impacts on the equilibrium level of investment and whether the

equilibrium level under the options approach is more or less socially optimal than under the NPV approach.

Denote the ex-ante expected returns of fast trading in the market (by the overall population of HFTs) and slow trading (by the overall slow trader population) by $V^f(\alpha)$ and $V^s(\alpha)$, respectively. Recall that a single trader's initial return is $X_0 = (1 + \alpha)^{-1}x$. For clarity, I adapt the notation slightly here. Let a single trader i 's initial return be denoted by $X_0^i = (1 + \alpha)^{-1}x^i$. If $x^i < (1 + \alpha)X^*$, then i is a slow trader, and otherwise he is fast (see Theorem 1). There are a total of N traders in the market such that $N \in [1, \infty)$ and x^i , for $i = \{1, \dots, N\}$, is i.i.d. across traders and is continuously distributed on $[0, \bar{x}]$, where $\bar{x} > (1 + \alpha)X^*$, and has density function $g(\cdot)$. Hereafter in this section (and in the accompanying appendices) I drop the i superscript for convenience, but x is equivalent to x^i .

Therefore

$$V^f(\alpha) = \int_{(1+\alpha)X^*}^{\bar{x}} \frac{(1+\alpha)^{-1}x}{\delta} g^f(x) dx \quad (16)$$

and

$$V^s(\alpha) = \int_0^{(1+\alpha)X^*} \left(\frac{\lambda(1+\alpha)^{-1}x}{(r - \ln((1-\lambda) + \lambda e^\mu))} + \left(\frac{(1+\alpha)^{-1}x}{X^*} \right)^{\beta_1} F(X^*) \right) g^s(x) dx, \quad (17)$$

where $g^f(\cdot)$ and $g^s(\cdot)$ are the scaled versions of the density function pertaining to fast and slow trading, respectively.

The equilibrium level of fast trading α^* is the level at which traders are indifferent between being fast and slow; i.e.,

$$V^s(\alpha^*) = V^f(\alpha^*) - I. \quad (18)$$

I denote the relative advantage of fast trading in the market by $\Delta(\alpha)$, where

$$\Delta(\alpha) := V^f(\alpha) - V^s(\alpha).$$

The socially optimal level of fast trading is the level of α that maximises utilitarian welfare, where utilitarian welfare is given by

$$\begin{aligned} W(\alpha) &= \alpha (V^f(\alpha) - I) + (1 - \alpha)V^s(\alpha) \\ &= \alpha (\Delta(\alpha) - I) + V^s(\alpha). \end{aligned} \tag{19}$$

Denoting the socially optimal level of fast trading by α^{SO} , then

$$W'(\alpha^{SO}) = 0, \tag{20}$$

where $W'(\alpha^{SO}) = \frac{\partial W(\alpha)}{\partial \alpha} \Big|_{\alpha=\alpha^{SO}}$.

To ascertain these levels of fast trading for traders whose investment decision is a now or never one, X^* is replaced by X_{NPV} in the above analysis, such that

$$F(X_{NPV}) = 0$$

for $F(\cdot)$ given by (9).

Before proceeding with the results, I explain the notation I will use in the remainder of the section. X^* and X_{NPV} denote the investment thresholds in the real options case and the NPV case, respectively. α_{RO}^* and α_{NPV}^* denote the equilibrium levels of fast trading in the options and the NPV cases, respectively. α^* without a subscript implies that I refer to the equilibrium level of fast trading in either/both cases. Whenever I replace the $*$ with the SO superscript, this means I refer to the socially optimal level of the parameter rather than the equilibrium level. Finally, $\Delta^{RO}(\cdot)$ and $\Delta^{NPV}(\cdot)$ refer to the relative advantage of being fast in the respective options and NPV case, while $\Delta(\cdot)$ implies I mean in both cases.

Proposition 4. *There is a unique equilibrium level of fast trading under both the options approach and the NPV approach. The level in both cases is such that (i) $\alpha^* = 0$ if $\Delta(0) < I$; (ii) $\alpha^* = 1$ if $\Delta(1) > I$; and (iii) $0 < \alpha^* < 1$ if $\Delta(1) < I < \Delta(0)$.*

Proof. See Appendix D. ■

This result that there is *always* a unique equilibrium level of HFT is novel. In Biais et al. [2] multiple equilibria can arise because the relative value of being fast, at market level, is not decreasing in α for some values of α . The difference between the two models which drives the result is as follows. In this model, $V^s(\alpha)$ increases in α (this is shown in Appendix D), whereas in their paper it decreases. In both models, the value from being slow decreases in α for an individual trader. However, this paper examines how the optimal policy I derive influences the equilibrium level of investment in the market. The *cumulative* value from being slow over the entire population of slow traders increases in α because the higher the level of α , the more traders will be discouraged from investing (cf. Proposition 1). Therefore, while the returns of individual slow traders decrease with an increase in α , there will be more of them in the market owing to the adherence to the optimal policy. The cumulative effect is that the expected value of slow trading in the market increases in α . However, in the model of Biais et al. [2], the cumulative effect and the individual effects are equivalent because the range of integration over all slow traders is unaffected by α . In both models, the expected value from being fast in the market decreases in α (see Appendix D). Moreover, in both models the relative advantage of being fast at market level is defined by $\Delta(\alpha) := V^f(\alpha) - V^s(\alpha)$ (but in Biais et al. [2] their $V^f(\alpha)$ and $V^s(\alpha)$ are defined differently to here). Hence, clearly, it must always hold in this model that $\Delta(\alpha)$ decreases in α . However, since $\partial V^s(\alpha)/\partial\alpha < 0$ in Biais et al. [2], $\Delta(\alpha)$ will increase in α , leading to multiple equilibria, if slow traders lose more than fast traders when α increases.

Proposition 5. *In both the options case and the NPV case, the equilibrium level of fast trading increases in the degree of market fragmentation.*

Proof. See Appendix E. ■

This result is intuitive and the pathway leading to the result is as follows. An increase in the level of market fragmentation leads to an increase in the expected present value from investing, and this makes it optimal for the traders to adopt the technology sooner, i.e., X^* is low (cf. Proposition 2). When X^* is low, $V^f(\alpha)$ and $V^s(\alpha)$ are, respectively,

high and low because the range of integration is, respectively, wide and narrow. Therefore, $\Delta(\alpha)$ is high and, presumably, higher than I . Since $\Delta(\alpha)$ decreases in α (see Appendix D), the equilibrium level of α required so that $\Delta(\alpha) \downarrow I$ must be high.

Corollary 1. *The equilibrium level of fast trading under the real options approach is never higher than under the NPV approach.*

The reasoning for the above corollary is as follows. When $\Delta^{RO}(1) - I > 0$ and $\Delta^{NPV}(1) - I > 0$, then $\alpha_{RO}^* = \alpha_{NPV}^* = 1$. But, owing to the value of waiting, for any given value of λ , $\Delta^{NPV}(1) > \Delta^{RO}(1)$ which implies that as λ increases, $(\Delta^{RO}(\alpha) - I) \downarrow 0$ faster than $(\Delta^{NPV}(\alpha) - I) \downarrow 0$. Hence, α_{RO}^* takes an interior value while α_{NPV}^* is still unity. Since both $\Delta^{RO}(\alpha)$ and $\Delta^{NPV}(\alpha)$ both decrease in λ (see Appendix E), if α_{NPV}^* takes an interior value, it will never be lower than α_{RO}^* . Furthermore, this also implies that $\alpha_{RO}^* = 0$ before $\alpha_{NPV}^* = 0$. However, the result is also intuitive because it always holds that $X_{NPV} < X^*$ owing to the value of waiting. Hence, under the NPV case, more traders will have adopted the technology than under the option case implying that $\alpha_{RO}^* < \alpha_{NPV}^*$. Therefore, accounting for the option to invest in evaluating the optimal investment strategy reduces the market equilibrium level of investment in the HFT technology.

Proposition 6. *When the market is highly fragmented, the equilibrium level of fast trading is never lower than the socially optimal level. However, for sufficiently low levels of market fragmentation, the equilibrium level will be lower than the socially optimal level. Moreover, there is a market level of fragmentation in the interior region such that the equilibrium level and the socially optimal level coincide. This is true in both the options case and the NPV case.*

Proof. See Appendix F. ■

The result that there is a level of market fragmentation such that the two levels coincide in the interior region is novel. In the model of Biais et al. [2], there is always an excess of investment in equilibrium, relative to what is socially optimal. The reason there

is a level such that they coincide in this paper is owing to the fact that $V^s(\alpha)$ increases in α , as discussed previously. A clear explanation for why this is the reason is provided in Appendix F, so I do not repeat it here.

The following follows trivially from the proof of Proposition 6:

Corollary 2. *The level of market fragmentation at which they coincide (in both cases) is the level of λ such that the following is satisfied:*

$$\frac{\partial V^f(\alpha)}{\partial V^s(\alpha)} = 1 - \alpha^{-1} \quad (21)$$

where $\alpha = \alpha^* = \alpha^{SO}$.

Figs. 3 and 4 depicts the result stated in Proposition 6 for the real options case and the NPV case, respectively. I assume the following parameter values hold: $r = 0.05$, $\mu = 0.02$, $\sigma = 0.2$, $I = 10$, and $\bar{x} = 12$. Moreover, I assume that x follows a uniform distribution on $[0, \bar{x}]$. Hence, appropriate scaling of the density function gives

$$g^s(x) = \frac{1}{\bar{x}} \left[\frac{\bar{x} - \phi(\bar{x} - (1 + \alpha)X^*)}{(1 + \alpha)X^*} \right]$$

and

$$g^f(x) = \frac{\phi}{\bar{x}},$$

for $\phi \in \left[1, \frac{\bar{x}}{\bar{x} - (1 + \alpha)X^*} \right]$. In the example, I let $\phi = 1.5$.

Corollary 3. *The real options approach to investment yields an equilibrium level of HFT which is less socially optimal than the equilibrium level obtained by adherence to the NPV approach.*

It is clear from Figs. 3 and 4 that α_{NPV}^* and α_{NPV}^{SO} are more closely aligned with each other than α_{RO}^* and α_{RO}^{SO} are. The reason why this arises in the model is as follows.

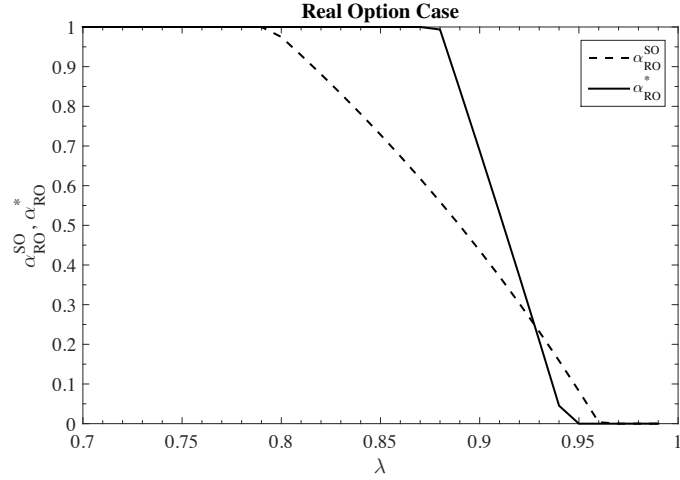


Figure 3: Effect of λ on α_{RO}^* and α_{RO}^{SO} .

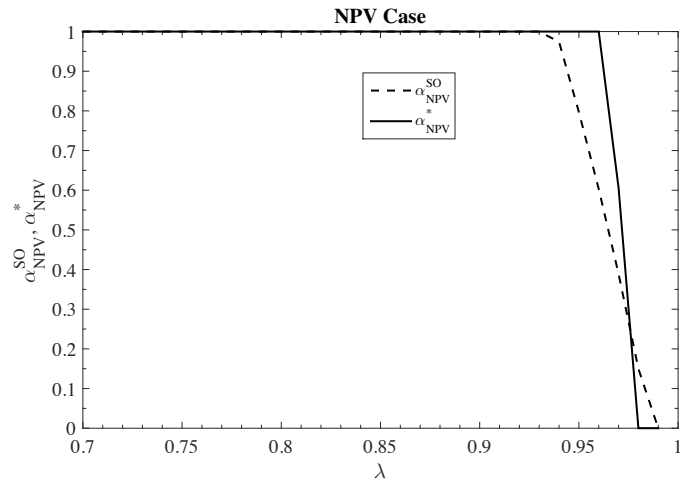


Figure 4: Effect of λ on α_{NPV}^* and α_{NPV}^{SO} .

The magnitude of social cost of HFT, in both cases, is captured by

$$\alpha^{SO} \left| \frac{\partial V^f(\alpha)}{\partial \alpha} \Big|_{\alpha=\alpha^{SO}} \right| + (1 - \alpha^{SO}) \frac{\partial V^s(\alpha)}{\partial \alpha} \Big|_{\alpha=\alpha^{SO}} > 0. \quad (22)$$

since this value is zero if $\alpha^* = \alpha^{SO}$ (cf. Appendix F).

In the real options case, the value of Eq. (22) incorporates the value of waiting, but in the NPV case, the value of waiting is zero. The value of waiting is impacted by α , and therefore, if we expand Eq. (22) more fully, the real options case would have an additional term that would be missing from the NPV case, owing to the value of waiting.

This additional term is primarily captured by (cf. Eq. (D.1))

$$(1 - \alpha_{RO}^{SO}) \left[X^* g^s((1 + \alpha_{RO}^{SO})X^*) - \beta_1 (1 + \alpha_{RO}^{SO})^{-\beta_1 - 1} (X^*)^{-\beta_1} \int_0^{(1 + \alpha_{RO}^{SO})X^*} x^{\beta_1} g^s(x) dx \right] F(X^*), \quad (23)$$

which, as I show in Appendix D, is positive.⁴ Hence, the value of Eq. (22) in the real options case is greater than in the NPV case implying that the social cost of HFT is greater when the value of waiting is accounted for.

4.1 Implications for Policy

If we consider the case where both the equilibrium level of HFT and the socially optimal level take interior values, it is true that the more fragmented the market, the higher is the social cost of HFT. Hence, the implications for policy arising from the results in this paper is that it should be aimed at reducing market fragmentation. However, eliminating market fragmentation completely is also not optimal (as evident from Fig. 3 where the social cost is zero (i.e., the levels coincide) for some, say $\hat{\lambda} < 1$, but is greater than zero again for $\lambda > \hat{\lambda}$.)

The debate surrounding market fragmentation is broad and multi-faceted. On one hand, according to O’Hara and Ye [16], “market fragmentation has increased competition and thereby reduced the trading charges and fees typically imposed by traditional exchanges. Moreover, it has fostered innovations such as greater latency and more sophisticated crossing networks by providing a wealth of trading options to the trading community. However, a large concern in the debate is how market fragmentation affects market quality via its effect on liquidity”. They (O’Hara and Ye [16]) investigate this concern by examining fragmentation in US equity markets and they conclude that market fragmentation does not appear to harm market quality, and that this should influence regulatory issues relating to market fragmentation.

⁴Eq. (23) does not represent the entire difference between the cost in the real options case and the NPV case because $X^* > X_{NPV}$ implying that the ranges of integration for $V^f(\alpha)$ and $V^s(\alpha)$ are different, but the difference owing to that is small relative to the difference captured by (23).

On the other hand, to cope with market fragmentation, there has been a surge in HFT activity of late. This induces a search cost on traders without access to the expensive technology and, as I show in this paper, high levels of market fragmentation is not in the interest of the trading community as a whole. This is because the relative advantage of being fast increases in the level of market fragmentation which implies that at market level, the social cost induced by this effect is high when markets are highly fragmented. Hence, the result in this paper suggests that regulatory issues surrounding market fragmentation should also be concerned with keeping abreast of the resulting market practices that emerge to cope with fragmentation, and the cost these practices are imposing both on traders with access to the technology and on those without. If this cost is too high, regulatory policy should be aimed at reducing the level of market fragmentation and thereby lessen the need for traders to adopt these costly practices in response.

5 Conclusion

In recent years, the state of market microstructure has changed considerably. There are many ways in which these changes have come about (see O'Hara [15] for a detailed description), but one of the biggest changes is that markets have become highly fragmented. When markets are fragmented, traders must search across many markets for venues which will execute their orders at their specified prices. This can result in delayed or partial execution which is costly. In response to the increase in market fragmentation, there has been a demand for speed by traders, and various types of expensive technologies have been developed. Such technologies enable traders to compare all trading venue instantaneously or obtain a glimpse of the true state of the market before everyone else.

In this paper I derive a dynamic model, using techniques from real options analysis, which provides an optimal timing strategy for slow traders considering investment in a high frequency trading technology. The model prescribes waiting longer to invest if the level of high frequency trading in the market increases, and it prescribes earlier adoption

if the market level of fragmentation increases.

I then assume that the optimal timing strategy derived for a single trader is adhered to by all traders in the market, and thereby derive the equilibrium level of fast trading in the market as well as the socially optimal level. I also derive these same levels of fast trading by assuming all traders adhere to the classical net present value rule of investment such that the value of waiting is zero. From this analysis, the following results emerge. There is always a unique equilibrium level of fast trading in the market, and this level increases in the degree of market fragmentation. There is also a unique level of market fragmentation such that the equilibrium level and the socially optimal level of fast trading coincide. These results pertain to both the options and the NPV case. However, I also show that incorporating the value of waiting into the evaluation of the optimal investment strategy leads to a lower market equilibrium level of HFT investment and, moreover, this equilibrium level is less socially optimal than if the value of waiting is not accounted for.

Finally, in light of the results, I discuss some potential implications for regulatory policy in this area.

Appendix

A Proof that $\Omega(r, \mu, \lambda) > 0$

$$\Omega(r, \mu, \lambda) = \frac{1}{\delta} - \frac{\lambda}{(r - \ln((1 - \lambda) + \lambda e^\mu))} > 0 \quad (\text{A.1})$$

If $\lambda = 1$, then $\Omega(\cdot) = 0$. Hence, if $\Omega(\cdot)$ decreases in λ , then it must be positive.

$$\frac{\partial \Omega(\cdot)}{\partial \lambda} < 0 \iff r - \ln((1 - \lambda) + \lambda e^\mu) + \frac{\lambda e^\mu}{(1 - \lambda) + \lambda e^\mu} > 0. \quad (\text{A.2})$$

This is true if $r - \ln((1 - \lambda) + \lambda e^\mu) > 0$.

Once again, if $\lambda = 1$, then the latter expression becomes δ which is positive by

assumption. If the expression decreases in λ , then it is positive and the result for $\Omega(r, \mu, \lambda)$ is proved. ■

$$\frac{\partial}{\partial \lambda} (r - \ln((1 - \lambda) + \lambda e^\mu)) = \frac{1 - e^\mu}{(1 - \lambda) + \lambda e^\mu} < 0.$$

B Proof of Theorem 1

The derivation of the optimal threshold uses well-developed standard techniques from real options theory (see, for example, Dixit and Pindyck [4]). Hence, to preserve space, I provide a only brief sketch of the derivation here.

Once the trader invests in the technology, he obtains a flow of returns of $X^F \equiv X$ forever. Thus, the net present value from investing is given by

$$V^A(X_0) = E^0 \left[\int_0^\infty e^{-rt} X_t dt \right] - I = \frac{X_0}{\delta} - I, \quad (\text{B.1})$$

where $V^A(X_0)$ indicates that it is the current expected value obtained after investing.

Prior to investing, the trader obtains a flow of returns X^S , as well as having the option to invest. Letting $V^B(X_0)$ denote the current value to the trader before investing, using standard arguments from Dixit and Pindyck [4], $V^B(X_0)$ solves

$$\frac{1}{2} \sigma^2 X_0^2 (V^B)''(X_0) + (r - \delta) X_0 (V^B)'(X_0) - r V^B(X_0) + X^S(X_0) = 0. \quad (\text{B.2})$$

It is usual in these models that the value to the trader before investing is comprised of the expected discounted payoff if he were never to invest, corrected for the fact that he has the option to invest should his expected return become sufficiently large to warrant the fixed investment cost. The former value is given by (cf. Eq. (4))

$$\begin{aligned} E^0 \left[\int_0^\infty e^{-rt} X_t^S dt \right] &= \lambda X_0 \int_0^\infty e^{-rt} ((1 - \lambda) + \lambda e^\mu)^t dt \\ &= \frac{\lambda X_0}{r - \ln((1 - \lambda) + \lambda e^\mu)}. \end{aligned} \quad (\text{B.3})$$

In the search procedure, we did not obtain an expression for X_t^S directly. However, the value of X_t^S for which Eq. (B.3) holds is as follows:

$$X_t^S = \frac{\lambda}{r - \ln((1 - \lambda) + \lambda e^\mu)} \delta X_t. \quad (\text{B.4})$$

Therefore, the following expression satisfies equation (B.2)

$$V^B(X_0) = A_1(X_0)^{\beta_1} + A_2(X_0)^{\beta_2} + \frac{\lambda X_0}{r - \ln((1 - \lambda) + \lambda e^\mu)}, \quad (\text{B.5})$$

where A_1 and A_2 are constants, and $\beta_1 > 1$ and $\beta_2 < 0$ are the roots of the following quadratic equation

$$\frac{1}{2} \sigma^2 \beta(\beta - 1) + (r - \delta)\beta - r = 0.$$

Hence, the value to the trader before investing is comprised of the expected discounted trading return if he were never to invest, corrected for the fact that he has the option to invest should his expected return become sufficiently large to warrant the fixed investment cost.

Applying the following boundary, value-matching, and smooth-pasting conditions yields the required result for X^* :

$$V^B(0) = 0; \quad V^B(X^*) = V^A(X^*); \quad \text{and} \quad (V^B)'(X^*) = (V^A)'(X^*). \quad (\text{B.6})$$

Finally, since $X_0 := (1 + \alpha)^{-1}x$, the result for the value function which solves the optimal stopping problem can be verified. ■

C Proof of Proposition 3

$$\frac{\partial \Omega(r, \mu, \lambda)}{\partial r} > 0 \iff \lambda \delta^2 - (r - \ln((1 - \lambda) + \lambda e^\mu))^2 > 0. \quad (\text{C.1})$$

If $\lambda = 1$, then $\partial \Omega / \partial r = 0$. However, if $[\lambda \delta^2 - (r - \ln((1 - \lambda) + \lambda e^\mu))^2]$ decreases in λ ,

then the condition must hold.

$$\begin{aligned} \frac{\partial}{\partial \lambda} [\lambda \delta^2 - (r - \ln((1 - \lambda) + \lambda e^\mu))^2] &< 0 \\ \iff \delta^2 + 2(e^\mu - 1) \frac{(r - \ln((1 - \lambda) + \lambda e^\mu))}{1 - \lambda + \lambda e^\mu} &< 0, \end{aligned} \quad (C.2)$$

which does not hold. Therefore, $\Omega(r, \mu, \lambda)$ decreases in r .

$$\frac{\partial \Omega(r, \mu, \lambda)}{\partial \mu} > 0 \iff (1 - \lambda + \lambda e^\mu) (r - \ln((1 - \lambda) + \lambda e^\mu))^2 - \delta^2 \lambda^2 e^\mu > 0. \quad (C.3)$$

If $\lambda = 1$, then $\partial \Omega / \partial \mu = 0$. However, if $[(1 - \lambda + \lambda e^\mu) (r - \ln((1 - \lambda) + \lambda e^\mu))^2 - \delta^2 \lambda^2 e^\mu]$ decreases in λ , the condition in Eq. (C.3) holds.

$$\begin{aligned} \frac{\partial}{\partial \lambda} [(1 - \lambda + \lambda e^\mu) (r - \ln((1 - \lambda) + \lambda e^\mu))^2 - \delta^2 \lambda^2 e^\mu] &< 0 \\ \iff (e^\mu - 1) (r - \ln((1 - \lambda) + \lambda e^\mu)) [r - \ln((1 - \lambda) + \lambda e^\mu) - 2] - 2\lambda \delta^2 e^\mu &< 0, \end{aligned} \quad (C.4)$$

which holds since $r < 1$. Therefore, $\Omega(r, \mu, \lambda)$ increases in μ . ■

D Proof of Proposition 4

First I use Leibniz rule for differentiating integrals to show that $\Delta(\alpha)$ decreases in α . I prove it holds for the option case and the NPV case follows trivially. Moreover, for ease

of exposition, I drop the “ RO ” notation throughout the proof.

$$\begin{aligned}
\frac{\partial \Delta(\alpha)}{\partial \alpha} &= \frac{\partial V^f(\alpha)}{\partial \alpha} - \frac{\partial V^s(\alpha)}{\partial \alpha} \\
&= -\frac{1}{\delta(1+\alpha)^2} \int_{(1+\alpha)X^*}^{\bar{x}(\alpha)} x g^f(x) dx - \frac{(X^*)^2}{\delta} g^f((1+\alpha)X^*) \\
&\quad - \left[-\frac{\lambda}{(1+\alpha)^2 (r - \ln((1-\lambda) + \lambda e^\mu))} \int_0^{(1+\alpha)X^*} x g^s(x) dx \right. \\
&\quad + \frac{\lambda(X^*)^2}{(r - \ln((1-\lambda) + \lambda e^\mu))} g^s((1+\alpha)X^*) \\
&\quad - \beta_1(1+\alpha)^{-\beta_1-1} (X^*)^{-\beta_1} F(X^*) \int_0^{(1+\alpha)X^*} x^{\beta_1} g^s(x) dx \\
&\quad \left. + X^* F(X^*) g^s((1+\alpha)X^*) \right], \tag{D.1}
\end{aligned}$$

where the term in the square brackets is $\partial V^s(\alpha)/\partial \alpha$.

It is clear from the equation that $\partial V^f(\alpha)/\partial \alpha < 0$. Then it is clear that $\partial \Delta(\alpha)/\partial \alpha < 0$ if $\partial V^s(\alpha)/\partial \alpha > 0$.

$\partial V^s(\alpha)/\partial \alpha > 0$ if

$$(X^*)^2 g^s((1+\alpha)X^*) > \frac{1}{(1+\alpha)^2} \int_0^{(1+\alpha)X^*} x g^s(x) dx \tag{D.2}$$

and if

$$X^* g^s((1+\alpha)X^*) > \beta_1(1+\alpha)^{-\beta_1-1} (X^*)^{-\beta_1} \int_0^{(1+\alpha)X^*} x^{\beta_1} g^s(x) dx. \tag{D.3}$$

Using the integration by parts technique to evaluate the integrals in the latter two equations, it is sufficient to approximate the integrals using only the “ uv ” term in the standard formula to show the conditions hold (since the other term is negative anyway). Letting $u = g^s(x)$ and $dv = x dx$ or $dv = x^{\beta_1} dx$, then $[uv]_0^{(1+\alpha)X^*} = \frac{1}{2}(1+\alpha)^2 (X^*)^2 g^s((1+\alpha)X^*)$ for Eq. (D.2), and $[uv]_0^{(1+\alpha)X^*} = \frac{1}{\beta_1+1} (1+\alpha)^{\beta_1+1} (X^*)^{\beta_1+1} g^s((1+\alpha)X^*)$ for (D.3).

Eq. (D.2) becomes

$$(X^*)^2 g^s((1+\alpha)X^*) > \frac{1}{2} (X^*)^2 g^s((1+\alpha)X^*), \tag{D.4}$$

which clearly holds, and Eq. (D.3) becomes

$$X^*g^s((1 + \alpha)X^*) > \frac{\beta_1}{\beta_1 + 1}X^*g^s((1 + \alpha)X^*), \quad (\text{D.5})$$

which also clearly holds.

Therefore, $\partial V^s(\alpha)/\partial \alpha > 0$ and $\Delta(\alpha)$ decreases in α .

From this, it is clear then that the equilibrium condition

$$\Delta(\alpha^*) - I = 0 \quad (\text{D.6})$$

is satisfied if $\Delta(0) - I > 0$ and $\Delta(1) - I < 0$ so that there is a unique zero point. If both are true, then the condition must hold for some $0 < \alpha^* < 1$.

If $\Delta(0) - I < 0$, then the condition can only possibly hold for some negative value of α . But, since this is implausible, the equilibrium condition must be that $\alpha^* = 0$ whenever $\Delta(0) - I < 0$.

On the other hand, if $\Delta(1) - I > 0$, then the condition can only hold for some α greater than unity. Again, this is implausible, so the only equilibrium that can hold is $\alpha^* = 1$ when $\Delta(1) - I > 0$. ■

E Proof of Proposition 5

I prove this result for the option case (but drop the “*RO*” notation for ease) and the result for the NPV case follows trivially.

First I determine how the relative advantage of being fast in the market is affected by λ .

$$\frac{\partial \Delta}{\partial \lambda} = \frac{\partial V^f}{\partial \lambda} - \frac{\partial V^s}{\partial \lambda}. \quad (\text{E.1})$$

We use Liebniz rule once more and recall that $\partial X^*/\partial\lambda > 0$:

$$\begin{aligned}
\frac{\partial\Delta}{\partial\lambda} = & - (1 + \alpha) \frac{\partial X^*}{\partial\lambda} \frac{X^*}{\delta} g^f((1 + \alpha)X^*) \\
& - \left(\int_0^{(1+\alpha)X^*} \left[(1 + \alpha)^{-1} \frac{x}{(r - \ln((1 - \lambda) + \lambda e^\mu))^2} \left(r - \ln((1 - \lambda) + \lambda e^\mu) \right. \right. \right. \\
& \left. \left. \left. + \frac{\lambda(e^\mu - 1)}{(1 - \lambda) + \lambda e^\mu} \right) - \beta_1 ((1 + \alpha)^{-1}x)^{\beta_1} (X^*)^{-\beta_1 - 1} \frac{\partial X^*}{\partial\lambda} F(X^*) \right] g^s(x) dx \right. \\
& \left. + (1 + \alpha) \frac{\partial X^*}{\partial\lambda} \left[\frac{\lambda X^*}{(r - \ln((1 - \lambda) + \lambda e^\mu))} + F(X^*) \right] g^s((1 + \alpha)X^*) \right). \tag{E.2}
\end{aligned}$$

This expression is definitely negative if

$$\begin{aligned}
& (1 + \alpha) \frac{\partial X^*}{\partial\lambda} \left[\frac{\lambda X^*}{(r - \ln((1 - \lambda) + \lambda e^\mu))} + F(X^*) \right] g^s((1 + \alpha)X^*) \\
& > \beta_1 (1 + \alpha)^{-\beta_1} (X^*)^{-\beta_1 - 1} \frac{\partial X^*}{\partial\lambda} F(X^*) \int_0^{(1+\alpha)X^*} x^{\beta_1} g^s((1 + \alpha)x) dx. \tag{E.3}
\end{aligned}$$

As in Appendix D, it is sufficient to approximate $\int_0^{(1+\alpha)X^*} x^{\beta_1} g^s((1 + \alpha)x) dx$ by $(\beta_1 + 1)^{-1} (1 + \alpha)^{\beta_1} (X^*)^{\beta_1} g^s((1 + \alpha)X^*)$. Replacing for this in the latter equation gives

$$\frac{\lambda(1 + \alpha)(X^*)^2}{(r - \ln((1 - \lambda) + \lambda e^\mu))} > \left(\frac{\beta_1}{\beta_1 + 1} - (1 + \alpha)X^* \right) F(X^*), \tag{E.4}$$

which clearly holds since the expression in the brackets on the right-hand side of the equation is negative. Therefore, this implies that $\Delta(\alpha)$ *decreases* in λ .

Say $\lambda = 1$. Then $X^* = \infty$ and $V^f(\alpha) = 0$. Hence $\Delta(\alpha) - I < 0$ for any α . From Proposition 4, the only equilibrium where this holds is when $\alpha^* = 0$. If λ decreases, then X^* decreases and $\Delta(\alpha) - I$ tends towards positive values. This can only result for positive α^* . Therefore, α^* *decreases* in λ which gives the result that α^* increases in the degree of market fragmentation $1 - \lambda$. ■

F Proof of Proposition 6

Once again I prove the result for the options case (dropping the “RO” notation) and the NPV result follows trivially.

First, from Eqs. (19) and (20), the socially optimal level of fast trading in the market solves

$$\Delta(\alpha^{SO}) - I + \alpha^{SO} \frac{\partial V^f(\alpha)}{\partial \alpha} \Big|_{\alpha=\alpha^{SO}} + (1 - \alpha^{SO}) \frac{\partial V^s(\alpha)}{\partial \alpha} \Big|_{\alpha=\alpha^{SO}} = 0. \quad (\text{F.1})$$

Say $\alpha^* = \alpha^{SO} = 1$. Then Eq. (F.1) becomes

$$\Delta(1) - I + \frac{\partial V^f(\alpha)}{\partial \alpha} \Big|_{\alpha=1} \geq 0$$

and the equilibrium condition is

$$\Delta(1) - I > 0.$$

For $\alpha^* = \alpha^{SO} = 1$, $\lambda \approx 0$. However, as λ increases, the $\Delta(\alpha^*) - I$ will move towards the interior equilibrium region (i.e., towards zero) and so too will $\Delta(\alpha^{SO}) - I$ (cf. Appendix E). But $\partial V^f(\alpha)/\partial \alpha < 0$ (cf. Appendix D) which implies that the socially optimal level condition will get to the interior region faster and that α^{SO} will be less than unity before α^* .

On the other hand, if $\alpha^* = \alpha^{SO} = 0$ (so $\lambda \approx 1$), then the equilibrium condition is

$$\Delta(0) - I < 0$$

and the socially optimal condition is

$$\Delta(0) - I + \frac{\partial V^s(\alpha)}{\partial \alpha} \Big|_{\alpha=0} \leq 0,$$

since $\partial V^s(\alpha)/\partial \alpha > 0$ (cf. Appendix D). This implies that the equilibrium level will move out of the interior region faster than the socially optimal level implying $\alpha^* = 0$ and $\alpha^{SO} > 0$ for some high values of λ .

Finally in the interior region, where $0 < \alpha < 1$, α^* solves

$$\Delta(\alpha^*) - I = 0 \quad (\text{F.2})$$

and α^{SO} solves Eq. (F.1). Clearly, the levels of α would coincide if

$$\alpha^{SO} \frac{\partial V^f(\alpha)}{\partial \alpha} \Big|_{\alpha=\alpha^{SO}} + (1 - \alpha^{SO}) \frac{\partial V^s(\alpha)}{\partial \alpha} \Big|_{\alpha=\alpha^{SO}} = 0. \quad (\text{F.3})$$

This is possible because $\frac{\partial V^f(\alpha)}{\partial \alpha} < 0$ and $\frac{\partial V^s(\alpha)}{\partial \alpha} > 0$ (see Appendix D). For low values of λ , $V^f(\alpha)$ is high relative to $V^s(\alpha)$. Therefore, the overall value of (F.3) will be negative. However, as λ increases, $V^f(\alpha)$ gets smaller relative to $V^s(\alpha)$ and, hence, for λ sufficiently high, the expression in (F.3) will be positive. Therefore, there must be some level of λ such that the expression is zero. ■

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