International Spillovers of Monetary Policy to Asset Prices*

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Abstract

We study a two-country general equilibrium model about how monetary policy, particularly interest rate instrument, affects stock prices. Using standard macroeconomic framework, New Keynesian model under price rigidity, we explore optimal monetary policy between advanced and emerging countries. We are still building up the theory that leads to the following prediction: due to exchange rate peg, monetary policy of emerging country is largely affected by advanced country's action and thus her output and stock price have experienced weaker recovery path after crisis.

Keywords: Asset Pricing, Risk Premia, Exchange Rate, International Trade, Home bias, Monetary Policy, Interest rate rule

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1 Introduction

The central banks around the world have been conducting expansionary monetary policy, conventional or not, since global financial crisis in 2008. So far, the economic recovery in large economy bodies such as USA, EU, and China has been disappointing. In the end of 2015, the US Fed announced an interest rate hike. Two months later, while keeping the interest rate flat, Fed Chair Yellen indicated that "uncertainty over Chinese economic growth have raised risks to the U.S. economy," reported by *Reuters* on 11 February 2016.

The international spillover effect is complicated but it is indeed an important question to study. John Taylor used the chart provided by Norges Bank to illustrate this point. The major contributor to the policy decision is the variation in foreign interest rate. For an open economy, it is impossible to conduct monetary policy without taking into consideration the decisions in other countries although central banks are special as they are independent from the political forces. Implicitly, it seems like that central banks are coordinating to maintain low interest rate policy although in reality the central bankers are constrained when they choose the optimal policy rate. On the other hand, since central bank is the agency that manages the foreign currency reserves in one country, exchange rate intervention, as documented in BIS report, becomes a main instrument for many central banks as well. Large economy like Russia and China have used both monetary policy and exchange rate policy together to boost economic growth. Therefore, exchange rate becomes a channel to transmit the spillover effect of easing policy from one country to another. Meanwhile, such spillover effect has create some extra variations on the exchange rate dynamic and brings the world into a new currency war that leads to real consequences in several countries.

To better understand how monetary policy affects the economic growth with an international perspective, we study a two-tree model of pure exchange, general equilibrium, production economy. To start with, we build up a frictionless model without central banks' presence. In this economy, the source of risk is the stochastic technology shocks which shall follow different but perhaps correlated stochastic processes. In Home and Foreign country respectively, there is a representative household that can choose between work and consumption which is expressed as a consumption bundle of locally-produced and imported goods. Furthermore, there is a firm with linear technology. If representative household in Home country is lazy, inclining to work less and consume more, the output of Home goods will then be smaller. His demand of consumption goods can be supplied by Foreign goods but if Foreign household is lazy as well, then they will have to work harder to satisfy their consumption demands.

One extra parameter to determine the optimal import is the ratio of social planner's weights on each country or we can interpret that as the relative size of population in one country. Larger country tends to have larger consumption at equilibrium. Finally, together with market clearing condition, we can express the optimal labour used in production as a function of home bias, country size, and Frisch elasticity of labour supply. Larger per unit of labour supply (thus larger output) is related to smaller country size, more industrious labour, larger demands of locally-produced goods expressed as weighted-average of home bias and the extent of openness in the other country.

The real exchange rate in this economy is determined by three types of discrepancy across countries: technology, size of country and the aggregate demands of production goods. The real wage rate mainly depends on the technology one country has. The influence of exporting goods on real wage can be summarized by the openness of the other country and the exchange rate (terms of trade). Eventually, the social welfare is determined by technology, home bias, country size and Frisch elasticity of labour supply. If high-tech country is also relatively larger, aggregate social welfare is higher when home bias is low. In the case where both countries have equal size, the social welfare gains from technology and international trade and loses from the labour's laziness (Frisch elasticity).

Next, we introduce imperfect competition by assuming a continuum of monopolistic firms in each country. Equilibrium output is smaller in presence of monopolistic competition. However, the real wage markup over marginal product is not time-varying so the influence of the imperfection competition is absent in inter-temporal production. When we derive stochastic discount factor and thus nominal interest rate in both countries, only the exogenous rates of technology progress remain. The equilibrium nominal rate is composed of time discount rate, country-level inflation (price evolution) and home-bias-weighted average rates of technology progress. Furthermore, uncovered interest rate parity (UIP) holds in the current setup.

We also introduce price rigidity in both countries. Since our international trade market is frictionless, the Phillips Curves for both countries are equivalent to the closed economy model. On the other hand, the dynamic investment and saving curves demonstrate the importance of exchange rate on top of the firms' price rigidity. It is worth noting that exchange rate is embedded in the country-level inflation. When the interest rate rule is exogenously determined based on inflation rate and output gap, we can show that exchange rate has been implicitly considered in such monetary policy. However, central bank of an emerging countries would choose to pay more attention to exchange rate. The heterogeneity on interest rate rule specification across countries can give us some space to discuss the interaction of monetary policies and the impact on stock price dynamic in a more realistic environment. We aim to demonstrate with numerical result, which is still working in progress, of the influence from optimal international monetary policy on stock and foreign exchange markets.

2 The Baseline Model: Frictionless Real Economy

2.1 Households and Social Planner

We study a two-country model with production economy. The household utility functions are

$$U_1 = \log C_1 - \frac{N_1^{1+\varphi}}{1+\varphi} = \alpha \log C_{1,H} + (1-\alpha)\log C_{1,F} - \frac{N_1^{1+\varphi}}{1+\varphi}, \tag{1}$$

$$U_2 = \log C_2 - \frac{N_2^{1+\varphi}}{1+\varphi} = (1-\alpha) \log C_{2,H} + \alpha\log C_{2,F} - \frac{N_2^{1+\varphi}}{1+\varphi}. \tag{2}$$

The subscript $j = 1, 2$ indicates consumption by household j and N_j is labour hours supplied by household j. On the other hand, N_i is the labour demand by firms in each country i and the subscript $i = H, F$ indicates production by country i, i.e. home and foreign countries respectively. The home bias parameter is α and $(1-\alpha)$ indicates the openness of each country. The consumption bundle for each representative agent is expressed as Cobb-Douglas functions of domestic and imported goods. The Frisch elasticity φ is measuring how much agent is averse to work. We assume homogeneous elasticity of labour supply across countries.

There is a global social planner with weight η who has the following utility

$$U_s = \eta U_1 + (1-\eta)U_2, \tag{3}$$

which is contained the following market clearing conditions,

$$\begin{aligned} N_H =& N_1, N_F = N_2, \\ Y_H =& A_H N_H = C_{1,H} + C_{2,H}, \\ Y_F =& A_F N_F = C_{1,F} + C_{2,F}, \end{aligned}$$

where A_i represents technology used in country i. We shall think about the case of heterogeneous technology across countries. We interpret the social planner's weight as country size in terms of population. η is the fraction of global population living in Home country and $(1-\eta)$ in Foreign country.

To solve for the optimal consumption of different goods, $C_{1,H}, C_{2,H}, C_{1,F}, C_{2,F}$, first we have to write down the Lagrangian,

$$\mathcal{L} = U_s - \phi_H(C_{1,H} + C_{2,H} - A_H N_H) - \phi_F(C_{1,F} + C_{2,F} - A_F N_F), \tag{4}$$

where ϕ_i can be understood as the unit price of the tradable goods produced in country i.

2.2 Optimal consumption for Home-produced goods

To solve for $C_{1,H}$, we need to think about the trade-off with regards to work N_H,

$$N_H = \frac{1}{A_H}(C_{1,H} + C_{2,H}); \frac{\partial N_H}{\partial C_{1,H}} = \frac{1}{A_H}, \tag{5}$$

FOC w.r.t. $C_{1,H}$,

$$\eta\alpha C_{1,H}^{-1} - \eta N_H^{\varphi}\frac{1}{A_H} = 0. C_{1,H} = \alpha\frac{A_H}{N_H^{\varphi}}.$$

To solve for $C_{2,H}$, similarly, we are thinking about the trade-off between home production and export,

$$N_H = \frac{1}{A_H}(C_{1,H} + C_{2,H}); \frac{\partial N_H}{\partial C_{2,H}} = \frac{1}{A_H}. \tag{6}$$

FOC w.r.t. $C_{2,H}$,

$$-\eta N_H^{\varphi}\frac{1}{A_H} + (1-\eta)(1-\alpha)C_{2,H}^{-1} = 0,$$
$$C_{2,H} = (1-\alpha)\frac{1-\eta}{\eta}\frac{A_H}{N_H^{\varphi}}.$$

Next we solve for N_H. The corresponding FOC leads us to obtain the unit price of the home-produced goods.

$$-\eta N_H^{\varphi} + \phi_H A_H = 0,$$
$$\phi_H = \eta\frac{N_H^{\varphi}}{A_H}.$$

2.3 Optimal consumption for Foreigner-produced goods

Similarly, to solve for $C_{1,F}$, we need to use the market condition of foreign goods.

$$N_F = \frac{1}{A_F}(C_{1,F} + C_{2,F}); \frac{\partial N_F}{\partial C_{1,F}} = \frac{1}{A_F}.$$

FOC w.r.t. $C_{1,F}$ is

$$\eta(1-\alpha)C_{1,F}^{-1} - (1-\eta)N_F^{\varphi}/A_F = 0,$$
$$C_{1,F} = (1-\alpha)\frac{\eta}{1-\eta}\frac{A_F}{N_F^{\varphi}}.$$

Next we solve for $C_{2,F}$. Again, we are considering the trade-off between the foreign production and consumption,

$$N_F = \frac{1}{A_F}(C_{1,F} + C_{2,F}); \frac{\partial N_F}{\partial C_{2,F}} = \frac{1}{A_F}$$

FOC w.r.t. $C_{2,F}$ is

$$(1-\eta)\alpha C_{2,F}^{-1} - (1-\eta)\frac{N_F^{\varphi}}{A_F} = 0,$$
$$C_{2,F} = \alpha \frac{A_F}{N_F^{\varphi}}.$$

Now we solve for N_F. The corresponding FOC can help us to obtain the unit price of foreign goods,

$$-(1-\eta)N_F^{\varphi} + A_F \phi_F = 0,$$
$$\phi_F = \eta(1-\alpha)C_{1,F}^{-1} = (1-\eta)\frac{N_F^{\varphi}}{A_F}.$$

Lemma 2.1. *The unit prices of home/foreign goods are*

$$\phi_H = \eta \frac{N_H^{\varphi}}{A_H} = \frac{\eta}{A_H}\left[\alpha + (1-\alpha)\frac{1-\eta}{\eta}\right]^{\frac{\varphi}{1+\varphi}},$$
$$\phi_F = (1-\eta)\frac{N_F^{\varphi}}{A_F} = \frac{1-\eta}{A_F}\left[\alpha + (1-\alpha)\frac{\eta}{1-\eta}\right]^{\frac{\varphi}{1+\varphi}}.$$

Proposition 2.1. *In the frictionless production economy, the optimal quantities of tradable goods are functions of production factors (i.e. technology and labour), home bias and country size. Furthermore, tradable goods which are produced and consumed within the same country are independent of social planner's weight. Tradable goods which are produced and consumed in different countries depend on the ratio of social planner's weight between consumer and producer country. The optimal allocation for different tradable goods are summarized as following*

$$C_{1,H} = \alpha \frac{A_H}{N_H^{\varphi}};$$
$$C_{2,H} = (1-\alpha)\frac{1-\eta}{\eta}\frac{A_H}{N_H^{\varphi}};$$
$$C_{1,F} = (1-\alpha)\frac{\eta}{1-\eta}\frac{A_F}{N_F^{\varphi}};$$
$$C_{2,F} = \alpha \frac{A_F}{N_F^{\varphi}}.$$

From the optimal tradable goods, we can see the trade-off between consumption and work. Longer working hours mean less consumption. At equilibrium, regardless of the social planner's weight, the rep. household locally chooses labour supply and the tradable goods that are produced and consumed within the same country. The home bias parameter determines the fraction to be consumed locally or to be exported. The optimal tradable goods that are consumed and produced in different countries is adjusted by using social planner's weight. When two countries have different size, without loss of generality, say Home country is larger, imported goods in Home country shall be more than that in Foreign country. Intuitively, Home country has larger population. Therefore, their total consumption, expressed as representative household's consumption, is natural to be larger than Foreign country.

Lemma 2.2. *For each producer country $\{H, F\}$, the optimal allocation between tradable goods that are consumed within the same country and in different country (exports)depends on the ratio of her home bias relative to her counterpart's openness and relative country size,*

$$\begin{aligned}\frac{C_{1,H}}{C_{2,H}} &= \frac{\alpha}{1-\alpha}\frac{\eta}{1-\eta}, \\ \frac{C_{2,F}}{C_{1,F}} &= \frac{\alpha}{1-\alpha}\frac{1-\eta}{\eta}.\end{aligned}$$

The fraction to be consumed within the same country or in different country is determined by exogenous variables $\{\alpha, \eta\}$. The first component is the same across countries. It is the ratio between home bias of the producer economy and the openness of the consumer economy. The second component is the ratio of country sizes between producer and consumer economies so they are reciprocals of each other. The relatively larger country will have larger consumption share of tradable goods produced in both countries.

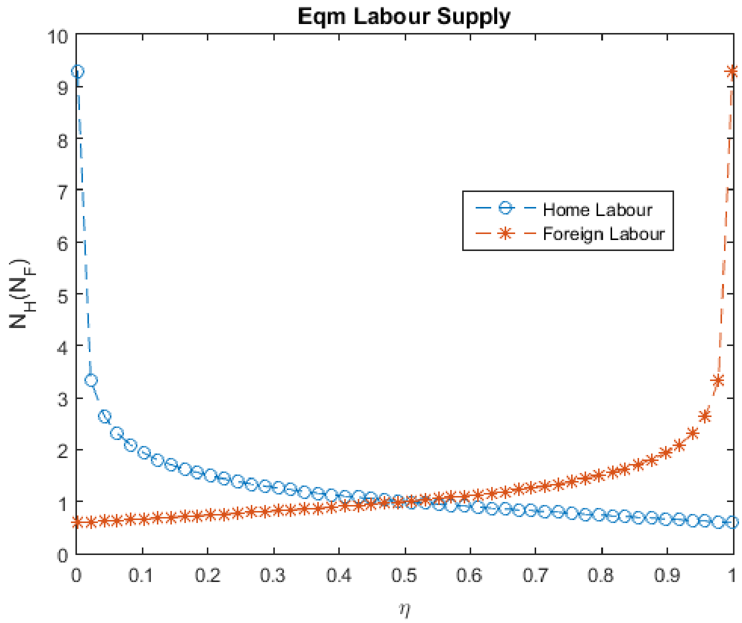

Figure 1: Equilibrium labour supply

We plot the equilibrium labour supply for two countries respectively. When the size of country is very small relative to the other country, the labour supply have to be high in order to satisfy the demand from the other country. The labour supply falls when the relative country size gets larger. Parameter values: $\alpha = 0.2, \varphi = 2$.

2.4 Equilibrium

After obtaining the optimal tradable goods in terms of labour demands, we now impose the market clearing condition to close our model.

$$A_H N_H = C_{1,H} + C_{2,H} = \alpha \frac{A_H}{N_H^{\varphi}} + (1-\alpha)\frac{1-\eta}{\eta}\frac{A_H}{N_H^{\varphi}},$$
$$A_F N_F = C_{1,F} + C_{2,F} = (1-\alpha)\frac{\eta}{1-\eta}\frac{A_F}{N_F^{\varphi}} + \alpha\frac{A_F}{N_F^{\varphi}}.$$

Divide both sides with technology term A_i for $i = \{H, F\}$ and move labour demands to the same side. We then obtain the optimal labour demand/supply in terms of household's home bias, social planner's weight and Frisch labour supply elasticity.

Proposition 2.2. *When tradable goods market is cleared, we obtain the optimal labour demand. At equilibrium, labour demand equals to supply. Labour supply, in the unit of hours worked, depends on the weighted-average of consumer country size (i.e. social planner's weight) normalized by producer country.*

$$N_H = \left[\frac{\eta\alpha + (1-\eta)(1-\alpha)}{\eta}\right]^{\frac{1}{1+\varphi}},$$
$$N_F = \left[\frac{(1-\eta)\alpha + \eta(1-\alpha)}{1-\eta}\right]^{\frac{1}{1+\varphi}}.$$

There are three exogenous parameters determining the optimal labour supply including home bias (openness), social planner's weight (relative country size), and the Frisch elasticity of labour supply. The extent that households are averse to work are homogeneous so we ignore the exponential and focus on the base number. Its numerator is weighted-average between home bias and foreign openness normalized by the her country size. Therefore, labour demand per unit of population is determined by the average demands of tradable goods produced in a specific country and household's work aversion.

2.5 Terms of trade

The terms of trade is defined as the unit price of Home goods in terms of the unit price of the Foreign goods,

$$\begin{aligned}
Q =& \frac{\phi_H}{\phi_F} = \frac{\eta}{1-\eta}\left(\frac{N_H}{N_F}\right)^{\varphi}\frac{A_F}{A_H}, \\
=& \frac{\eta}{1-\eta}\frac{A_F}{A_H}\left[\frac{\alpha + (1-\alpha)\frac{1-\eta}{\eta}}{\alpha + (1-\alpha)\frac{\eta}{1-\eta}}\right]^{\frac{\varphi}{1+\varphi}}, \\
=& \frac{\eta}{1-\eta}\frac{A_F}{A_H}\left[\frac{1-\eta}{\eta}\frac{\alpha\eta + (1-\alpha)(1-\eta)}{\alpha(1-\eta) + (1-\alpha)\eta}\right]^{\frac{\varphi}{1+\varphi}}, \\
=& \frac{A_F}{A_H}\frac{\eta}{1-\eta}^{\frac{1}{1+\varphi}}\left[\frac{\alpha\eta + (1-\alpha)(1-\eta)}{\alpha(1-\eta) + (1-\alpha)\eta}\right]^{\frac{\varphi}{1+\varphi}}.
\end{aligned}$$

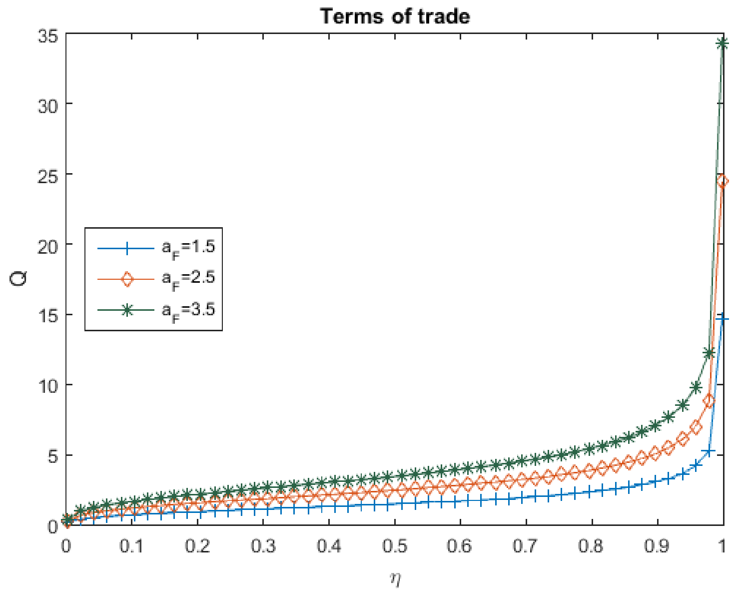

Figure 2: Terms of trade

We plot the terms of trade between two countries. The terms of trade increase (appreciation for Foreign country) in the technology and the size of Foreign country. Parameter values: $A_H = 1, A_F = \{1.5, 2.5, 3.5\}, \alpha = 0.2, \varphi = 2$.

Proposition 2.3. *The terms of trade is defined as the unit price of Home goods in terms of the unit price of Foreign goods,*

$$Q = \frac{A_F}{A_H} \frac{\eta}{1-\eta}^{\frac{1}{1+\varphi}} \left[\frac{\alpha\eta + (1-\alpha)(1-\eta)}{\alpha(1-\eta) + (1-\alpha)\eta} \right]^{\frac{\varphi}{1+\varphi}}. \tag{7}$$

Technology difference is the first component in determining the terms of trade. The second one is the relative country size whose importance can be reduced if the household very much dislike to work. The last component is the ratio of weighted-average home bias and foreign openness across countries. In other words, it can be interpreted as the ratio of tradable goods demands between Home and Foreign goods. If households very much dislike to work, the relative demand would be dominant in determining the terms of trade.

2.6 Real wage

Real wage, in terms of real goods, is to compensate the labour's dis-utility due to work hours. Take household 1 for instance, when he thinks about the trade-off between consumption and labour supply, *the consumption* must be a consumption bundle including Home- and Foreign-produced goods.

If the households only consider locally-produced goods (i.e. $C_{1,H}, C_{1,F}$) when allocating between consumption and work, the real wage is just equal to marginal product of labour in each firms, $w_1 = A_H, w_2 = A_F$.

The decentralized Lagrangian is

$$\mathcal{L}_1 = U_1 - \phi_H(C_{1,H} + C_{2,H} - A_H N_H).$$

Derive the FOCs of household in Home country,

$$\begin{aligned} \frac{\partial U_1}{\partial C_1} =& -\phi_H \left(\frac{\partial C_{1,H} + C_{2,H}}{\partial C_1} \right), \\ \frac{\partial U_1}{\partial N_1} =& \phi_H A_H. \end{aligned}$$

Note that $C_1 = C_{1,H}^{\alpha} C_{1,F}^{1-\alpha}$ so $\frac{\partial C_{2,H}}{\partial C_1} = 0$. Then the Home real wage is

$$
\begin{aligned}
-\frac{\partial U_1/\partial N_1}{\partial U_1/\partial C_1} =& \frac{A_H}{\frac{\partial C_{1,H}}{\partial C_1}}, \\
=& A_H \frac{\partial C_1}{\partial C_{1,H}} = A_H \alpha \left(\frac{C_{1,F}}{C_{1,H}}\right)^{1-\alpha}, \\
=& A_H \alpha \left[\frac{(1-\alpha)\frac{\eta}{1-\eta}\frac{A_F}{N_F^{\varphi}}}{\alpha \frac{A_H}{N_H^{\varphi}}}\right]^{1-\alpha} = (A_H\alpha)^{\alpha} \left[(1-\alpha)\frac{\eta}{1-\eta} A_F\right]^{1-\alpha} \frac{N_H}{N_F}^{\varphi(1-\alpha)}, \\
=& (A_H\alpha)^{\alpha}[(1-\alpha)A_F]^{1-\alpha} \left(\frac{\eta}{1-\eta}\right)^{1-\alpha} \left[\left(\frac{\eta}{1-\eta}\right) \frac{\eta\alpha + (1-\eta)(1-\alpha)}{(1-\eta)\alpha + \eta(1-\alpha)}\right]^{\frac{\varphi(1-\alpha)}{1+\varphi}}, \\
=& (A_H\alpha)^{\alpha}[(1-\alpha)A_F]^{1-\alpha} \left(\frac{\eta}{1-\eta}\right)^{\frac{1-\alpha}{1+\varphi}} \left[\frac{\eta\alpha + (1-\eta)(1-\alpha)}{(1-\eta)\alpha + \eta(1-\alpha)}\right]^{\frac{\varphi(1-\alpha)}{1+\varphi}}.
\end{aligned}
$$

Moreover, we can express the real wage in terms of trade,

$$
(A_H\alpha)^{\alpha}[(1-\alpha)A_F]^{1-\alpha} \left(Q\frac{A_H}{A_F}\right)^{1-\alpha} = \alpha^{\alpha}(1-\alpha)^{1-\alpha} A_H Q^{1-\alpha}.
$$

The decentralized Lagrangian for Foreign country is

$$
\mathcal{L}_2 = U_2 - \phi_F(C_{1,F} + C_{2,F} - A_F N_F),
$$

Similarly, the FOCs of household in Foreign country are,

$$
\begin{aligned}
\frac{\partial U_2}{\partial C_2} =& -\phi_F \left(\frac{\partial C_{1,F} + C_{2,F}}{\partial C_2}\right), \\
\frac{\partial U_2}{\partial N_2} =& \phi_F A_F.
\end{aligned}
$$

Note that $C_2 = C_{2,H}^{1-\alpha} C_{2,F}^{\alpha}$ so $\frac{\partial C_{1,F}}{\partial C_2} = 0$. Then, the Foreign real wage is

$$
\begin{aligned}
-\frac{\partial U_2/\partial N_2}{\partial U_2/\partial C_2} =& \frac{A_F}{\frac{\partial C_{2,F}}{\partial C_2}}, \\
=& A_F \frac{\partial C_2}{\partial C_{2,F}} = A_F \alpha \left(\frac{C_{2,H}}{C_{2,F}} \right)^{1-\alpha}, \\
=& A_F \alpha \left[\frac{(1-\alpha)\frac{1-\eta}{\eta}\frac{A_H}{N_H^{\varphi}}}{\alpha \frac{A_F}{N_F^{\varphi}}} \right]^{1-\alpha} = (A_F \alpha)^{\alpha} \left[(1-\alpha)\frac{1-\eta}{\eta} A_H \right]^{1-\alpha} \frac{N_F}{N_H}^{\varphi(1-\alpha)}, \\
=& (A_F\alpha)^{\alpha}[(1-\alpha)A_H]^{1-\alpha} \left(\frac{1-\eta}{\eta} \right)^{1-\alpha} \left[\left(\frac{\eta}{1-\eta} \right) \frac{(1-\eta)\alpha + \eta(1-\alpha)}{\eta\alpha + (1-\eta)(1-\alpha)} \right]^{\frac{\varphi(1-\alpha)}{1+\varphi}}, \\
=& (A_F\alpha)^{\alpha}[(1-\alpha)A_H]^{1-\alpha} \left(\frac{1-\eta}{\eta} \right)^{\frac{1-\alpha}{1+\varphi}} \left[\frac{(1-\eta)\alpha + \eta(1-\alpha)}{\eta\alpha + (1-\eta)(1-\alpha)} \right]^{\frac{\varphi(1-\alpha)}{1+\varphi}}, \\
=& (A_F\alpha)^{\alpha}[(1-\alpha)A_H]^{1-\alpha} \left(\frac{1}{Q} \frac{A_F}{A_H} \right)^{1-\alpha}, \\
=& \alpha^{\alpha}(1-\alpha)^{1-\alpha} \frac{A_F}{Q^{1-\alpha}}.
\end{aligned}
$$

Proposition 2.4. *Real wages are marginal utility of substitution between consumption bundle and labour hours for Home and Foreign firms respectively. Note that the unit for Home (Foreign) wage rate is $C_1 = C_{1,H}^{\alpha} C_{1,F}^{1-\alpha}$ $(C_2 = C_{2,H}^{1-\alpha} C_{2,F}^{\alpha})$. Terms of trade, technology, and home bias are three factors in determining real wages.*

$$
\begin{aligned}
\frac{W_1}{P_1} =& \alpha^{\alpha}(1-\alpha)^{1-\alpha} A_H Q^{1-\alpha}, \\
\frac{W_2}{P_2} =& \alpha^{\alpha}(1-\alpha)^{1-\alpha} \frac{A_F}{Q^{1-\alpha}}.
\end{aligned}
$$

More advanced technology, depreciation of terms of trade, and smaller home bias will result in higher real wage rate.

2.7 Social Welfare

First of all, we compute the welfare for households in both countries respectively. Then, we will compute the social welfare by using social planner's weight. The Home welfare is

$$\begin{aligned}
U_1^* =&\alpha \log\left(\alpha \frac{A_H}{N_H^{\varphi}}\right) + (1-\alpha)\log\left[(1-\alpha)\frac{\eta}{1-\eta}\frac{A_F}{N_F^{\varphi}}\right] - \frac{\alpha + (1-\alpha)(1-\eta)/\eta}{1+\varphi}, \\
=&\alpha a_H - \alpha\varphi n_H + (1-\alpha)a_F - (1-\alpha)\varphi n_F + \alpha\log\alpha + (1-\alpha)\log\left[(1-\alpha)\frac{\eta}{1-\eta}\right] \\
&- \frac{\alpha + (1-\alpha)(1-\eta)/\eta}{1+\varphi},
\end{aligned}$$

where the lower-case indicates the log-term. Similarly, the Foreign welfare is

$$\begin{aligned}
U_2^* =&(1-\alpha)\log\left((1-\alpha)\frac{1-\eta}{\eta}\frac{A_H}{N_H^{\varphi}}\right) + \alpha\log\left[\alpha\frac{A_F}{N_F^{\varphi}}\right] - \frac{\alpha + (1-\alpha)\eta/(1-\eta)}{1+\varphi}, \\
=&(1-\alpha)a_H - (1-\alpha)\varphi n_H + \alpha a_F - \alpha\varphi n_F + \alpha\log\alpha + (1-\alpha)\log\left[(1-\alpha)\frac{1-\eta}{\eta}\right] \\
&- \frac{\alpha + (1-\alpha)\eta/(1-\eta)}{1+\varphi},
\end{aligned}$$

Lemma 2.3. *We denote β_i for country i as the aggregate demand of tradable goods across countries. It has the functional form of weighted-average home bias and openness of foreign economy.*

$$\begin{aligned}
\beta_H =&\eta\alpha + (1-\eta)(1-\alpha), \\
\beta_F =&\eta(1-\alpha) + (1-\eta)\alpha.
\end{aligned}$$

Now we can apply social planner's weight and compute the social welfare in three steps. First, we arrange the consumption-related terms,

$$\begin{aligned}
&\eta[\alpha a_H - \alpha\varphi n_H + (1-\alpha)a_F - (1-\alpha)\varphi n_F] + (1-\eta)[\alpha a_H - \alpha\varphi n_H + (1-\alpha)a_F - (1-\alpha)\varphi n_F], \\
=&[\eta\alpha + (1-\eta)(1-\alpha)](a_H - \varphi n_H) + [\eta(1-\alpha) + (1-\eta)\alpha](a_F - \varphi n_F), \\
=&\beta_H a_H + \beta_F a_F - [\eta\alpha + (1-\eta)(1-\alpha)]\varphi n_H - [\eta(1-\alpha) + (1-\eta)\alpha]\varphi n_F, \\
=&\beta_H a_H + \beta_F a_F - \beta_H \frac{\varphi}{1+\varphi}\log\left(\frac{\beta_H}{\eta}\right) - \beta_F\frac{\varphi}{1+\varphi}\log\left(\frac{\beta_F}{1-\eta}\right), \\
=&\beta_H\left[a_H - \frac{\varphi}{1+\varphi}\log\left(\frac{\beta_H}{\eta}\right)\right] + \beta_F\left[a_F - \frac{\varphi}{1+\varphi}\log\left(\frac{\beta_F}{1-\eta}\right)\right].
\end{aligned}$$

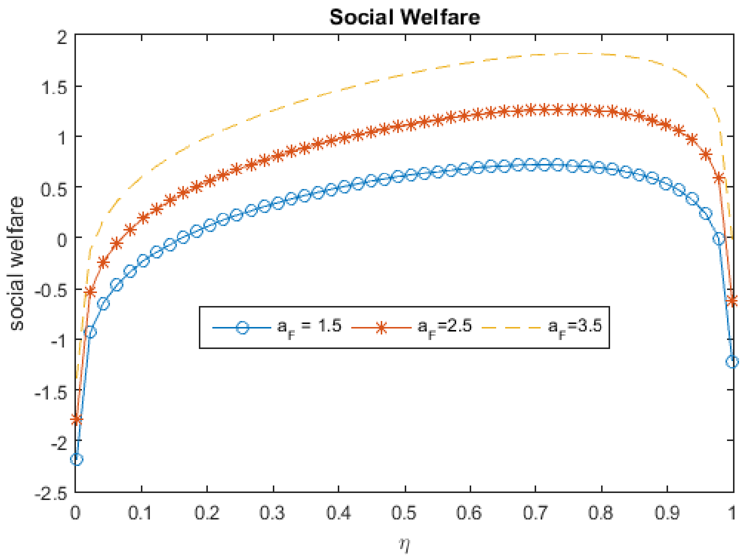

Figure 3: Social welfare with heterogeneous technology

We plot the social welfare with heterogeneous technology. The Home country technology is normalized to 1 and the social welfare is larger when Foreign technology is more advanced relative to Home technology. The social welfare is particularly high when Foreign country is large in size. Parameter values: $A_H = 1, A_F = \{1.5, 2.5, 3.5\}, \alpha = 0.2, \varphi = 2$.

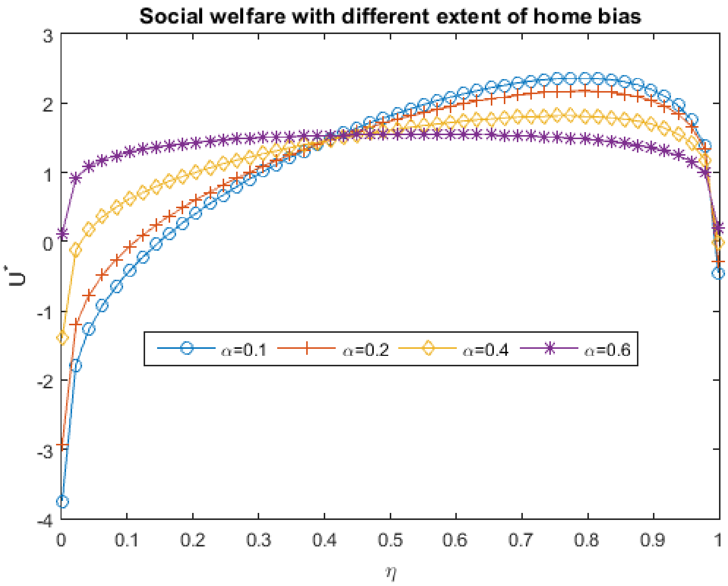

Figure 4: Social welfare with home bias

We plot the social welfare with different extent of home bias. When home bias is smaller than 0.5, generally, social welfare increases in size of Home country. The increase is sharp when home bias is small. In the case when home bias is 0.6, the social welfare remains flat except for the case when one country is extremely small compared with the other. Parameter values: $a_H = 1, a_F = 3.5, \alpha = \{0.1, 0.2, 0.4, 0.6\}, \varphi = 2$.

Secondly, we arrange the constant terms,

$$
\begin{aligned}
&\eta\left(\alpha\log\alpha+(1-\alpha)\log\left[(1-\alpha)\frac{\eta}{1-\eta}\right]\right)+(1-\eta)\left(\alpha\log\alpha+(1-\alpha)\log\left[(1-\alpha)\frac{1-\eta}{\eta}\right]\right),\\
=&(\eta+1-\eta)\alpha\log\alpha+(1-\alpha)[\eta\log(1-\alpha)+\eta\log\eta-\eta\log(1-\eta)\\
&+(1-\eta)\log(1-\alpha)+(1-\eta)\log(1-\eta)-(1-\eta)\log\eta],\\
=&\alpha\log\alpha+(1-\alpha)\left[\log(1-\alpha)+\underbrace{(1-2\eta)\log\frac{1-\eta}{\eta}}_{\text{(+) Deviation from }\eta=0.5}\right].
\end{aligned}
$$

Finally, we can arrange the labour dis-utility terms

$$
\begin{aligned}
&-\eta\left(\frac{\alpha+(1-\alpha)(1-\eta)/\eta}{1+\varphi}\right)-(1-\eta)\left(\frac{\alpha+(1-\alpha)\eta/(1-\eta)}{1+\varphi}\right),\\
&=-\frac{1}{1+\varphi}\left[\eta\alpha+(1-\alpha)(1-\eta)+(1-\eta)\alpha+(1-\alpha)\eta\right],\\
&=-\frac{1}{1+\varphi}[\alpha+(1-\alpha)]=-\frac{1}{1+\varphi}.
\end{aligned}
$$

Proposition 2.5. *The equilibrium social welfare is*

$$
\begin{aligned}
U_s^*=&\beta_H\left[a_H-\frac{\varphi}{1+\varphi}\log\left(\frac{\beta_H}{\eta}\right)\right]+\beta_F\left[a_F-\frac{\varphi}{1+\varphi}\log\left(\frac{\beta_F}{1-\eta}\right)\right] &(8)\\
&+\alpha\log\alpha+(1-\alpha)\left[\log(1-\alpha)+(1-2\eta)\log\frac{1-\eta}{\eta}\right]-\frac{1}{1+\varphi}. &(9)
\end{aligned}
$$

If both countries have the same size $\eta = 0.5$, then the social welfare is generated from technology, the openness for international trade, and the dis-utility due to work.

$$U_s^{**}=a_H+a_F+\alpha\log\alpha+(1-\alpha)\log(1-\alpha)-\frac{1+\varphi\log 4}{1+\varphi}. \qquad (10)$$

3 Nominal Economy

Before introducing monopolistic competition among firms, let me re-write the pure exchange economy into a nominal one. As in pure exchange economy, there is a social planner who attribute weights to Home and Foreign countries respectively. Hence, the optimal allocation of tradable goods shall remain unchanged so we can match the terms to obtain the price or wage explicitly.

We start by writing down the budget constraint for each agent,

$$P_H C_{i,H} + P_F C_{i,F} \leq W_i N_i, \tag{11}$$

where the subscript $i = 1, 2$ indicates the consumption, labour supply and nominal wage by household i. $P_{i,H}, P_{i,F}$ are the nominal prices of Home and Foreign goods for agent $i = 1, 2$. Assume labour immobility, labour market clearing conditions are

$$N_1 = N_H; N_2 = N_F.$$

The social planner attributes the weight η to Home agent and $1 - \eta$ to Foreign agent and chooses the optimal allocation with the budget constraints from both country. The Lagrangian is

$$\mathcal{L} = \int_0^\infty e^{-\delta t}[\eta U_1 + (1-\eta)U_2] - \lambda_1(P_{1,H}C_{1,H} + P_{1,F}C_{1,F} - W_1 N_H) \tag{12}$$

$$-\lambda_2(P_{2,H}C_{2,H} + P_{2,F}C_{2,F} - W_2 N_F), \tag{13}$$

where P_1, P_2 are the price level for each country. We then choose the Lagrange multipliers as

$$\lambda_1 = \kappa\eta \frac{\Lambda_1}{P_1},$$

$$\lambda_2 = \kappa(1-\eta)\frac{\Lambda_2}{P_2},$$

where Λ_i is the real stochastic discount factor for agent i and κ is some constant.

Note that the Consumer Price Indices are

$$P_1 C_1 = P_{1,H}C_{1,H} + P_{1,F}C_{1,F},$$

$$P_2 C_2 = P_{2,H}C_{2,H} + P_{2,F}C_{2,F}.$$

FOCs w.r.t. N_H and C_1

$$\eta C_1^{-1} - \kappa\eta \frac{\Lambda_1}{P_1} P_1 = 0 \Rightarrow C_1^{-1} = \kappa \Lambda_1,$$

$$\eta N_H^{\varphi} - \kappa\eta \frac{\Lambda_1}{P_1} W_1 = 0 \Rightarrow \frac{W_1}{P_1} = C_1 N_H^{\varphi}.$$

FOC w.r.t. $C_{1,H}$

$$\begin{aligned} \eta\alpha C_{1,H}^{-1} - \eta\kappa\frac{\Lambda_1}{P_1}P_{1,H} = 0, \\ \Rightarrow P_1C_1 = \alpha^{-1}P_{1,H}C_{1,H} = \frac{W_1}{N_H^{\varphi}}. \end{aligned}$$

Real wage for Home labour shall be equal to marginal product of labour (MPL) so the nominal wage is $W_1 = A_H P_{1,H}$. Substitute into the above equation, the optimal $C_{1,H}$ is aligned with the pure exchange economy case,

$$\begin{aligned} \alpha^{-1}P_{1,H}C_{1,H} = \frac{A_H P_{1,H}}{N_H^{\varphi}}, \\ C_{1,H} = \alpha A_H N_H^{-\varphi}. \end{aligned}$$

Similarly, the optimal $C_{2,F} = \alpha\frac{W_2}{P_{2,F}N_F^{\varphi}}$. With equilibrium nominal wage $W_2 = A_F P_{2,F}$, again we have optimal $C_{2,F}$ same as in pure exchange economy, $C_{2,F} = \alpha A_F N_F^{-\varphi}$.

FOC w.r.t. $C_{1,F}$

$$\begin{aligned} \eta(1-\alpha)C_{1,F}^{-1} = \kappa\eta\frac{\Lambda_1}{P_1}P_{1,F} = 0, \\ P_1C_1 = (1-\alpha)^{-1}P_{1,F}C_{1,F} = \frac{A_H P_{1,H}}{N_H^{\varphi}}, \\ \Rightarrow C_{1,F} = \frac{1-\alpha}{N_H^{\varphi}}\frac{W_1}{P_{1,F}}. \end{aligned}$$

Note that in pure exchange economy, optimal $C_{1,F}$ is a function of Foreign labour N_F. In this nominal economy, Home labour has to use the nominal wage earned to exchange for Foreign goods. As there is no other friction yet, after introducing terms of trade $S = P_{1,F}/P_{1,H}$ we shall be able to align optimal $C_{1,F}$ with the case in pure exchange economy.

Recall optimal $C_{1,F} = \frac{\eta(1-\alpha)}{1-\eta}\frac{A_F}{N_F^{\varphi}}$ in pure exchange economy. Matching optimal consumption from both economy, terms of trade is solved to be

$$\begin{aligned} \frac{\eta(1-\alpha)}{1-\eta}\frac{A_F}{N_F^{\varphi}} = \frac{1-\alpha}{N_H^{\varphi}}\frac{A_H P_{1,H}}{P_{1,F}}, \\ \frac{\eta}{1-\eta}\frac{A_F}{N_F^{\varphi}} = \frac{1}{N_H^{\varphi}}\frac{A_H}{S}, \\ \Rightarrow S = \frac{1-\eta}{\eta}\frac{A_H}{A_F}\left(\frac{N_F}{N_H}\right)^{\varphi}. \end{aligned}$$

This ($S = 1/Q$) is the same as Proposition 2.3.

FOC w.r.t. $C_{2,H}$

$$
\begin{aligned}
(1-\alpha)(1-\eta)C_{2,H}^{-1} &= (1-\eta)\frac{P_{2,H}}{P_2 C_2}, \\
P_2 C_2 = P_{2,H} C_{2,H}(1-\alpha)^{-1} &= W_2 N_F^{-\varphi}, \\
C_{2,H} = \frac{W_2}{N_F^{\varphi}}\frac{1-\alpha}{P_{2,H}} = \frac{A_F P_{2,F}}{N_F^{\varphi}}\frac{1-\alpha}{P_{2,H}} &= (1-\alpha)\frac{A_F}{N_F^{\varphi}} S.
\end{aligned}
$$

Using Proposition 2.3, optimal $C_{2,H}$ is aligned with the case in pure exchange economy,

$$
C_{2,H} = (1-\alpha)\frac{A_F}{N_F^{\varphi}}\left[\frac{1-\eta}{\eta}\frac{A_H}{A_F}\left(\frac{N_F}{N_H}\right)^{\varphi}\right] = (1-\alpha)\frac{1-\eta}{\eta}\frac{A_H}{N_H^{\varphi}}.
$$

Moreover, combining the optimal consumption goods within one country, we can establish the following sharing rules,

$$
\frac{C_{1,F}}{C_{1,H}} = \frac{1-\alpha}{\alpha}\frac{P_{1,H}}{P_{1,F}}, \tag{14}
$$

$$
\frac{C_{2,H}}{C_{2,F}} = \frac{1-\alpha}{\alpha}\frac{P_{1,F}}{P_{1,H}}. \tag{15}
$$

With total consumption expenditure in Home country ($P_1 C_1$), we can derive the duo price indices

$$
\begin{aligned}
P_1 C_1 =& P_{1,H} C_{1,H} + P_{1,F} C_{1,F} = P_1 C_{1,H}^{\alpha} C_{1,F}^{1-\alpha}, \\
P_1 \left(\frac{C_{1,F}}{C_{1,H}}\right)^{1-\alpha} =& P_{1,H} + P_{1,F}\frac{C_{1,F}}{C_{1,H}}, \\
P_1 \left(\frac{1-\alpha}{\alpha}\frac{P_{1,H}}{P_{1,F}}\right)^{1-\alpha} =& P_{1,H} + P_{1,F}\frac{1-\alpha}{\alpha}\frac{P_{1,H}}{P_{1,F}} = \left(1+\frac{1-\alpha}{\alpha}\right)P_{1,H}, \\
P_1 =& \tau P_{1,H}^{\alpha} P_{1,F}^{(1-\alpha)},
\end{aligned}
$$

where $\tau = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}$ is some constant. Therefore, the price indices for Home and Foreign countries are

$$
P_1 = \tau P_{1,H}^{\alpha} P_{1,F}^{(1-\alpha)}, \tag{16}
$$

$$
P_2 = \tau P_{2,H}^{1-\alpha} P_{2,F}^{\alpha}. \tag{17}
$$

The relationship between price level and term of trade is more convenient for later use,

$$\frac{P_1}{P_{1,H}} = \tau S^{1-\alpha}, \tag{18}$$

$$\frac{P_2}{P_{2,F}} = \tau Q^{1-\alpha}. \tag{19}$$

For each agent, the real wage rate shall reflect her purchasing power. With total consumption expenditure, we can easily convert $W_1/P_{1,H}$ to W_1/P_1.

$$\frac{W_1}{P_1} = \frac{W_1}{P_{1,H}} \frac{P_{1,H}}{P_1} = A_H \tau^{-1} Q^{1-\alpha}, \tag{20}$$

$$\frac{W_2}{P_2} = \frac{W_2}{P_{2,F}} \frac{P_{2,F}}{P_2} = A_F \tau^{-1} S^{1-\alpha}, \tag{21}$$

which are aligned with Proposition 2.4. The solution for labours can be obtained by market clearing condition.

3.1 Monopolistic Firms

We assume there is a continuum of firms in each country. Each firm produces differentiated goods because consumers have preference for goods variety. Assuming the preferences for goods variety are symmetric, we then define duo quantity and price indices,

$$C_H = \left[\int_0^1 C_H(j)^{1-\frac{1}{\epsilon}} dj\right]^{\frac{\epsilon}{\epsilon-1}}; P_H = \left[\int_0^1 P_H(j)^{1-\epsilon} dj\right]^{\frac{1}{1-\epsilon}},$$

$$C_F = \left[\int_0^1 C_F(i)^{1-\frac{1}{\epsilon}} di\right]^{\frac{\epsilon}{\epsilon-1}}; P_F = \left[\int_0^1 P_F(i)^{1-\epsilon} di\right]^{\frac{1}{1-\epsilon}}.$$

The optimal allocation is

$$C_H(j) = \left[\frac{P_H(j)}{P_H}\right]^{-\epsilon} C_H, \forall j \in [0,1],$$

$$C_F(i) = \left[\frac{P_F(i)}{P_F}\right]^{-\epsilon} C_F, \forall i \in [0,1].$$

The profit maximization function for Home firm j is

$$\sup_{P_{H,t}(j)} \int_0^\infty \exp\left[-\int_0^u i_{1,s} ds\right] [P_{H,u}(j) Y_{H,u}(j) - W_{1,u} N_{H,u}(j)] du. \tag{22}$$

Each period, monopolistic firm j adjust price to maximize her profit. Re-write the profit flow with optimal allocation of consumption goods,

$$\pi_t(j) = P_{H,t}(j)^{1-\epsilon} P_{H,t}^{\epsilon} C_{H,t} - \frac{W_{1,t}}{A_{H,t}} \left[\frac{P_{H,t}(j)}{P_{H,t}} \right]^{-\epsilon} C_{H,t}, \forall t$$

FOCs w.r.t. $P_{H,t}(j)$

$$(1-\epsilon) \left[\frac{P_{H,t}(j)}{P_{H,t}} \right]^{-\epsilon} C_{H,t} = -\epsilon \frac{W_{1,t}}{A_{H,t}} \left[\frac{P_{H,t}(j)}{P_{H,t}} \right]^{-\epsilon} P_{H,t}(j)^{-1} C_{H,t},$$
$$\Rightarrow P_{H,t}(j) = \frac{\epsilon}{\epsilon - 1} \frac{W_{1,t}}{A_{H,t}}.$$

At a symmetric equilibrium, all firms choose the same price in each period t

$$P_{H,t} = P_{H,t}(j) = \frac{\epsilon}{\epsilon-1} \frac{W_{1,t}}{A_{H,t}}; P_{F,t} = P_{F,t}(i) = \frac{\epsilon}{\epsilon-1} \frac{W_{2,t}}{A_{F,t}}. \tag{23}$$

Define nominal exchange rate $\mathcal{E}$ as the ratio of Home prices to Foreign ones. We also assume the consumption goods are traded in the global market and law of one price so the nominal exchange rate is unique across prices of tradable goods.

$$P_{1,H} = \mathcal{E} P_{2,H}; P_{1,F} = \mathcal{E} P_{2,F}. \tag{24}$$

Combining (16) and (17) together with the definition of nominal exchange rate (24), we can express nominal exchange rate as a function of terms of trade and the ratio of price levels between two countries,

$$\mathcal{E} = \frac{P_1}{P_2} S^{2\alpha-1}. \tag{25}$$

Real exchange rate is defined as nominal exchange rate multiplied by the ratio of price indices between two countries,

$$E = \frac{C_1}{C_2} = \frac{\mathcal{E} P_2}{P_1} = S^{2\alpha-1}, \tag{26}$$

which is a function of terms of trade and home bias. Note that the difference between real exchange rate (E) and terms of trade (S) is the following: the former is of the country-wide consumption bundles, i.e. C_1 vs. C_2 and the latter is of the product level, i.e. $P_{j,F}$ vs. $P_{j,H}$ for $j = \{H, F\}$. Not surprisingly, the effect of prices is absent.

The nominal exchange rate is the ratio of nominal values of aggregate consumption by the representative households in two countries,

$$\mathcal{E} = \frac{P_1}{P_2} E = \frac{P_1 C_1}{P_2 C_2}. \tag{27}$$

Market clearing condition for Home goods

$$\begin{aligned}
Y_H(j) =& C_{1,H}(j) + C_{2,H}(j) = \left[\frac{P_{H,t}(j)}{P_{H,t}}\right]^{-\epsilon} \left[\alpha \left(\frac{P_{1,H}}{P_1}\right)^{-1} C_1 + (1-\alpha)\left(\frac{P_{2,H}}{P_2}\right)^{-1} C_2\right], \\
Y_H =& \left[\int_0^1 Y_H(j)^{1-\frac{1}{\epsilon}} dj\right]^{\frac{\epsilon}{\epsilon-1}} = \left[\alpha \left(\frac{P_{1,H}}{P_1}\right)^{-1} C_1 + (1-\alpha)\left(\frac{P_{1,H}}{\mathcal{E} P_2}\right)^{-1} C_2\right], \\
=& \frac{P_1 C_1}{P_{1,H}} \left[\alpha + (1-\alpha)\mathcal{E}\frac{P_2 C_2}{P_1 C_1}\right] = C_1 S^{1-\alpha}\tau \left[\alpha + (1-\alpha)\mathcal{E}\frac{1}{\mathcal{E}}\right] = C_1 S^{1-\alpha}\tau.
\end{aligned}$$

Market clearing condition for Foreign goods,

$$\begin{aligned}
Y_F(i) =& C_{1,F}(i) + C_{2,F}(i) = \left[\frac{P_{F,t}(i)}{P_{F,t}}\right]^{-\epsilon} \left[(1-\alpha) \left(\frac{P_{1,F}}{P_1}\right)^{-1} C_1 + \alpha\left(\frac{P_{2,F}}{P_2}\right)^{-1} C_2\right], \\
Y_F =& \left[\int_0^1 Y_F(i)^{1-\frac{1}{\epsilon}} di\right]^{\frac{\epsilon}{\epsilon-1}} = \left[(1-\alpha) \left(\frac{\mathcal{E} P_{2,F}}{P_1}\right)^{-1} C_1 + \alpha\left(\frac{P_{1,F}}{P_2}\right)^{-1} C_2\right], \\
=& \frac{P_2 C_2}{P_{2,F}} \left[(1-\alpha)\frac{P_1 C_1}{P_2 C_2}\frac{1}{\mathcal{E}} + \alpha\right] = C_2 Q^{1-\alpha}\tau \left[(1-\alpha)\mathcal{E}\frac{1}{\mathcal{E}} + \alpha\right] = C_2 Q^{1-\alpha}\tau.
\end{aligned}$$

Lemma 3.1. *When Home and Foreign goods market is cleared, we can express country-wide production as a function of consumption baskets and terms of trade with home bias.*

$$\begin{aligned}
Y_H &= C_1 S^{1-\alpha}\tau, \\
Y_F &= C_2 Q^{1-\alpha}\tau.
\end{aligned}$$

Note that the price dispersion of differentiated goods is constant across time so its effect vanishes in the calculation of production growth,

$$\begin{aligned}
d\ln Y_H =& d\ln C_1 + (1-\alpha) d\ln S, \\
d\ln Y_F =& d\ln C_2 + (1-\alpha) d\ln Q.
\end{aligned}$$

3.2 Stochastic Discount Factor

The (nominal) stochastic discount factor by Home and Foreign agents

$$\frac{\Lambda^{\$}_{1,t+dt}}{\Lambda^{\$}_{1,t}} = e^{-\delta dt}\left(\frac{Y_{H,t+dt}}{Y_{H,t}}\right)^{-\alpha}\left(\frac{Y_{F,t+dt}}{Y_{F,t}}\right)^{-(1-\alpha)}\frac{P_{1,t}}{P_{1,t+dt}},$$
$$\frac{\Lambda^{\$}_{2,t+dt}}{\Lambda^{\$}_{2,t}} = e^{-\delta dt}\left(\frac{Y_{H,t+dt}}{Y_{H,t}}\right)^{-(1-\alpha)}\left(\frac{Y_{F,t+dt}}{Y_{F,t}}\right)^{-\alpha}\frac{P_{2,t}}{P_{2,t+dt}}.$$

Nominal price of a Home bond paying off at time $t + dt$

$$\begin{aligned}
e^{-i_{1,t}dt} =& \frac{\Lambda^{\$}_{1,t+dt}}{\Lambda^{\$}_{1,t}}, \\
1 - i_{1,t}dt =& 1 + \frac{d\Lambda^{\$}_{1,t}}{\Lambda^{\$}_{1,t}}, \\
i_{1,t}dt =& -\left(\frac{d\Lambda_{1,t}}{\Lambda_{1,t}} - \frac{dP_{1,t}}{P_{1,t}}\right) = \delta dt + \frac{dC_{1,t}}{C_{1,t}} + \frac{dP_{1,t}}{P_{1,t}}, \\
\frac{dC_{1,t}}{C_{1,t}} =& (i_{1,t} - \delta)dt - \frac{dP_{1,t}}{P_{1,t}}.
\end{aligned}$$

Lemma 3.2. *On the demand side, the growth rates of consumption baskets for Home and Foreign countries are nominal rate net of time discount rate and inflation rate,*

$$\begin{aligned}
\frac{dC_{1,t}}{C_{1,t}} =& (i_{1,t} - \delta)dt - \frac{dP_{1,t}}{P_{1,t}}, \\
\frac{dC_{2,t}}{C_{2,t}} =& (i_{2,t} - \delta)dt - \frac{dP_{2,t}}{P_{2,t}}.
\end{aligned}$$

By using Lemma 3.1 and 3.2, we can combine the demand with supply sides and the evolution of Home and Foreign output

$$\begin{aligned}
d\ln Y_H =& (i_1 - \delta)dt - d\ln P_1 + (1-\alpha)d\ln S, \\
d\ln Y_F =& (i_2 - \delta)dt - d\ln P_2 + (1-\alpha)d\ln Q.
\end{aligned}$$

There are two inputs, technology and labour, of the country-wide production. The equilibrium labour is a function of home bias and country size, which do not vary with time. Hence, the evolution of output can be expressed as the evolution of technology progress, i.e. $d\ln Y_j = d\ln A_j$ for country j. Similarly, the evolution of terms of trade can be expressed by the difference of technology progress between two countries, i.e. $d\ln Q = d\ln A_F - d\ln A_H$ and $d\ln S = d\ln A_H - d\ln A_F$. Now we can simplify the above SDEs for the evolution of price levels in both countries,

$$\alpha d\ln A_H + (1-\alpha)d\ln A_F = (i_1 - \delta)dt - d\ln P_1, \tag{28}$$

$$\alpha d\ln A_F + (1-\alpha)d\ln A_H = (i_2 - \delta)dt - d\ln P_2. \tag{29}$$

Proposition 3.1. *The equilibrium nominal interest rate is composed of time discount rate, country-wide inflation rate and weighted average of technology progress between two countries,*

$$i_1 dt = [\delta + \alpha\mu_H + (1-\alpha)\mu_F]dt + d\ln P_1, \tag{30}$$

$$i_2 dt = [\delta + \alpha\mu_F + (1-\alpha)\mu_H]dt + d\ln P_2, \tag{31}$$

given the technology progress in each country follows an exogenous deterministic process,

$$d\ln A_H = \mu_H dt; d\ln A_F = \mu_F dt.$$

With the specification of interest rate rules, we can solve for the evolution of the price levels in both countries.

The Uncovered Interest Rate Parity holds in the complete market,

$$(i_1 - i_2)dt = (2\alpha - 1)(\mu_H - \mu_F)dt + d\ln P_1 - d\ln P_2 = d\ln \mathcal{E}. \tag{32}$$

4 Price Adjustment under Calvo Pricing

Each firm may reset price with probability $1-\theta$ in any period t. On the other hand, a fraction θ of firms keep their prices unchanged. Therefore, θ is to measure the extent of price stickiness. We start from using variables in Home country. Note that the corresponding variables and derivation in Foreign country are similar. We first derive the optimal price included two components: a markup from monopolistic competition plus weighted discounted value of expected marginal costs and nominal prices. With aggregate price dynamics, we can establish the relation between inflation and marginal costs. Then, we are ready to derive New Keynesian Phillips Curve (NKPC) and dynamic investment and saving curve (DIS) in next subsection.

Aggregate price index of Home products

$$P_{H,t} = \left[\theta P_{H,t-1}^{1-\epsilon} + (1-\theta) P_{H,t}^{*1-\epsilon}\right]^{\frac{1}{1-\epsilon}}.$$

The inflation (price evolution) of Home products is

$$\begin{aligned}\Pi_{H,t}^{1-\epsilon} &= \left(\frac{P_{H,t}}{P_{H,t-1}}\right)^{1-\epsilon} = \left(\left[\theta P_{H,t-1}^{1-\epsilon} + (1-\theta) P_{H,t}^{*1-\epsilon}\right]^{\frac{1}{1-\epsilon}} \frac{1}{P_{H,t-1}}\right)^{1-\epsilon}, \\ &= \theta + (1-\theta)\left(\frac{P_{H,t}^{*}}{P_{H,t-1}}\right)^{1-\epsilon}.\end{aligned}$$

Now we approximate the above equation around the zero inflation steady state $\Pi_{H,t} = \Pi_H = 1$ and $P_{H,t} = P_{H,t-1} = P_H$. Hereafter, we express log variables in lower case letters. Define the log deviation of any variable in the following manner

$$x_t - x = \frac{X_t - X}{X} \Rightarrow \begin{cases} \pi_{H,t} - \pi_H = \dfrac{\Pi_{H,t} - \Pi_H}{\Pi_H}, \\ p_{H,t-1} - p_H = \dfrac{P_{H,t-1} - P_H}{P_H}, \\ p_{H,t}^{*} - p_H = \dfrac{P_{H,t}^{*} - P_H}{P_H}. \end{cases}$$

Using Taylor expansion up to first order, we can approximate the inflation

$$\begin{aligned}\Pi_H^{1-\epsilon} + (1-\epsilon)\Pi_H^{-\epsilon}\Pi_H\pi_{H,t} = &\left[\theta + (1-\theta)\left(\frac{P_H}{P_H}\right)^{1-\epsilon}\right] \\ &+ (1-\theta)(1-\epsilon)\left(\frac{P_H}{P_H}\right)^{-\epsilon}\left[\frac{1}{P_H}P_H(p_{H,t}^{*} - p_H) - \frac{P_H}{P_H^2}P_H(p_{H,t-1} - p_H)\right], \\ 1 + (1-\epsilon)\pi_{H,t} = &1 + (1-\theta)(1-\epsilon)(p_{H,t}^{*} - p_H - p_{H,t-1} + p_H).\end{aligned}$$

The log-linearised inflation of Home product price is

$$\pi_{H,t} = (1-\theta)(p_{H,t}^{*} - p_{H,t-1}). \tag{33}$$

Next, we look into the individual firm's decision on price setting. Each firm chooses price to maximize the current value of future profit flows

$$\max_{P_{H,t}^{*}(j)} \sum_{k=0}^{\infty} \theta^k E_t \left\{\frac{\Lambda_{t+k}^{\$}}{\Lambda_t^{\$}}[P_{H,t}^{*}(j) Y_{H,t+k|t}(j) - \Psi_{H,t+k|t}(Y_{H,t+k|t}(j))]\right\}, \tag{34}$$

subject to the optimal output of firm j under monopolistic competition

$$Y_{H,t+k|t}(j) = \left(\frac{P^*_{H,t}(j)}{P_{H,t+k}} \right)^{-\epsilon} Y_{H,t+k}. \tag{35}$$

Note that the subscript $H, t+k|t$ indicates the variables are from Home country with price flexibility at time t.

Here, I would like to make a comment on the stochastic discount factor (SDF) used for price setting. In a closed economy model, there is only one consumption good and one price level which are the main ingredients of SDF for monopolistic firms. However, in an open economy model, consumption goods produced in domestic country (Y_H) is only a fraction of consumption basket (C_1) in one country. Similarly, price of domestically-produced goods (P_H) deviate from the country-wide price level (P_1). Since we have a frictionless market of international trade, with equation (16), (17) and Lemma 3.1, we show that the total consumption expenditure by household is equivalent to total value of output in corresponding country.

$$Y_{H,t} = C_{1,t} S_t^{1-\alpha} \tau = C_{1,t} \frac{P_{1,t}}{P_{H,t}} \Rightarrow C_{1,t} P_{1,t} = Y_{H,t} P_{H,t} \ \forall t.$$

Therefore, the SDF can be expressed in the following equivalent ways

$$\frac{\Lambda^\$_{t+k}}{\Lambda^\$_t} = e^{-\delta k} \left(\frac{C_{1,t+k}}{C_{1,t}} \right)^{-1} \frac{P_{1,t}}{P_{1,t+k}} = e^{-\delta k} \left(\frac{Y_{H,t+k}}{Y_{H,t}} \right)^{-1} \frac{P_{H,t}}{P_{H,t+k}}.$$

We showed that SDF by Household is equivalent to SDF by firms. In this subsection, we choose to focus on the firms' SDF for the ease of derivation later on.

By inserting the optimal allocation constraint and the expression of SDF, we can re-write firm's objective function as an unconstrained maximization problem,

$$\sum \theta^k E_t \left\{ e^{-\delta k} \frac{Y_{H,t}}{Y_{H,t+k}} \frac{P_{H,t}}{P_{H,t+k}} \left[P^*_{H,t}(j) \left(\frac{P^*_{H,t}(j)}{P_{H,t+k}} \right)^{-\epsilon} Y_{H,t+k} - \Psi_{H,t+k|t}(\cdot) \right] \right\}, \tag{36}$$

where the nominal total cost is a function of output from firm j given price flexibility in period t: $\Psi_{H,t+k|t}(\cdot) = \Psi_{H,t+k|t} \left(\left[\frac{P^*_{H,t}(j)}{P_{H,t+k}} \right]^{-\epsilon} Y_{H,t+k} \right)$.

Taking FOC,

$$\sum \theta^k E_t \left(e^{-\delta k} \frac{Y_{H,t}}{Y_{H,t+k}} \frac{P_{H,t}}{P_{H,t+k}} \left[(1-\epsilon) \left(\frac{P^*_{H,t}(j)}{P_{H,t+k}} \right)^{-\epsilon} Y_{H,t+k} \right.\right.$$
$$\left.\left. + \ \Psi'_{H,t+k|t}(\cdot)\epsilon \left(\frac{P^*_{H,t}(j)}{P_{H,t+k}} \right)^{-\epsilon-1} \frac{1}{P_{H,t+k}} Y_{H,t+k} \right] \right) = 0.$$

Move marginal-cost term to the RHS of equation,

$$\sum \theta^k E_t \left[e^{-\delta k} \frac{Y_{H,t}}{Y_{H,t+k}} \frac{P_{H,t}}{P_{H,t+k}} \left(\frac{P^*_{H,t}(j)}{P_{H,t+k}} \right)^{-\epsilon} Y_{H,t+k} \right]$$
$$= \frac{\epsilon}{\epsilon-1} \sum \theta^k E_t \left[e^{-\delta k} \frac{Y_{H,t}}{Y_{H,t+k}} \frac{P_{H,t}}{P_{H,t+k}} \Psi'_{H,t+k|t}(\cdot) \left(\frac{P^*_{H,t}(j)}{P_{H,t+k}} \right)^{-\epsilon-1} \frac{Y_{H,t+k}}{P_{H,t+k}} \right].$$

Cancel out the common terms on both numerator and denominator,

$$\sum \theta^k E_t \left[e^{-\delta k} \frac{1}{P_{H,t+k}} \left(\frac{1}{P_{H,t+k}} \right)^{-\epsilon} \right] = \frac{\epsilon}{\epsilon-1} \sum \theta^k E_t \left[e^{-\delta k} \frac{1}{P_{H,t+k}} \Psi'_{H,t+k|t}(\cdot) P^*_{H,t}(j)^{-1} \left(\frac{1}{P_{H,t+k}} \right)^{-\epsilon} \right],$$
$$\Rightarrow P^*_{H,t}(j) \sum \theta^k E_t (e^{-\delta k} P^{\epsilon-1}_{H,t+k}) = \frac{\epsilon}{\epsilon-1} \sum \theta^k E_t [e^{-\delta k} \Psi'_{H,t+k|t}(\cdot) P^{\epsilon-1}_{H,t+k}].$$

So far, the total cost is of nominal term. It is convenient to use "*real*" marginal cost in our calculation. Nominal marginal cost deflated by price of good is real marginal cost.

$$\Psi'_{H,t+k|t,real}(\cdot) = \frac{\Psi'_{H,t+k|t,nominal}(\cdot)}{P_{H,t}} \tag{37}$$

With a slight abuse of notation, we ignore the subscript "*real*" and hereafter we denote the real marginal cost as $\Psi'_{H,t+k|t,real}(\cdot) = \Psi'_{H,t+k|t}(\cdot)$. Hence, the re-written optimal price set by firm j at period t shall be

$$P^*_{H,t}(j) = \frac{\epsilon}{\epsilon-1} \frac{E_t \sum \theta^k e^{-\delta k} \Psi'_{H,t+k|t}(\cdot) P^{\epsilon}_{H,t+k}}{E_t \sum \theta^k e^{-\delta k} P^{\epsilon-1}_{H,t+k}}.$$

4.1 Log-linearisation Approximation

Now we shall approximate the optimal price around zero inflation steady state. First let us list the relevant four variables when inflation stays at zero:

$$\begin{aligned} Y_{H,t+k|t} =& Y_{H,t|t}, \\ \frac{\Lambda^{\$}_{t+k}}{\Lambda^{\$}_{t}} =& e^{-\delta k}, \\ \Psi'_{H,t+k}(\cdot) =& \Psi'_{H,t}(\cdot) = \Psi'_{H}(\cdot), \\ \Pi_{H,t} =& \frac{P^*_{H,t}(j)}{P_{H,t-1}} = \frac{P^*_{H,t}(j)}{P_{H,t}} = \frac{P^*_{H,t}(j)}{P_{H,t+k}} = 1, \\ & \Rightarrow P^*_{H,t}(j) = P_{H,t-1} = P_{H,t} = P_{H,t+k} = P_H. \end{aligned}$$

Divide both sides by $P_{H,t-1}$,

$$\frac{P^*_{H,t}(j)}{P_{H,t-1}} E_t \sum \theta^k e^{-\delta k} P^{\epsilon-1}_{H,t+k} = \frac{\epsilon}{\epsilon-1} E_t \sum \theta^k e^{-\delta k} \Psi'_{H,t+k|t}(\cdot) \frac{P^{\epsilon}_{H,t+k}}{P_{H,t-1}}$$

The Taylor expansion of LHS is with respect to three dimensions $P^*_{H,t}(j)$, $P_{H,t-1}$, $P_{H,t+k}$,

$$\begin{aligned} LHS : \sum \theta^k e^{-\delta k} P^{\epsilon-1}_{H} + & \frac{1}{P_H} \sum \theta^k e^{-\delta k} P^{\epsilon-1}_{H} (P^*_{H,t}(j) - P_H) \\ & - \frac{P_H}{P^2_H} \sum \theta^k e^{-\delta k} P^{\epsilon-1}_{H} (P_{H,t-1} - P_H) \\ & + E_t \sum \theta^k e^{-\delta k} (\epsilon-1) P^{\epsilon-2}_{H} (P_{P_{H,t+k}} - P_H). \end{aligned}$$

By definition of log deviation, we introduce logarithm terms expressed in lower case letters,

$$\begin{aligned} & E_t \sum \theta^k e^{-\delta k} P^{\epsilon-1}_{H} [1 + (p^*_{H,t}(j) - p_H) - (p_{H,t-1} - p_H) + (\epsilon-1)(p_{H,t+k} - p_H)], \\ & = P^{\epsilon-1}_{H} E_t \sum \theta^k e^{-\delta k} [1 + p^*_{H,t}(j) - p_{H,t-1} + (\epsilon-1)(p_{H,t+k} - p_H)]. \end{aligned}$$

Similarly, the Taylor expansion to RHS is three-dimensioned too $\Psi'_{H,t+k}(\cdot)$, $P_{H,t-1}$, $P_{H,t+k}$,

$$\begin{aligned} RHS : \frac{\epsilon}{\epsilon-1} \sum \theta^k e^{-\delta k} P^{\epsilon-1}_{H} \Psi'_{H}(\cdot) + & \frac{\epsilon}{\epsilon-1} E_t \sum \theta^k e^{-\delta k} P^{\epsilon-1}_{H} [\Psi'_{H,t+k}(\cdot) - \Psi'_{H}(\cdot)] \\ & + \frac{\epsilon}{\epsilon-1} E_t \sum \theta^k e^{-\delta k} P^{\epsilon-1}_{H} \Psi'_{H}(\cdot) \epsilon P^{\epsilon-2}_{H} (P_{H,t+k} - P_H) \\ & - \frac{\epsilon}{\epsilon-1} \sum \theta^k e^{-\delta k} P^{\epsilon-1}_{H} \Psi'_{H}(\cdot) P^{\epsilon-2}_{H} (P_{H,t-1} - P_H). \end{aligned}$$

Note that the real marginal cost and the log deviation of real marginal cost are

$$\Psi'_H(\cdot) = \frac{\epsilon - 1}{\epsilon},$$
$$\psi'_{H,t+k}(\cdot) - \psi'_H(\cdot) = \frac{\Psi'_{H,t+k}(\cdot) - \Psi'_H(\cdot)}{\Psi'_H(\cdot)}.$$

Re-arrange the equation and introduce logarithm terms,

$$\frac{\epsilon}{\epsilon - 1}\Psi'_H(\cdot)P_H^{\epsilon-1}E_t\sum\theta^k e^{-\delta k}[1 + [\psi'_{H,t+k}(\cdot) - \psi'_H(\cdot)] + \epsilon(p_{H,t+k} - p_H) - (p_{H,t-1} - p_H)],$$
$$= P_H^{\epsilon-1}E_t\sum\theta^k e^{-\delta k}[1 + [\psi'_{H,t+k}(\cdot) - \psi'_H(\cdot)] + \epsilon(p_{H,t+k} - p_H) - (p_{H,t-1} - p_H)].$$

Now we can combine LHS with RHS,

$$E_t\sum\theta^k e^{-\delta k}[1 + p^*_{H,t}(j) - p_{H,t-1} + (\epsilon - 1)(p_{H,t+k} - p_H)]$$
$$= E_t\sum\theta^k e^{-\delta k}[1 + [\psi'_{H,t+k}(\cdot) - \psi'_H(\cdot)] + \epsilon(p_{H,t+k} - p_H) - (p_{H,t-1} - p_H)].$$

Cancel out the common terms,

$$E_t\sum\theta^k e^{-\delta k}[p^*_{H,t}(j) + (-1)(p_{H,t+k})] = E_t\sum\theta^k e^{-\delta k}[\psi'_{H,t+k}(\cdot) - \psi'_H(\cdot)],$$
$$\Rightarrow E_t\sum\theta^k e^{-\delta k}p^*_{H,t}(j) = \frac{p^*_{H,t}(j)}{1 - \theta e^{-\delta}} = E_t\sum\theta^k e^{-\delta k}[p_{H,t+k} + \psi'_{H,t+k}(\cdot) - \psi'_H(\cdot)].$$

Then the optimal log price for firm j shall be

$$p^*_{H,t}(j) = -\psi'_H(\cdot) + (1 - \theta e^{-\delta})E_t\sum\theta^k e^{-\delta k}[p_{H,t+k} + \psi'_{H,t+k}(\cdot)].$$

We can calculate the log value of real marginal cost as we know $\Psi'_H(\cdot) = \frac{\epsilon-1}{\epsilon}$,

$$-\psi'_H(\cdot) = -\ln\frac{\epsilon - 1}{\epsilon} = \ln\frac{\epsilon}{\epsilon - 1}$$
$$= \ln\left[1 + \frac{\epsilon - (\epsilon - 1)}{\epsilon - 1}\right] \approx \frac{1}{\epsilon - 1}.$$

4.2 Deriving NKPC and DIS

In this subsection, we first relate current inflation to a functional form of expected future inflation and real marginal cost. When labour and goods markets are cleared, real marginal cost can be related to domestic output. Thus NKPC is obtained, telling us that current inflation depends on future inflation and output gap. With earlier derived expression of actual nominal rate plus the natural rate, we can derive DIS. Output gap path is determined given exogenous nominal rate and real rate.

We start from subtracting both sides by $p_{H,t-1}$

$$\begin{aligned} p^*_{H,t}(j) - p_{H,t-1} =& -\psi'_H(\cdot) - \frac{p_{H,t-1}}{1-\theta e^{-\delta}} + (1-\theta e^{-\delta})E_t \sum \theta^k e^{-\delta k}[p_{H,t+k} + \psi'_{H,t+k}(\cdot)], \\ =& -\psi'_H(\cdot) + (1-\theta e^{-\delta})E_t \sum \theta^k e^{-\delta k}[p_{H,t+k} - p_{H,t-1} + \psi'_{H,t+k}(\cdot)]. \end{aligned}$$

Let us re-write the RHS in terms of log inflation

$$\begin{aligned} p^*_{H,t}(j) - p_{H,t-1} = -\psi'_H(\cdot) &+ (1-\theta e^{-\delta})E_t \sum \theta^k e^{-\delta k}\psi'_{H,t+k}(\cdot) \\ &+ (1-\theta e^{-\delta})E_t[(p_{H,t} - p_{H,t-1}) \\ &+ \theta e^{-\delta}(p_{H,t+1} - p_{H,t} + p_{H,t} - p_{H,t-1}) \\ &+ (\theta e^{-\delta})^2(p_{H,t+2} - p_{H,t+1} + p_{H,t+1} - p_{H,t} + p_{H,t} - p_{H,t-1}) + \cdots] \end{aligned}$$

Collecting terms of marginal cost and distributing sum of log prices to $(1-\theta e^{-\delta})$, we can largely simplify the above equation

$$\begin{aligned} p^*_{H,t}(j) - p_{H,t-1} =& (1-\theta e^{-\delta})E_t \sum \theta^k e^{-\delta k}[\psi'_{H,t+k}(\cdot) - \psi'_H(\cdot)] \\ &+ E_t[\pi_{H,t} + \theta e^{-\delta}(\pi_{H,t+1} + \pi_{H,t}) + (\theta e^{-\delta})^2(\pi_{H,t+2} + \pi_{H,t+1} + \pi_{H,t}) + \cdots] \\ &- E_t[\theta e^{-\delta}\pi_{H,t} + (\theta e^{-\delta})^2(\pi_{H,t+1} + \pi_{H,t}) + (\theta e^{-\delta})^3(\pi_{H,t+2} + \pi_{H,t+1} + \pi_{H,t}) + \cdots], \\ =& (1-\theta e^{-\delta})E_t \sum \theta^k e^{-\delta k}\tilde{\psi}'_{H,t+k}(\cdot) + E_t[\pi_{H,t} + \theta e^{-\delta}\pi_{H,t+1} + (\theta e^{-\delta})^2\pi_{H,t+2} + \cdots], \end{aligned}$$

where the tilde over marginal cost $\tilde{\psi}'_{H,t+k}(\cdot) = \psi'_{H,t+k}(\cdot) - \psi'_H(\cdot)$ indicates the log deviation of marginal cost from its steady state value.

To re-write the expression of inflation in more compact (recursive) form, we separate the current terms ($k = 0$) from the summation,

$$\begin{aligned} p^*_{H,t}(j) - p_{H,t-1} =& (1-\theta e^{-\delta})E_t \sum_{k=1}^{\infty} \theta^k e^{-\delta k}\tilde{\psi}'_{H,t+k}(\cdot) + E_t \sum_{k=1}^{\infty} \theta^k e^{-\delta k}\pi_{H,t+k} + \underbrace{(1-\theta e^{-\delta})\tilde{\psi}'_{H,t}(\cdot) + \pi_{H,t}}_{\text{when } k=0}, \\ =& \theta e^{-\delta}E_t(p^*_{H,t+1}(j) - p_{H,t}) + (1-\theta e^{-\delta})\tilde{\psi}'_{H,t}(\cdot) + \pi_{H,t}. \end{aligned}$$

Recall the aggregate price dynamic in Home country (33)

$$\frac{\pi_{H,t}}{1-\theta} = p^*_{H,t}(j) - p_{H,t-1}.$$

Inflation at period t can be expressed in term of expected inflation and log deviation of marginal cost,

$$\begin{aligned}\frac{\pi_{H,t}}{1-\theta} - \pi_{H,t} = \frac{1-(1-\theta)}{1-\theta}\pi_{H,t} = \theta e^{-\delta} E_t\left(\frac{\pi_{H,t+1}}{1-\theta}\right) + (1-\theta e^{-\delta})\tilde{\psi}'_{H,t}(\cdot),\\ \Rightarrow \pi_{H,t} = e^{-\delta} E_t \pi_{H,t+1} + \frac{(1-\theta e^{-\delta})(1-\theta)}{\theta}\tilde{\psi}'_{H,t}(\cdot).\end{aligned}$$

Note that we observe no variables characterizing international trade in NKPC. Thanks to frictionless foreign exchange market, we can exclude foreign variables from firm's price setting problem by substituting aggregate consumption with domestic production. Exchange rate only reflects home bias, country size and technology difference between two countries.

Next, we further arrange log deviation of marginal cost. The goal is to come up with output gap from log deviation of marginal cost. Then NKPC is completed.

$$\begin{aligned}\psi'_{H,t}(\cdot) =& \frac{\epsilon-1}{\epsilon} = w_{1,t} - p_{H,t} - a_{H,t},\\ =&(w_{1,t} - p1,t) + (p_{1,t} - p_{H,t}) - a_{H,t},\\ =&(c_{1,t} + \psi n_{H,t}) + [\ln\tau + (1-\alpha)s_t] - a_{H,t},\\ =&[c_{2,t} + (2\alpha - 1)s_t] + \psi(y_{H,t} - a_{H,t}) + [\ln\tau + (1-\alpha)s_t] - a_{H,t},\\ =&[y_{F,t} + (1-\alpha)s_t - \ln\tau] + (\alpha s_t + \ln\tau) + \psi y_{H,t} - (1+\psi)a_{H,t},\\ =&y_{F,t} + s_t + \psi y_{H,t} - (1+\psi)a_{H,t}.\end{aligned}$$

The real marginal cost can be expressed as a function of Home and Foreign output, exchange rate, and domestic technology. With Lemma 3.1 and due to log utility assumption, we can again exclude the foreign output from the expression of marginal cost,

$$\psi'_{H,t}(\cdot) = (1+\psi)(y_{H,t} - a_{H,t}). \tag{38}$$

With flexible price, domestic output attains natural level, denoted as $\overline{y}_{H,t}$

$$\psi'_H(\cdot) = (1+\psi)(\overline{y}_{H,t} - a_{H,t}). \tag{39}$$

Thus, log deviation of marginal cost is

$$\tilde{\psi}'_{H,t}(\cdot) = (1+\psi)(y_{H,t} - \overline{y}_{H,t}) = (1+\psi)\tilde{y}_{H,t}. \tag{40}$$

NKPC is

$$\pi_{H,t} = e^{-\delta} E_t \pi_{H,t+1} + \frac{(1-\theta e^{-\delta})(1-\theta)(1+\psi)}{\theta} \tilde{y}_{H,t}. \tag{41}$$

Recall nominal interest rate from Proposition 3.1 and write it in discrete time

$$\begin{aligned} i_{1,t} =& \delta + E_t \pi_{1,t+1} + \Delta c_{1,t+1}, \\ =& \delta + E_t \pi_{1,t+1} + [\Delta y_{H,t+1} - (1-\alpha)\Delta s_{t+1}], \end{aligned}$$

The nature rate is

$$r^n_{1,t} = \delta + E_t(\Delta c^n_{t+1}) = \delta + E_t(\Delta y^n_{H,t+1}) - (1-\alpha)E_t(\Delta s_{t+1}).$$

Combine the above two equations,

$$i_{1,t} - E_t(\pi_{1,t+1}) - r^n_{1,t} = E_t(\Delta \tilde{y}_{H,t+1}).$$

Then, DIS is

$$\tilde{y}_{H,t} = E_t(\tilde{y}_{H,t+1}) - [i_{1,t} - E_t(\pi_{1,t+1}) - r^n_{1,t}],$$

Replacing $\pi_{1,t+1}$ with $\pi_{H,t+1}$, we introduce rate of exchange growth into DIS

$$\tilde{y}_{H,t} = E_t(\tilde{y}_{H,t+1}) - [i_{1,t} - E_t(\pi_{H,t+1}) - (1-\alpha)E_t(\Delta s_{t+1}) - r^n_{1,t}], \tag{42}$$

where $\Delta s_{t+1} = 0$ only if both Home and Foreign countries have homogeneous growth in technology development, i.e. $\mu_H = \mu_F$. In that case, DIS degenerates to a closed economy equilibrium.

We now write NKPC in terms of dt and take the limit $dt \to 0$. The continuous forms of NKPC is

$$\begin{aligned}
\pi_{H,t} =& e^{-\delta dt} E_t(\pi_{H,t} + d\pi_{H,t}) + \frac{(1 - e^{\ln\theta dt} e^{-\delta dt})(1-\theta)(1+\psi)}{\theta} \tilde{y}_{H,t}, \\
=& (1 - \delta dt)\pi_{H,t} + (1 - \delta dt) E_t(d\pi_{H,t}) + \frac{[1 - e^{-(\delta - \ln\theta)dt}](1-\theta)(1+\psi)}{\theta} \tilde{y}_{H,t}, \\
0 =& -\delta \pi_{H,t} dt + E_t(d\pi_{H,t}) + \frac{[(\delta - \ln\theta)dt](1-\theta)(1+\psi)}{\theta} \tilde{y}_{H,t} + O(dt), \\
dt \to 0 : & E_t\left(\frac{d\pi_{H,t}}{dt}\right) = \delta\pi_{H,t} - \frac{1}{\theta}(1-\theta)(\delta - \ln\theta)(1+\psi)\tilde{y}_{H,t}.
\end{aligned}$$

Similarly, DIS in continuous time is

$$\begin{aligned}
E_t(\tilde{y}_{H,t+1}) - \tilde{y}_{H,t} + (1-\alpha)[E_t(s_{t+1}) - s_t] + \pi_{H,t} = i_{1,t} - E_t(\pi_{H,t+1}) - r^n_{1,t} + \pi_{H,t}, \\
E_t[d\tilde{y}_{H,t} + (1-\alpha)ds_t] = -\pi_{H,t}dt + i_{1,t}dt - E_t(d\pi_{H,t}) - r^n_{1,t}dt, \\
E_t[d\tilde{y}_{H,t} + (1-\alpha)ds_t + d\pi_{H,t}] = (i_{1,t} - r^n_{1,t} - \pi_{H,t})dt.
\end{aligned}$$

Proposition 4.1. *Non-policy building block of New Keynesian Model with an open economy can be summarized by Home-NKPC and DIS,*

$$\begin{aligned}
E_t\left(\frac{d\pi_{H,t}}{dt}\right) &= \delta\pi_{H,t} - \frac{1}{\theta}(1-\theta)(\delta - \ln\theta)(1+\psi)\tilde{y}_{H,t}; \\
E_t\left[\frac{d\tilde{y}_{H,t}}{dt} + (1-\alpha)\frac{ds_t}{dt} + \frac{d\pi_{H,t}}{dt}\right] &= i_{1,t} - r^n_{1,t} - \pi_{H,t}.
\end{aligned}$$

Furthermore, the corresponding NKPC and DIS in Foreign country

$$\begin{aligned}
E_t\left(\frac{d\pi_{F,t}}{dt}\right) &= \delta\pi_{F,t} - \frac{1}{\theta}(1-\theta)(\delta - \ln\theta)(1+\psi)\tilde{y}_{F,t}; \\
E_t\left[\frac{d\tilde{y}_{F,t}}{dt} - (1-\alpha)\frac{ds_t}{dt} + \frac{d\pi_{F,t}}{dt}\right] &= i_{2,t} - r^n_{2,t} - \pi_{F,t}.
\end{aligned}$$

Together with an exogenously specified interest rate rule, we develop 3 equations for a standard New Keynesian model in each country.

5 Concluding Remark

We have built up a New Keynesian model with price rigidity of international trade, a nominal economy. As labours are assumed to be immobile across countries, the Phillips Curves in both countries are independent of foreign variables. The dynamic investment and saving curves shows the influence of exchange rate evolution on the output gap. Together with interest rate rule specification, we will be able to solve for the dynamic of output gap and inflation in both countries. Furthermore, with the optimal price chosen under the price rigidity, we can derive the stock price on country level.

5.1 Next Step

So far we have derived NKPC and DIS for both countries. Following standard approach in macroeconomic, next step is to specify monetary policy in terms of an interest rate rule (i.e. a Taylor rule) in each country. Then we are ready to solve for the evolution of output gap and country-level inflation. We would like to stress the heterogeneity on the choice of interest rate rule. There are several ways to address the heterogeneity: first of all, on the target interest rate, secondly, on the Taylor rule coefficient and on the components of Taylor rule. There is a large fraction of countries, more than 50% in terms of world GDP, manipulating their exchange rate. It is common for the central banks to peg their currency to major trade partners. We would like to introduce the exchange rate target in one of our two countries to address this fact. The goal is to show that monetary policy will be rigid for pegging country and that will lead to weaker economic recovery.