

A Macro-Finance Model for Crude Oil Futures
Prices in the Real Economy

Haicheng Shu
The University of York

THIS PAPER IS WORK IN PROGRESS,
PARAGRAPHS, MODEL SPECIFICATIONS,
EMPIRICAL EVIDENCES,
ARE ALL SUBJECT TO CHANGES

June 3, 2016

Abstract

This paper presents a macro-finance model of the economy and crude oil markets. This model allows us to study interactions between the convenience yield, the spot and futures markets, monetary policy and macroeconomic indicators. We use the Kalman filter to represent latent variables that handle the effect of exogenous shocks to inflation, the oil price and convenience yield, and to deal with missing observations. Traditional models use latent variables with little economic meaning to explain commodity futures, while this model also makes the effect of macroeconomic variables explicit. This model will be of interest to Central Banks and monetary policy makers, since the discussions in macro-finance models for commodity futures has been neglected by the practitioners and the literatures for many years.

Contents

1	Introduction	5
2	The arbitrage relationship	8
3	Macro-finance model	10
3.1	The state variable dynamics under the probability measure P	10
3.1.1	The latent factor dynamics	10
3.1.2	The observable variable dynamics	11
3.2	The state variable dynamic under the probability measure Q	14
3.3	Cross sectional parameters under the probability measure Q	16
4	The Kalman Filter and the likelihood function using the measure P	18
4.1	The state space representation	18
4.2	Kalman filter and the maximum likelihood estimation	19
5	Empirical implementation	21
5.1	Data	21
6	Findings and results	26
6.1	The state variable estimates	26
6.2	The parameter estimates	32
6.3	Impulse response functions	37
6.4	Variance decomposition	42
7	Concluding remarks	45
8	Appendix	47
8.1	The companion form under probability measure P and Q . . .	47
8.2	The state space representation	49
8.3	Change of probability measure	50

8.4	Analysing risk premium using Return Forecasting Regression .	53
8.5	Kalman filter and the maximum likelihood estimation	55
8.6	Missing observations	58

1 Introduction

This paper presents a macro-finance model of the economy and crude oil markets. This model allows us to study interactions between the convenience yield, the spot and futures markets, monetary policy and macroeconomic indicators.

Theoretical literature on commodity futures pricing models dates back to the early 1980s. (see Schwartz (1982), Brennan and Schwartz (1985), Gibson and Schwartz (1990), Brennan (1991), Cortazar and Schwartz (1994), Ross (1995), Schwartz (1997), Casassus and Collin-Dufresne (2003)). These models suggest that the term structure model of commodity futures prices employs three latent variables, and is similar to the yield factor model of the interest rate term structure which also employs latent variables. Although the three latent variables can explain most of the variation in the commodity futures term structure, these traditional futures pricing model is silent about the nature of these latent variables and their links with the macro economy.

Central Banks and policy makers are interested in the interaction between the macro economy and the financial markets. Conventionally, when evaluating how their policies affect the financial market, their attention is primarily on the bond and equity markets. However the oil futures market is also important, and central bank researchers have been looking at them recently (see Chin and Liu (2015), Millard and Shakir (2013), Elekdag et al (2007), Jo (2012), Bank of England (2015a), Bank of England (2015b)). Oil prices affect industrial costs, consumer price inflation and thus consumer spending power. We have seen how oil prices shocks hit inflation and output and affect monetary policy. We have also witnessed how the demand pressure created by the global economic expansion during the past fifteen years drove the oil price to a historically high level. More recently, we have seen how the slowdown in the in emerging markets such as China, has combined with a surge in production in the US, Russia, Saudi Arabia to collapse the oil price.

Macro-finance models have been developed to study the interactions between macroeconomic variables and term structure dynamics following the pioneering research of Ang and Piazzesi (2003). They successfully introduced macroeconomic indicators into the interest rate term structure model. They found that although these indicators provide a good description of the behaviour of short rates, it was necessary to retain latent variables to model long term rates. Subsequent research suggests that the main latent variable represents the exogenous shocks to central bank inflation target or underlying rate of inflation (see Kozicki and Tinsley (2005), Dewachter and Lyrio (2006), Dewachter, Lyrio and Maes (2006)).

We apply this methodology to the oil futures market. We follow the work on the term structure of interest rates in adopting the “central bank model” developed by Svensson (1999), Rudebusch (2002), Smets (1999), Kozicki and Tinsley (2005) and others. This model represents the behaviour of the macroeconomy in terms of real US output gap (g_t), inflation (π_t) and the short term interest rate (r_t). We also model the underlying inflation rate as a latent variable using the Kalman filter. Meanwhile, we add the real oil price and its convenience yield to explore the links with the economy. We use two latent variables to handle exogenous supply side and other shocks to the oil price and convenience yield. We impose equilibrium restrictions demonstrated in Spencer (2008) and Spencer and Liu (2010) to describe the long run relationship between the observed and latent variables.

We made two modifications to this model in order to extend it to the oil futures market. First, crude oil futures contracts are specified in nominal terms, while the macro model is naturally specified in terms of the real oil price. Thus we need to apply a price level adjustment to get the nominal oil price. The resulting model is linear homoscedastic and so the log futures prices are affine in these variables. Second, we use a long run of data to identify the links between the oil price and the macro economy, including the period of the 1970s oil shocks. However, oil future prices are only available

for more recent periods. We used the Kalman filter to resolve the resulting missing observation problem.

The empirical results are consistent with the existing economic literature and intuition. Impulse response functions are in line with our preliminary macro analysis, and support general economic theories such as the Taylor rule. The long run inflation asymptote is modelled as a latent variable, and clearly identifies historical macroeconomic events such as the Volker deflation and the corresponding monetary regime switch at the beginning of the 1980s. The inflation asymptote also clearly reveals impacts from the intensified upward moving trend due to the oil shocks in the 1970s, and the downward sloping trend after the monetary regime switch. Furthermore, we find that the dynamic of a long term factor, which represents shocks affecting the oil inventory, picks up historical events such as the Gulf war in the year 1990 to 1991, the internet bubble and Asian financial crisis roughly in the year 1997 to 2000, and the September 11 attack and the subsequent Iraq invasion at the beginning of 2000s, the 2008 financial crisis, and the European debt crisis after the year 2010. The model of convenience yield suggests that crude oil inventory plays the role as a buffer damping the effect of oil and economic shocks on the real economy.

The remainder of this paper will be organized as follows. Section 2 introduces the log real spot oil price expression implied by the no arbitrage relationship with price level adjustment. Section 3 introduces the theoretical macro-finance model. Specifying the state variable dynamics under the real world measure and risk neutral measure, the risk premium specification and the change of probability measures, the cross sectional parameters solved from the affine term structure model. Section 4 and 5 demonstrates the empirical methodologies and also describes the data that we are using. Section 6 discusses the empirical findings and results. Section 7 concludes.

2 The arbitrage relationship

In this section we specify the arbitrage relationships relating the futures prices to the spot price, convenience yield and interest rate. We start with the well known property of futures prices: they follow a martingale under Q .

$$F_{\tau,t} = E_t^Q(F_{\tau-1,t+1}) = S_t e^{(r_t - \delta_t)\tau} \quad \tau \geq 1. \quad (1)$$

This is because these contracts do not yield dividends or convenience yields and do not have a cost of carry. We also know that the maturity value of the futures price will always equal the future spot price. So for the special case of $\tau = 1$:

$$F_{0,t+1} = S_{t+1}. \quad (2)$$

Taking expectations of both sides at time t under the risk neutral measure Q , noting that $F_{1,t}$ is known at that time:

$$F_{1,t} = E_t^Q(S_{t+1}). \quad (3)$$

The risk neutral spot oil price dynamics follow by combining this with the standard arbitrage condition for a *forward* price. Importantly for the special case of $\tau = 1$, the interest rate is known and so this also holds for the futures price:

$$F_{1,t} = S_t e^{(r_t - \delta_t)}, \quad (4)$$

where δ_t is the convenience yield and r_t the interest rate. Equating (3) and (4) and taking logs:

$$\ln E_t^Q(S_{t+1}) = s_t + r_t - \delta_t. \quad (5)$$

Finally, suppose that S_{t+1} is lognormal under Q so that taking logs again:

$$s_{t+1} = s_t + \mu_t + \epsilon_{s,t+1}^Q \quad \epsilon_{s,t+1}^Q \sim N(0, \sigma_s^2). \quad (6)$$

Importantly, μ_t is the expected value of the log price change so that:

$$E_t^Q(S_{t+1}) = S_t e^{(\mu_t + \frac{1}{2}\sigma_s^2)}. \quad (7)$$

Taking logs and substituting into (5) gives:

$$\mu_t = r_t - \delta_t - \frac{1}{2}\sigma_s^2. \quad (8)$$

Finally, substituting this back into equation (6) gives the dynamic equation for the spot price under probability measure Q :

$$s_{t+1} = s_t + r_t - \delta_t - \frac{1}{2}\sigma_s^2 + \epsilon_{s,t+1}^Q \quad \epsilon_{s,t+1}^Q \sim N(0, \sigma_s^2) \quad (9)$$

Clearly, it is the nominal futures price that we use in above derivations to obtain equation (9), which implies s_{t+1} to be the nominal log spot oil price factor, and equation (9) to be the nominal arbitrage relationship. However, the macro model works with the real oil price s_{t+1}^R , we can represent s_{t+1} as the sum of the log real oil price and the log CPI price level p_t . It follows another arbitrage relationship derived by adjusting this for inflation:

$$s_{t+1} = s_{t+1}^R + p_{t+1} = (s_t^R + p_t) + r_t - \delta_t - \frac{1}{2}\sigma_s^2 + \epsilon_{s,t+1}^Q, \quad (10)$$

to get the real arbitrage relationship:

$$s_{t+1}^R = s_t^R - \pi_{t+1} + r_t - \delta_t - \frac{1}{2}\sigma_s^2 + \epsilon_{s,t+1}^Q. \quad (11)$$

where $\pi_{t+1} = p_{t+1} - p_t$ is the inflation equal to the first difference of the price level taking natural logarithm.

3 Macro-finance model

3.1 The state variable dynamics under the probability measure P

In this section we set out the model of the real oil price and the world economy, which are assumed to be interdependent. This is specified under the real world measure P . Note that, for the convenience of notation, in this section, expectation, parameters and error terms are all defined under the real world measure P , without superscripts.

The macro economy is represented by four observed variables: g_t is US output gap based on the US GDP constant price series represents the business cycle, π_t is the US inflation rate calculated by implicit price deflator and r_t is the US interest rate. We also include the observed log spot crude oil price deducting implicit price deflator as one observed variable, denoted as s_t^R , this is to replace the spot oil price as a latent variable suggested in previous futures pricing literatures. Similarly, we include the implied convenience yield, δ_t , in the observed system. In another word, we are considering the observed vector, composed by the convenience yield and the four macro variables, $m_t = \begin{pmatrix} \delta_t & s_t^R & g_t & \pi_t & r_t \end{pmatrix}$ measured without measurement errors. We argue this vector is influenced by three latent variables. Respectively they are: a long term factor representing shocks affecting the oil inventory from the demand side, denoted as δ_t^* ; a real spot oil price trend (or the underlying real spot oil price), denoted as s_t^* ; and a long run inflation asymptote, as a policy indicator, denoted as π_t^* ; This conveniently allows us to estimate the parameters of the KVAR used to model the macro economy

3.1.1 The latent factor dynamics

The the long term convenience yield factor (δ_t^*), underlying real spot price trend (s_t^*), and inflation asymptote (π_t^*) follow their own stochastic processes.

Considering δ_t^* is a $I(0)$ variable, s_t^* and π_t^* are integrated of order one ($I(1)$), we define these processes as:

$$\delta_t^* = \kappa_{\delta^*} + \xi_{\delta^*} \delta_{t-1}^* + \epsilon_{\delta^*,t} \quad \epsilon_{\delta^*,t} \sim N(0, \sigma_{\delta^*}^2), \quad (12)$$

$$s_t^* = \kappa_{s^*} + s_{t-1}^* + \epsilon_{s^*,t} \quad \epsilon_{s^*,t} \sim N(0, \sigma_{s^*}^2), \quad (13)$$

$$\pi_t^* = \kappa_{\pi^*} + \pi_{t-1}^* + \epsilon_{\pi^*,t} \quad \epsilon_{\pi^*,t} \sim N(0, \sigma_{\pi^*}^2). \quad (14)$$

Stack equation (12), (13) and (14) together, putting the latent variables in the vector $z_t = \begin{pmatrix} \delta_t^* & s_t^* & \pi_t^* \end{pmatrix}$ we can write the latent factor dynamics compactly as:

$$z_t = K_z + \Upsilon_z z_{t-1} + \epsilon_{z,t} \quad \epsilon_{z,t} \sim (0, \Sigma_{z,t}), \quad (15)$$

This specification emphasizes that the three latent variables are assumed to be independent.

3.1.2 The observable variable dynamics

To impose long term trend of convenience yield (δ^*), we introduce the observed convenience yield (δ_t) implied by the no arbitrage futures pricing equation (1). Casassus and Collin-Dufresne (2003) suggests convenience yield is determined by the log spot price of commodity product and the interest rate. We argue a stationary variable represents the long term convenience yield factor, denoted as δ_t^* , also known as the underlying convenience yield, is affecting the convenience yield temporarily. Therefore, we define δ_t follows the stochastic process as:

$$\delta_t = \kappa_{\delta} + \theta_{\delta,\delta^*} \delta_t^* + \theta_{\delta,s^*} s_t^* + \theta_{\delta,\pi^*} \pi_t^* + \phi_{\delta,\delta} \delta_{t-1} + \phi_{\delta,s} s_{t-1} + \phi_{\delta,r} r_{t-1} + \epsilon_{\delta,t} \quad (16)$$

Furthermore, the macro variables, including real oil price, denoted as s^R , are interdependent with each other and the latent system, except for the

convenience yield and its long term trend, specified in the forms as:

$$s_t^R = \kappa_s + \theta_{s,\delta^*} \delta_t^* + \theta_{s,s^*} s_t^* + \theta_{s,\pi^*} \pi_t^* + \phi_{s,\delta} \delta_{t-1} + \phi_{s,s} s_{t-1}^R + \phi_{s,g} g_{t-1} + \phi_{s,\pi} \pi_{t-1} + \phi_{s,r} r_{t-1} + \epsilon_{s,t} \quad (17)$$

$$g_t = \kappa_g + \theta_{g,s^*} s_t^* + \theta_{g,\pi^*} \pi_t^* + \phi_{g,s} s_{t-1}^R + \phi_{g,g} g_{t-1} + \phi_{g,\pi} \pi_{t-1} + \phi_{g,r} r_{t-1} + \epsilon_{g,t} \quad (18)$$

$$\pi_t = \kappa_\pi + \theta_{\pi,s^*} s_t^* + \theta_{\pi,\pi^*} \pi_t^* + \phi_{\pi,s} s_{t-1}^R + \phi_{\pi,g} g_{t-1} + \phi_{\pi,\pi} \pi_{t-1} + \phi_{\pi,r} r_{t-1} + \epsilon_{\pi,t} \quad (19)$$

$$r_t = \kappa_r + \theta_{r,s^*} s_t^* + \theta_{r,\pi^*} \pi_t^* + \phi_{r,s} s_{t-1}^R + \phi_{r,g} g_{t-1} + \phi_{r,\pi} \pi_{t-1} + \phi_{r,r} r_{t-1} + \epsilon_{r,t} \quad (20)$$

Combining equation (16), (17), (18), (19), and (20), the observed variables follow the dynamic system under the measure P :

$$m_t = K_m^- + \Theta z_t + \Upsilon_m m_{t-1} + \epsilon_{m,t} \quad \epsilon_{m,t} \sim (0, \Sigma_{m,t}), \quad (21)$$

Notice that z_t in equation (21) is contemporary with left hand side m_t . We substitute equation (15) into equation (21) to introduce the one lag to z_t as:

$$\begin{aligned} m_t &= K_m^- + \Theta(K_z + \Upsilon_z z_{t-1} + \epsilon_{z,t}) + \Upsilon_m m_{t-1} + \epsilon_{m,t} \\ &= K_m^- + \Theta K_z + \Theta \Upsilon_z z_{t-1} + \Upsilon_m m_{t-1} + \eta_{m,t}, \end{aligned} \quad (22)$$

where:

$$\eta_{m,t} = \epsilon_{m,t} + \Upsilon_{z,m} \epsilon_{z,t} \quad \eta_{m,t} \sim N(0, \Sigma_{m,t}). \quad (23)$$

Stacking equation (15) and (21) gives the KVAR model with real oil price under measure P , which we can represent compactly as the following

companion form:

$$X_t = \begin{pmatrix} \delta_t^* & s_t^* & \pi_t^* & \delta_t & s_t^R & g_t & \pi_t & r_t \end{pmatrix}' = K + \Upsilon X_{t-1} + W_t \quad W_t \sim N(0, \Sigma_t), \quad (24)$$

Let $\Upsilon_{z,m} = \Theta \Upsilon_z$ and $K_m = K_m^- + \Theta K_z$, we have equation (24) in matrix form:

$$\begin{pmatrix} z_t \\ m_t \end{pmatrix} = \begin{pmatrix} K_z \\ K_m \end{pmatrix} + \begin{pmatrix} \Upsilon_z & 0_{3,5} \\ \Upsilon_{z,m} & \Upsilon_m \end{pmatrix} \begin{pmatrix} z_{t-1} \\ m_{t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{z,t} \\ \eta_{m,t} \end{pmatrix}. \quad (25)$$

W_t in equation (24) can be decomposed by ‘‘LDL’’ decomposition as:

$$W_t = LDV_t \quad V_t \sim N(0, I), \quad (26)$$

This is all we need to model the real world dynamics. However, because the cross sectional pricing relationships are nominal we need to add the log CPI price level p_t in to the state vector.

We are interested in the the long term relationship between the observed variables and the latent factors. As we know, the real spot oil price s_t^R depends on other macro variables. We are in a position to specify the long term parameters of the third latent factor, which is interpreted as the inflation asymptote π_t^* , to the real spot oil price and the macro vector.

As implied by Spencer (2008) and Spencer and Liu (2010) etc, in the steady state, the observed vector m_t , as in the second row of equation (25), has a central tendency in the long term:

$$X_m^* = \begin{pmatrix} \delta^* & s^{R*} & g^* & \pi^* & r^* \end{pmatrix}' = K_m^* + \Upsilon_m^* X_m^* + \Upsilon_{z,m}^* X_z^*. \quad (27)$$

rearranging equation (27) we have:

$$X_m^* = (I - \Upsilon_m^*)^{-1} K_m^* + (I - \Upsilon_m^*)^{-1} \Upsilon_{z,m}^* X_z^*, \quad (28)$$

let $(I - \Upsilon_m)^{-1}K_m = \varphi$ and $(I - \Upsilon_m)^{-1}\Upsilon_{z,m} = R$, we respecify K_m and $\Upsilon_{z,m}$ in (25) to employ the state variable central tendency in steady state as:

$$K_m^* = (I - \Upsilon_m^*)\varphi \quad (29)$$

$$\Upsilon_{z,m}^* = (I - \Upsilon_m^*)R, \quad (30)$$

hence, observed vector in equation (25) can be respecified as:

$$m_t = K_m^* + \Upsilon_{z,m}^*z_{t-1} + \Upsilon_m m_{t-1} + \epsilon_{m,t} \quad \epsilon_{m,t} \sim (0, \Sigma_{m,t}), \quad (31)$$

where:

$$\varphi = \begin{pmatrix} \varphi_\delta \\ \varphi_s \\ \varphi_g \\ \varphi_\pi \\ \varphi_r \end{pmatrix} \quad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad (32)$$

which represent the long term relationship between the latent variables and the observed vector. As in the long run, δ_t^* and s_t^* are imposed by the implied convenience yield and observed real spot oil price in the long run respectively. π_t^* is related to the inflation and interest rate jointly, we incorporate this by putting ones respectively in the R vector.

3.2 The state variable dynamic under the probability measure Q

In this section, we change to the risk neutral dynamics under the Q measure and use this to get the parameters of the cross section. Importantly, the first element in the state vector is still the real but not the nominal oil price used in the standard arbitrage equation (9). We do this using the essentially affine model of Duffee (2002) to change the measure. This redefines the

deterministic and stochastic parts of the VAR under measure P , in a way that ensures the expectation of W^Q under the Q measure is zero. This implies a system that is congruent with system (24). Specifically:

$$X_t = K^Q + \Upsilon^Q X_{t-1} + W^Q \quad W^Q \sim N(0, \Sigma_t), \quad (33)$$

The third row of Υ^Q specifies the relationship between oil price and the state variable under risk neutral measure, which is implied by the risk neutral arbitrage relationship of nominal oil price as equation (9), adjusting with inflation for the real oil price. This is also implied from our previous real arbitrage relationship, recall equation (11), that we can further specify as:

$$\begin{aligned} s_{t+1}^R &= s_t^R - \pi_{t+1} + r_t - \delta_t - \frac{1}{2}\sigma_s^2 + \epsilon_{s,t+1} \\ &= s_t^R - \Upsilon_\pi^Q X_t + r_t - \delta_t - \frac{1}{2}\sigma_s^2 + \epsilon_{s,t+1}. \end{aligned} \quad (34)$$

It can be shown that K^Q and Υ^Q in equation (33) under the measure Q are related to the K^P and Υ^P in equation (24) under the measure P by:

$$K^Q = K - LDD'\Lambda_1 \quad (35)$$

$$\Upsilon^Q = \Upsilon - L\Lambda_2 \quad (36)$$

where Λ_1 is a 8×1 vector, and Λ_2 is a 8×8 matrix, stand for the risk premium parameters, composed by $\Lambda_{1,z}$, $\Lambda_{1,m}$ and $\Lambda_{2,z}$, $\Lambda_{2,m}$, $\Lambda_{2,z,m}$ as:

$$\Lambda_1 = \begin{pmatrix} \Lambda_{1,z} \\ \Lambda_{1,m} \end{pmatrix} \quad \Lambda_2 = \begin{pmatrix} \Lambda_{2,z} & 0_{3,5} \\ \Lambda_{2,z,m} & \Lambda_{2,m} \end{pmatrix} \quad (37)$$

3.3 Cross sectional parameters under the probability measure Q

Having specified state vector under the Q measure in equation (33), we can start our derivation of cross sectional parameters under the risk neutral measure. First we adopt the trial solution:

$$f_{\tau,t} = \alpha_{\tau} + \Psi_{\tau}X_t + \psi_{p,\tau}p_t. \quad (38)$$

The initial condition, is implied by the special case when $\tau = 0$, that $f_{0,t} = s_t^R + p_t$, which gives the starting values for the first latent factor as $\psi_{s,0} = 1$, as we know this is restricted to the price level, hence we also have $\psi_{p,0} = 1$. Again, because of $f_{0,t} = s_t$, we can see other variables do not play any role when $\tau = 0$, therefore, the initial ψ for the rest of other variables are all equal to zero, as:

$$\psi_{s,0} = 1, \quad (39)$$

$$\psi_{p,0} = 1, \quad (40)$$

$$\psi_{\delta,0} = \psi_{\pi^*,0} = \psi_{g,0} = \psi_{\pi,0} = \psi_{r,0} = 0. \quad (41)$$

This makes the futures prices exponentially affine in the factors. To verify the trial solution (38) and find its parameters we take logs of equation (1) to get:

$$f_{\tau,t} = \ln E_t(F_{\tau-1,t+1}) = E_t(f_{\tau-1,t+1}) + \frac{1}{2}Var(f_{\tau-1,t+1}). \quad (42)$$

Substituting the specification of X_t and p_t as in equations (33) into equa-

tion (38) after incrementing t and reducing τ :

$$\begin{aligned} E_t(f_{\tau-1,t+1}) &= \alpha_{\tau-1} + \Psi_{\tau-1}E_t(X_{t+1}) + \psi_{p,\tau-1}E_t(p_{t+1}) \\ &= \alpha_{\tau-1} + \Psi_{\tau-1}(K^Q + \Upsilon^Q X_t) + \psi_{p,\tau-1}(\Upsilon_\pi^Q X_t + p_t), \end{aligned} \quad (43)$$

$$Var(f_{\tau-1,t+1}) = \Psi_{\tau-1}\Sigma_t\Psi'_{\tau-1}. \quad (44)$$

Substituting these into equation (42) using the starting valued as we discussed in equation (39) and (40) verifies the trial solution in equation (38) provided that:

$$\Psi_\tau = \Psi_{\tau-1}\Upsilon^Q + \Upsilon_\pi^Q, \quad (45)$$

$$\psi_{p,\tau} = \psi_{p,\tau-1} = 1, \quad (46)$$

$$\alpha_\tau = \alpha_{\tau-1} + \Psi_{\tau-1}K^Q + \frac{1}{2}\Psi_{\tau-1}\Sigma_t\Psi'_{\tau-1}, \quad (47)$$

where α is constant, equation (46) implying all ψ_p is equal to one, that is:

$$i = \left(\psi_{p,1} \quad \psi_{p,2} \quad \dots \quad \psi_{p,\tau} \right)' = \left(1 \quad 1 \quad \dots \quad 1 \right)'. \quad (48)$$

Finally, the affine equation, restricted to the price level p_t , can be defined as:

$$f_{\tau,t} = \alpha_\tau + \Psi'_\tau X_t + ip_t, \quad (49)$$

This gives the model of the cross section under measure Q , which shows how the futures price depends contemporaneously upon the state vector, the next section focus on the dynamic of the state vector under measure P .

Furthermore, our specification allows us to simplify equation (49). Rear-

range it we have:

$$f_{\tau,t} - ip_t = \alpha_\tau + \Psi'_\tau X_t; \quad (50)$$

$$h_{\tau,t} = \alpha_\tau + \Psi'_\tau X_t, \quad (51)$$

this gives us the relationship between the futures prices adjusted with the log implicit price deflater p_t , and state vector X_t with real log spot oil price.

4 The Kalman Filter and the likelihood function using the measure P

4.1 The state space representation

To complete the dynamic term structure model with KVAR settings, we need to first identify the state space representation so that the maximum likelihood function based on Kalman filter can be derived. Following our companion form (24) under measure P , and the affine specification (51). We define our state space representation as:

$$y_t = D + HX_t + e_t \quad e_t \sim N(0, Q) \quad (52)$$

$$X_t = A + BX_{t-1} + W_t \quad W_t \sim N(0, \Sigma_t). \quad (53)$$

where, equation (52) is the measurement equation, and equation (53) is the transition equation.

Specifically, implied by Dewachter and Lyrio (2006), Dewachter, Lyrio and Maes (2006), we allow the observed variables $m_t = \begin{pmatrix} \delta_t & s_t^R & g_t & \pi_t & r_t \end{pmatrix}$, to be identified by existing data series: $m_t^o = \begin{pmatrix} \delta_t^o & s_t^{R,o} & g_t^o & \pi_t^o & r_t^o \end{pmatrix}$, without any measurement error. This means the measurement equation (52) can be

written in matrix form as:

$$\begin{pmatrix} h_{1,t} \\ h_{2,t} \\ \vdots \\ h_{\tau,t} \\ m_t^o \end{pmatrix} = \begin{pmatrix} \alpha \\ j_0 \end{pmatrix} + \begin{pmatrix} \Psi \\ j_1 \end{pmatrix} \begin{pmatrix} z_t \\ m_t \end{pmatrix} + e_t \quad e_t \sim N(0, Q), \quad (54)$$

where α and Ψ are recursive parameters defined in equation (45), (46), and (47) $j_0 = 0_{5,1}$ is a 5×1 vector of zeros, j_1 is a 5×8 selection matrix picking out observed variables from the state vector X_t , and Q is a diagonal matrix with zeros in the last five diagonal elements.

We follow the specifications of state variables dynamics, as equation (24) in the previous section, to further specify the transition equation. Recall the transition equation (53):

$$X_t = A + BX_{t-1} + W_t \quad W_t \sim N(0, \Sigma_t), \quad (55)$$

where matrix A and B stand for matrix K and Υ in equation (24) incorporate with the state variable central tendency in steady state as specified in equation (29) and (30).

4.2 Kalman filter and the maximum likelihood estimation

Representing expectations conditional upon the available information with a ‘hat’ (so that $\hat{z}_t = E_t(z_t)$; $\hat{z}_{s|t} = E_t(z_s)$; $s \geq t$). Define the following state

variable covariance at time t as:

$$P_{zz} = E_t(z_t - \hat{z}_t)(z_t - \hat{z}_t)' = \hat{V}_t; \quad (56)$$

$$P_{mm} = E_t(m_{t+1} - \hat{m}_{t+1|t})(m_{t+1} - \hat{m}_{t+1|t})' = \Theta \hat{V}_{t+1|t} \Theta' + \Sigma_{m,t} \quad (57)$$

$$\begin{aligned} P_{hh} &= E_t(h_{t+1|t} - \hat{h}_{t+1|t})(h_{t+1|t} - \hat{h}_{t+1|t})' \\ &= (\Psi_{z,\tau} + \Psi_{m,\tau} \Theta) \hat{V}_{t+1|t} (\Psi'_{z,\tau} + \Psi'_{m,\tau} \Theta') + \Psi_{m,\tau} \Sigma_{m,t} \Psi_{m,\tau} + Q_t. \end{aligned} \quad (58)$$

$$P_{zh} = E_t(z_{t+1} - \hat{z}_{t+1|t})(h_{t+1} - \hat{h}_{t+1|t}) = (\Psi_{m,\tau} \Theta + \Psi_{z,\tau}) \hat{V}_{t+1|t}, \quad (59)$$

$$P_{hm} = E_t(h_{t+1} - \hat{h}_{t+1|t})(m_{t+1} - \hat{m}_{t+1|t}) = \Psi_{2,\tau} \Sigma_{m,t} + (\Psi_{2,\tau} \Theta + \Psi_{1,\tau}) \hat{V}_{t+1|t} \Theta', \quad (60)$$

$$P_{mz} = E_t(m_{t+1} - \hat{m}_{t+1|t})(z_{t+1} - \hat{z}_{t+1|t}) = \Theta \hat{V}_{t+1|t}, \quad (61)$$

The t -conditional covariance matrix for this $t + 1$ dated system is:

$$\begin{pmatrix} P_{hh} & P_{hm} & P_{hz} \\ P'_{hm} & P_{mm} & P_{mz} \\ P'_{hz} & P'_{mz} & P_{zz} \end{pmatrix} = E_t \begin{pmatrix} h_{t+1} - \hat{h}_{t+1|t} \\ m_{t+1} - \hat{m}_{t+1|t} \\ z_{t+1} - \hat{z}_{t+1|t} \end{pmatrix} \begin{pmatrix} h_{t+1} - \hat{h}_{t+1|t} & m_{t+1} - \hat{m}_{t+1|t} & z_{t+1} - \hat{z}_{t+1|t} \end{pmatrix}, \quad (62)$$

This allows the expectations to be updated as:

$$\hat{z}_{t+1} = \hat{z}_{t+1|t} + \begin{pmatrix} P_{zh} & P_{zm} \end{pmatrix} \begin{pmatrix} P_{hh} & P_{hm} \\ P_{mh} & P_{mm} \end{pmatrix}^{-1} \begin{pmatrix} h_{t+1} - \hat{h}_{t+1|t} \\ m_{t+1} - \hat{m}_{t+1|t} \end{pmatrix}, \quad (63)$$

$$\hat{V}_{t+1} = \hat{V}_{t+1|t} - \begin{pmatrix} P_{zh} & P_{zm} \end{pmatrix} \begin{pmatrix} P_{hh} & P_{hm} \\ P_{mh} & P_{mm} \end{pmatrix}^{-1} \begin{pmatrix} P_{zh} \\ P_{zm} \end{pmatrix}. \quad (64)$$

Let H_{t-1} refer to the hyper-parameter at the last time stage $t - 1$, and our target is to maximize the log likelihood function with respect to this

hyper-parameter H_{t-1} :

$$\begin{aligned} \ln \mathcal{L}(H_{t+1}) = & -\frac{nT}{2} \ln(2\pi) - \frac{1}{2} \ln \left(\text{Det} \begin{pmatrix} P_{hh} & P_{hm} \\ P_{mh} & P_{mm} \end{pmatrix} \right) \\ & - \frac{1}{2} \begin{pmatrix} h_{t+1} - h_{t+1|t} & m_{t+1} - m_{t+1|t} \end{pmatrix} \begin{pmatrix} P_{hh} & P_{hm} \\ P_{mh} & P_{mm} \end{pmatrix}^{-1} \begin{pmatrix} h_{t+1} - \hat{h}_{t+1|t} \\ m_{t+1} - \hat{m}_{t+1|t} \end{pmatrix} \end{aligned} \quad (65)$$

5 Empirical implementation

5.1 Data

We estimate the model using quarterly time series of the macro variables and crude oil futures. All data are downloaded from Thomson Reuters DataStream. Summary statistics are presented in table (1), we also present plots of our log real WTI oil futures prices in figure (1) and the four observed macro variables in figure (2).

The time period of the four variables: observed spot crude oil price, US output gap, US inflation and US Fed Fund rate, starts from Q1 1964 to Q4 2015. This allows us to include oil shocks during the 1970s in our research. The observed spot crude oil price is a spliced series composed by the Brent and West Texas Intermediate (WTI) spot oil price. Although we intended to use the pure WTI spot oil price series in the first place, the availability of WTI spot oil price in our data source only starts from Q1 1983, however, Brent spot oil price series dates back to Q1 1970. We make a plausible assumption that the differences between Brent and WTI spot oil price series dynamics during the time period Q1 1970 to Q1 1983 are minimal, and they only different in recent years because of new development in the oil industry. This allows us to construct a WTI spot oil price series dates back to Q1 1970. We assume oil price before the 1970s is constant over time, because before the 1970s, the oil market is monopolized by major US oil companies,

	Mean	Stdev	Skewness	Excess kurtosis	ADF p-value	Num of obs
Log real WTI futures prices						
h_1	-0.877	0.516	0.374	-1.021	0.287	128
h_2	-0.878	0.518	0.390	-1.061	0.311	128
h_3	-0.881	0.520	0.409	-1.091	0.330	128
h_6	-0.889	0.524	0.452	-1.154	0.370	128
h_9	-0.896	0.527	0.478	-1.197	0.399	128
h_{12}	-0.906	0.533	0.486	-1.244	0.423	128
h_{18}	-0.865	0.555	0.363	-1.504	0.296	106
h_{24}	-0.752	0.575	-0.084	-1.664	0.147	82
Observed variables						
δ^o	0.012	0.023	-0.034	2.020	0.001	128
$s^{R,o}$	-1.067	0.678	-0.204	-0.878	0.104	208
g^o	0.000	0.015	-0.360	0.523	0.001	208
π^o	0.009	0.006	1.240	0.831	0.404	208
r^o	0.014	0.009	0.708	0.830	0.249	208

Table 1: Summary statistics

as the oil price at that time is described by the phrase :“ take the price used by Exxon, add it to that used by Shell and divide the sum by two” (Carollo 2012). Therefore, although we are unable to find the spot oil price data for the time period between Q1 1964 to Q1 1970 in either Brent or WTI categories, this allows us to make another reasonable assumption that crude oil price is fixed for the period during the 1960s until Q1 1970. We generate the US output gap by applying the HP filter to the US GDP taking natural logarithm. US inflation is the log difference of the US implicit price deflater. And finally, we use Federal Fund rate as proxy for US interest rate.

In terms of the oil futures term structure, we use WTI light crude oil futures traded on New York Mercantile Exchange (NYMEX) in our research.

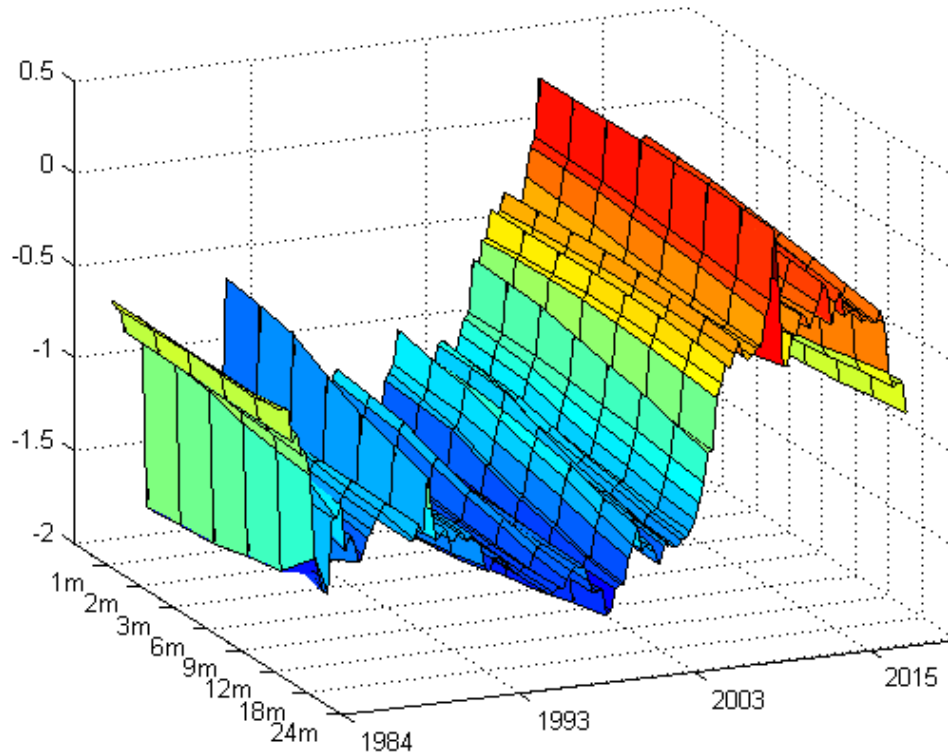


Figure 1: Log real WTI oil futures prices data

We choose oil futures contracts with 1, 2, 3, 6, 9, 12, 18 and 24 months to study, so that it covers two years of oil futures price dynamics. For oil futures contract with 1, 2, 3, 6, 9 and 12 months maturities, the series starts from Q1 1984. Oil futures contract with 18 months maturity starts from Q3 1989, and Oil futures contract with 24 months maturity starts from Q3 1995. Although we try to align our oil futures term structure with our observed variables, the availability of data from our data source only provides oil futures data starting from the year 1984, this is because oil futures contracts only start

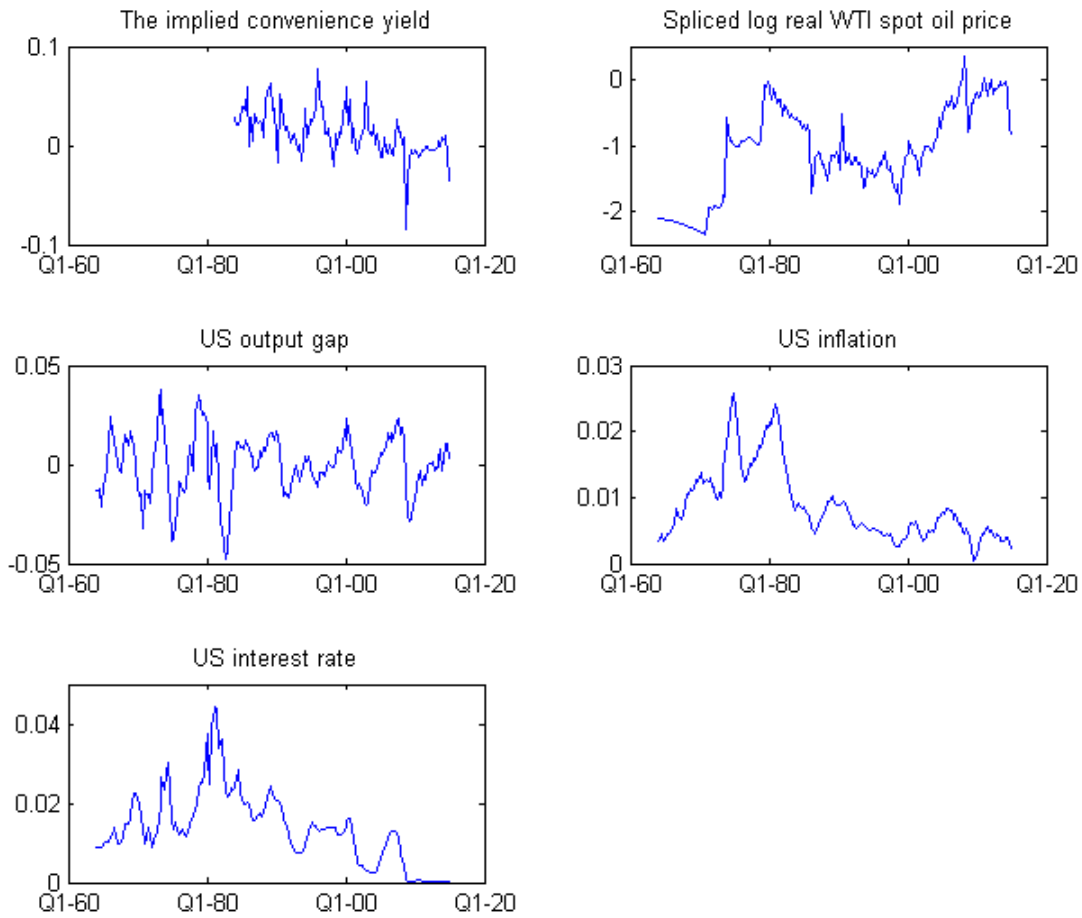


Figure 2: Observed variables

trading at that time.

From figure (2) we can see, the oil price only become marketable when Organization of Petroleum Exporting Countries (OPEC) took control of the crude oil market, starting from the oil embargo due to the Yom Kippur war in the early 1970s, causing the first severe oil shock to the real economy. Iranian revolution triggers another oil shock at the time around the year 1979, where the pro-west Iranian dynasty led by Mohammad Pahlavi is overthrown by

the national republic led by Muslim religious leader Ayatollah Khomeini. This causes another dramatic increase in the real oil price, because being heavily influence and supported by the west, Iran under King Pahlavi's rule has been a reliable main crude oil exporter to the western countries for many decades, the market was deeply worrying about the uncertain future crude oil production introduced by the newly established Islamic regime at the time. Another oil shock in our data happens in the early 1990s, caused by the Gulf war in the Middle East, where we can see that a obvious spike appears at that time. These historical events are pure shocks to the real economy because they are unexpected by the market. For the first oil shock, after a long period of fixed oil price due to the large American firms' monopoly since the World War II, the market had failed to evaluate the power of the newly founded OPEC, and its oil policies on the crude oil price from the supply side. The second and the third oil shocks are also unexpected by the market because the market is unable to correctly estimate the complicated economic and political conditions caused by sudden events such as the Iranian revolution and the Gulf war, thus the impacts toward the oil market. Although one may argue that, oil shocks are normally caused by these arbitrary events, and exogenous to the real economy, we also witness the persistent growth in oil price after the the year 2000, caused by the economic expansion in Europe and the emerging markets such as the BRICS. This creates a pressure from the demand side to pushes the oil price to increase, until the financial crisis happened in the year 2008, and substantially weakens the demand for oil form the industrial production. Moving on to the sharp decrease in oil price in the recent years, as we can see from the figure the oil price collapses at the end of the plotted line. This is caused by the excessive expansion of oil production by the main countries in the OPEC lately from the supply side. To be more implicit, we have seen Russia expands its oil production in order to finance for its conflicts in Crimea against Ukraine. Saudi Arabia also announcing oil production expansion, as they intend to decrease the global

oil price, in order to weaken Iran, their main opponent in the back of the recent Yemen conflicts in the short term. On the other side, Iran is also counting on the expansion of oil production in order to raise fund for them to get involve in the conflicts. In the long term, The OPEC expects that this predatory pricing strategy could create exogenous pressure for the newly develop shale oil refinery in the US, because the cost of shade oil refinery is significantly more expansive than traditional crude oil refinery technique. From the demand side, as the largest oil importer since the 1980s, the US demand for traditional crude oil import is reduced due to its own shale oil supply. Meanwhile, the economic expansion in the emerging markets has been significantly slowing down recently, this also results in the lower demand for oil in the relevant countries such as China and India.

6 Findings and results

6.1 The state variable estimates

Figure (3) shows our state variable estimates, table (2) presents the root mean squared error (RMSE) of the crude oil futures and observed variables. From the table, variables such as the output gap, interest rate, inflation and convenience yield, have very good fitting to their observed series, with only very small RMSE. Real spot oil price has slightly higher RMSE less than 15 basis point, this is in line with all other futures series.

		Root mean squared error						
$h_{\tau,t}$	1m	2m	3m	6m	9m	12m	18m	24m
	0.017	0.016	0.015	0.014	0.013	0.012	0.011	0.013
m_t^o	δ	s	g	π	r			
	0.002	0.013	0.000	0.000	0.000			

Table 2: The root mean squared error (RMSE) of estimations

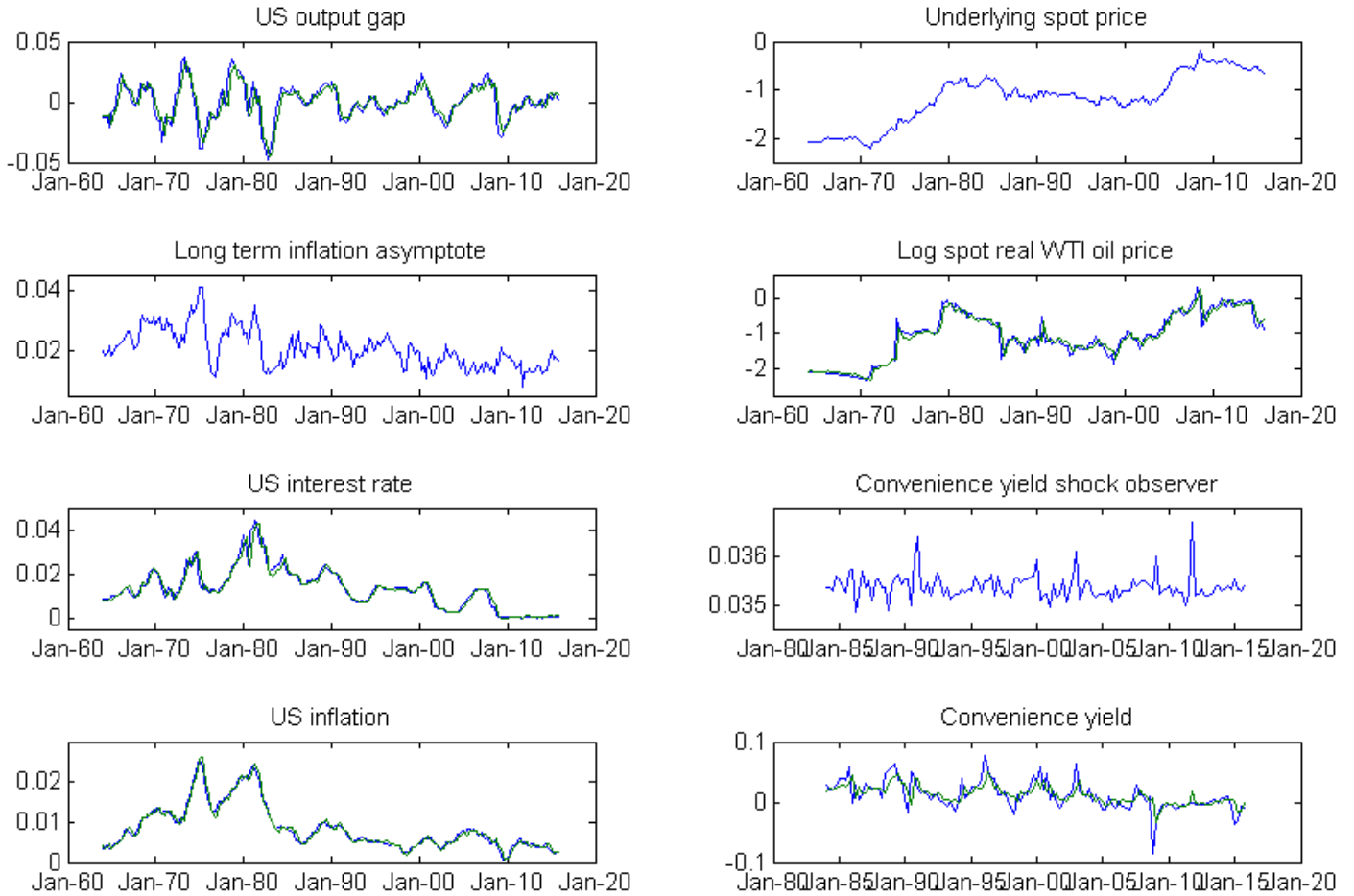


Figure 3: State variables: observations and estimations

The long term inflation asymptote as a latent variable, is restricted by the joint dynamics of the inflation and interest rate, picks up variations of these two variables together. We can see π^* is in line with Ireland (2008),

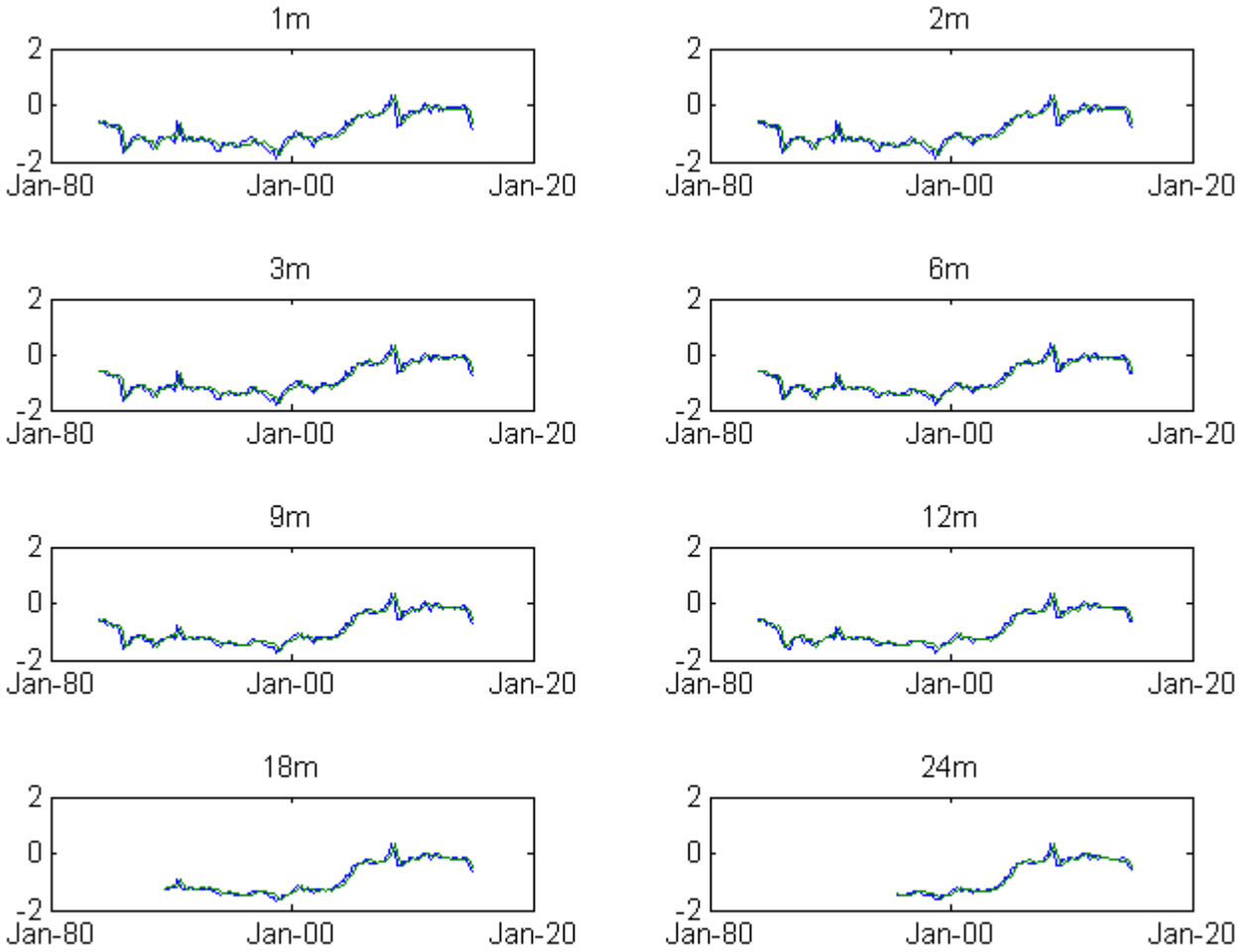


Figure 4: The WTI log real spot oil price

which slowly increases to a peak until around the mid 1970s, then it start to decrease gradually, particularly after the Volker's deflation in the early

1980s, which is an important monetary regime switching event at the time where both inflation and interest are clearly trying to point out. Convenience yield shock observer is restricted to the observed convenience yield variation in the long term. It is called as a “shock observer” because, interestingly, the convenience yield trend successfully identifies the past events that are considered to be influential to the real economy, including the oil shocks whichever have strong impact to the real output.



Figure 5: The comparison of the two series: the green line as the real spot oil price and the blue line as its underlying series

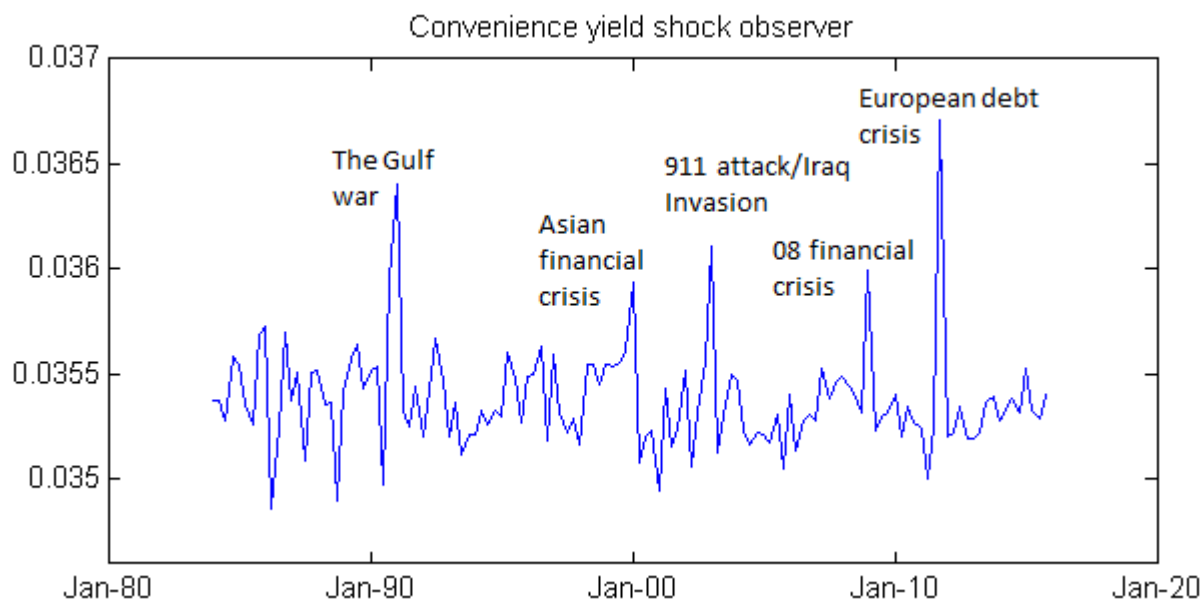


Figure 6: The convenience yield shock observer

We can see many spikes in this variable along the time line after the year 1984, where the crude oil futures price is introduced to the model. These spikes are clearly identifying: the Gulf War at the beginning of 1990s, when the crude oil price is doubled in six months. The joint impact from the Asian financial crisis and the Internet bubble in the end of the 1990s. The 2001 September 11 attack and consequently, the beginning of Iraq Invasion. The 2008 financial crisis, and the latest European debt crisis around 2012. However, putting these identified shocks aside, after the 2012 the European debt crisis, convenience yield shock observer become very flat and stable, there is however, a well acknowledged global real oil price collapse at the end of the period, real oil oil price and convenience yield at around 2015 are both pointing out with a cliff fall in their variations. We believe that the convenience yield shock observer fail to identify this event because unlike all

the other shocks we discussed, we have not seen the weak output, mainly caused by the weak demand, being boosted by the recent sharp decrease of the real oil price, also because of the weak demand for oil, the real oil price become much less relevant this time with respect to the previous events.

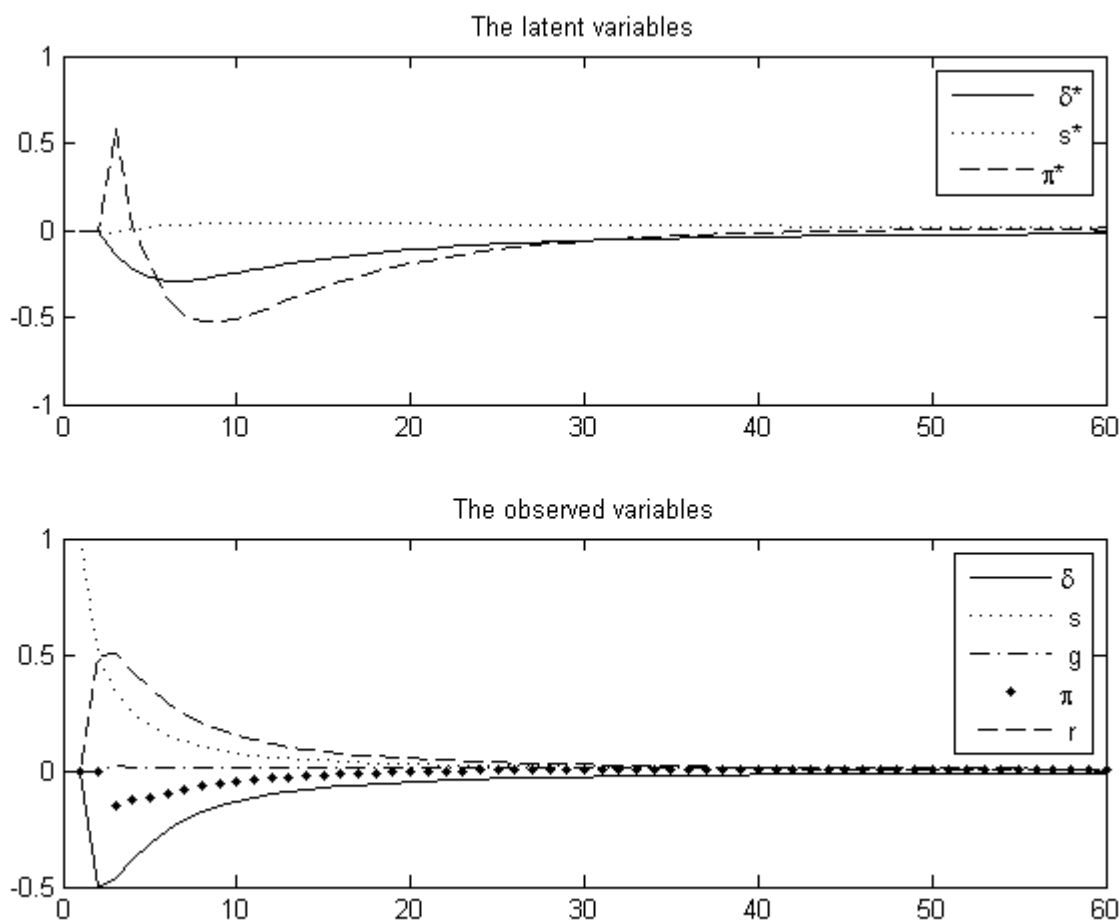


Figure 7: Factor loadings of the crude oil macro finance model

The underlying spot oil price as another latent variable, captures the

general trend of the real spot oil price across the time period from the figure (5). We have noticed that comparing with the two oil shocks in the 1970s, the steady increase of the underlying spot oil price appears to be relatively much more progressive and gradual. Indicating that although these two oil shocks push up the substantial oil price, there has been a raising pressure in the underlying oil price along the way, the sudden jumps in real oil price of oil shocks caused by exogenous events might only play the role as triggers to the potentially more persistent real oil price increase. Conversely, during the rather dramatic oil shock led by the Gulf war in the 1992, the underlying oil price however, unlike the oil shocks in the 1970s, happens to be rather flat and calm, implying that this oil shock could only has short term, rather than persistent effect to the real oil price at the end, as there has been no obvious increase in the underlying real price. This has further implications to other macro economic variables particularly the output gap. If we once again take a look at the figure (5), and map the two oil shocks in the 1970s, to the output gap series underneath, there are two dramatic falls in the output gap almost simultaneously to the 1970s oil shocks.

However, in terms of the 1992 oil shock, the fall in output has been much smaller and recovers much quicker. It seems that the Gulf war oil shock is unable to trigger another persistent oil price increase in the long term, hence its effect to output is very limited. In summary, it is obvious to us that, oil shocks caused by exogenous events are unable to affect the real economy substantially unless there has been a persistent raising pressure in the underlying oil price trend, and underlying oil price as a latent variable, plays the role as a very good indicator for the severity of exogenous oil shocks to the real economy

6.2 The parameter estimates

Table (4) gives parameters estimated under the measure P and table (5) gives parameters estimated under the measure Q . As the link between these two

probability measure, the risk premium parameter estimates are presented in table (6). The t-statistics of our parameter estimates are generated through the Hessian matrix of the log likelihood function, we can provide stand error of parameter estimates using the square root of the diagonal elements in the inverse of this Hessian matrix.

Granger causality test for observed variables		
Null hypothesis	F-statistics	Probability
Inflation \nRightarrow US output	1.217	0.298
US output \nRightarrow inflation	5.410	0.005
Interest rate \nRightarrow US output	5.600	0.001
US output \nRightarrow interest rate	9.107	0.000
Interest rate \nRightarrow inflation	3.854	0.022
Inflation \nRightarrow interest rate	3.335	0.037
Log spot oil price \nRightarrow US output	0.823	0.440
US output \nRightarrow log spot oil price	0.987	0.374
Log spot oil price \nRightarrow inflation	3.472	0.032
Inflation \nRightarrow log spot oil price	0.311	0.732
Log spot oil price \nRightarrow interest rate	0.367	0.544
Interest rate \nRightarrow log spot oil price	0.133	0.715

Table 3: Preliminary macro data analysis: Granger causality for the macro variables using unconstrained vector autoregressive model for the time period 1964 Q1 to 2015 Q4. Lag lengths are selected using Akaike information criterion.

From our tables, many key parameters under the measure P are statistically significant. $\phi_{s,\delta}$, $\phi_{s,g}$, $\phi_{s,\pi}$, are significant under 99% of confidence level, meaning that the real oil price is heavily driven by the real economy. $\phi_{\delta,s}$, $\phi_{\delta,r}$ are also significant under 99% confidence level, indicating that convenience yield is not standalone, real spot oil price and interest rate jointly affects its dynamics, which is in line with the Casassus and Collin-Dufresne (2003). The significant $\phi_{\pi,s}$ indicates that the influence from the real oil price on the real economy through the inflation is small but effective. Nearly all parameter estimates under the measure Q and the risk premium parameters

Parameters	Estimates	t-stat	Parameters	Estimates	t-stat
ξ_{δ^*}	0.038	0.604	$\phi_{s,\pi}$	4.607	38.219
k_{δ^*}	0.034	8.351	$\phi_{s,r}$	0.152	0.279
k_{s^*}	0.007	1.521	$\phi_{g,s}$	-0.002	-1.187
k_{π^*}	0.000	-0.024	$\phi_{g,g}$	0.888	168.433
φ_{δ}	-0.024	-39.415	$\phi_{g,\pi}$	0.094	0.633
φ_s	0.052	0.878	$\phi_{g,r}$	-0.140	-1.897
φ_g	-0.008	-4.366	$\phi_{\pi,s}$	0.001	4.504
φ_{π}	0.000	0.129	$\phi_{\pi,g}$	0.018	3.897
φ_r	-0.013	-226.645	$\phi_{\pi,\pi}$	0.903	164.049
$\phi_{\delta,\delta}$	0.585	28.061	$\phi_{\pi,r}$	0.002	0.180
$\phi_{\delta,s}$	-0.009	-11.068	$\phi_{r,s}$	0.000	0.780
$\phi_{\delta,r}$	0.254	10.738	$\phi_{r,g}$	0.042	4.402
$\phi_{s,\delta}$	-1.473	-124.911	$\phi_{r,\pi}$	0.014	1.013
$\phi_{s,s}$	0.808	18.747	$\phi_{r,r}$	0.922	95.394
$\phi_{s,g}$	1.276	11.827			

Table 4: Parameter estimates under the measure P

Parameters	Estimates	t-stat	Parameters	Estimates	t-stat
$v_{\delta,\delta}$	0.021	23.600	$v_{\pi,g}$	0.128	186.196
$v_{\delta,s}$	-0.008	-0.388	$v_{\pi,\pi}$	-0.137	-7.699
$v_{\delta,r}$	-1.473	-48.708	$v_{\pi,r}$	0.130	6.317
$v_{g,s}$	-0.404	-160.608	$v_{r,s}$	0.049	75.762
$v_{g,g}$	0.586	64.841	$v_{r,g}$	0.056	165.172
$v_{g,\pi}$	2.050	11.980	$v_{r,\pi}$	-0.447	-52.649
$v_{g,r}$	-0.703	-2.671	$v_{r,r}$	0.510	16.722
$v_{\pi,s}$	0.135	39.368			

Table 5: Parameter estimates under the measure Q

are significant even under the most rigorous confidence level. In comparison, parameters such as $\phi_{s,r}$, $\phi_{g,s}$, $\phi_{g,\pi}$, $\phi_{\pi,r}$, $\phi_{r,s}$, and $\phi_{r,\pi}$ under the measure P are insignificant even under 90% of confidence level. we can see more insights of this through Cochrane and Piazzesi (2008), which suggests that,

Parameters	Estimates	t-stat	Parameters	Estimates	t-stat
λ_{1,δ^*}	0.016	6.579	$\lambda_{2,g,s}$	-0.401	-94.519
$\lambda_{1,\delta}$	0.363	29.928	$\lambda_{2,g,g}$	-0.301	-78.826
λ_{1,s^*}	0.327	52.085	$\lambda_{2,g,\pi}$	1.956	11.914
λ_{1,π^*}	0.013	25.179	$\lambda_{2,g,r}$	-0.563	-9.042
$\lambda_{1,s}$	-0.049	-7.410	λ_{2,π,s^*}	-0.186	-174.617
$\lambda_{1,g}$	-0.428	-11.529	λ_{2,π,π^*}	6.114	58.988
$\lambda_{1,\pi}$	-0.194	-85.779	$\lambda_{2,\pi,s}$	0.134	134.183
$\lambda_{1,r}$	0.332	29.387	$\lambda_{2,\pi,g}$	0.110	24.119
$\lambda_{2,\delta^*,\delta^*}$	0.762	12.513	$\lambda_{2,\pi,\pi}$	-1.040	-167.190
λ_{2,s^*,s^*}	-0.002	-1.764	$\lambda_{2,\pi,r}$	0.128	10.387
λ_{2,π^*,π^*}	-0.130	-47.459	λ_{2,r,s^*}	-0.028	-295.687
$\lambda_{2,\delta,\delta}$	-0.212	-14.577	λ_{2,r,π^*}	1.429	51.639
$\lambda_{2,\delta,s}$	0.030	162.682	$\lambda_{2,r,s}$	0.049	382.193
$\lambda_{2,\delta,r}$	-0.263	-15.222	$\lambda_{2,r,g}$	0.014	1.411
λ_{2,g,s^*}	0.560	282.318	$\lambda_{2,r,\pi}$	-0.460	-25.616
λ_{2,g,π^*}	-18.922	-61.057	$\lambda_{2,r,r}$	-0.412	-156.728

Table 6: Risk premium parameter estimates

Parameters	Estimates	t-stat	Parameters	Estimates	t-stat
$c_{\delta,s}$	5.414	8.111	d_{s^*}	0.063	15.168
$c_{g,s}$	0.005	1.201	d_{π^*}	0.003	13.079
$c_{\pi,s}$	0.001	3.714	d_{δ}	0.017	16.637
$c_{\pi,g}$	0.000	0.000	d_s	0.144	18.737
$c_{r,s}$	0.001	0.755	d_g	0.007	19.416
$c_{r,g}$	0.120	6.023	d_{π}	-0.001	-12.112
$c_{r,\pi}$	0.376	1.700	d_r	0.002	17.571
d_{δ^*}	0.007	31.702			

Table 7: Volatility parameter estimates

Q parameters are precise because they come from the cross section which has “tiny” measurement errors, while the P parameters come from the VAR which has large forecast errors, therefore the parameters in the affine term structure models under the measure P are naturally less well defined than

those under the measure Q . More specifically, the term structure models use parameters under the risk neutral measure, instead of the real world measure, to determine the cross sectional factor loadings, hence the observed term structure can be described as a tractable linear combination of the factor loadings and the state variables. This implies that state dynamic under the measure Q , and the risk premium parameters, are normally accurate because of the restriction of the observed term structure data fitting. However, the state dynamic under the measure P , on the other hand, contains much larger forecast error, as it is determined by a time series VAR, which is much less restricted to the factor loading and the model data fitting, it is only indirectly determined via the bridge of risk premium parameters, which is also precise, and variance innovation. Therefore, the state dynamic parameters under the real world measure P will not be as well defined as the associated parameters under the risk neutral measure Q .

Null hypothesis From 1964 Q1 to	F-stat		Probability	
	2010 Q4	2015 Q4	2010 Q4	2015 Q4
Log spot oil price \nRightarrow US output	1.533	0.823	0.058	0.440
US output \nRightarrow log spot oil price	1.075	0.987	0.834	0.374

Table 8: Granger causality test for the real oil price and output gap pairwise using unconstrained vector autoregressive model, comparing for the time period 1964 Q1 to 2010 Q4, and 1964 Q1 to 2015 Q4. Lag lengths are selected using Akaike information criterion.

Regardless of this, from our preliminary data analysis, we might be able to accept $\phi_{r,s}$ and $\phi_{s,r}$ to be insignificant, meaning that real oil price is not very interacting with the interest rate, not directly at least. It might also be acceptable for $\phi_{g,s}$ to be insignificant, because on one hand, this is in line with our preliminary test. Table (8) shows the pairwise Granger causality test for real oil price and the output gap, for the period from 1964 Q1 to 2010 Q4 and 2015 Q4 respectively. We can see these two variables does not Granger causing each other under a unconstrained VAR system if we take the last

five years into consideration, the real oil price only weakly Granger causing the output gap variation if we exclude the last five years. On the other hand, this makes economic sense. Real oil price is not the only determining factor to the real output variation, thus, output does not always response to real oil price, especially if we take the recent dramatic crude oil price collapse into consideration, we can tell even if the crude oil price is shapely decreasing for nearly two years, the flat output mainly caused by weak global industrial production is unlikely to be boosted just because of that. Table 7 provides comparison of Granger causality test of the real oil price and output gap pairwise for two different time periods. To be more specific, it is true that we have witnessed high oil price introduced by serious oil shocks strikes the real economy and lower down the output. However, these oil shocks are only effective when the real economy is in demand and dependent upon sufficient oil supply. In the case when the real economy is encumbered by the slowing down output, although a oil shock which tightens the oil supply might worsen the situation, the real output is unlikely to be restored only because the crude oil price is falling due to its over production.

6.3 Impulse response functions

In figure (8) and (9), we present the impulse response functions. From which we can see that all our impulse responses are in line with the general macroeconomic literatures. The Taylor rule suggests that central bank targets inflation by adjusting interest rate in the market, in order to maintain lower level of inflation and stable output. It is of the central banks' concern to manage the ascending pressure of inflation as the result of overly extensive economic expansion. Therefore, higher interest rate will be announced by the central banks to increase cost of finance for excessive investment, in order to drag down the inflation rate. Such that excessive output can be finally restrained. Our model successfully captures its implications. If we look at the responses of US output gap and inflation to the shock from interest rate,

we will see they are both negative, meaning that the interest rate has been an effective tool for the policy makers to maintain stable inflation rate when the real output is overheating, by dragging both of them down. From the other side, we can also see the responses of interest rate to the shocks from real oil price, output gap, and inflation to be positive, which means the interest rate is alerted to economic condition in order to be adjusted accordingly. The policy makers take these factors into consideration when deciding on their interest rate policies.

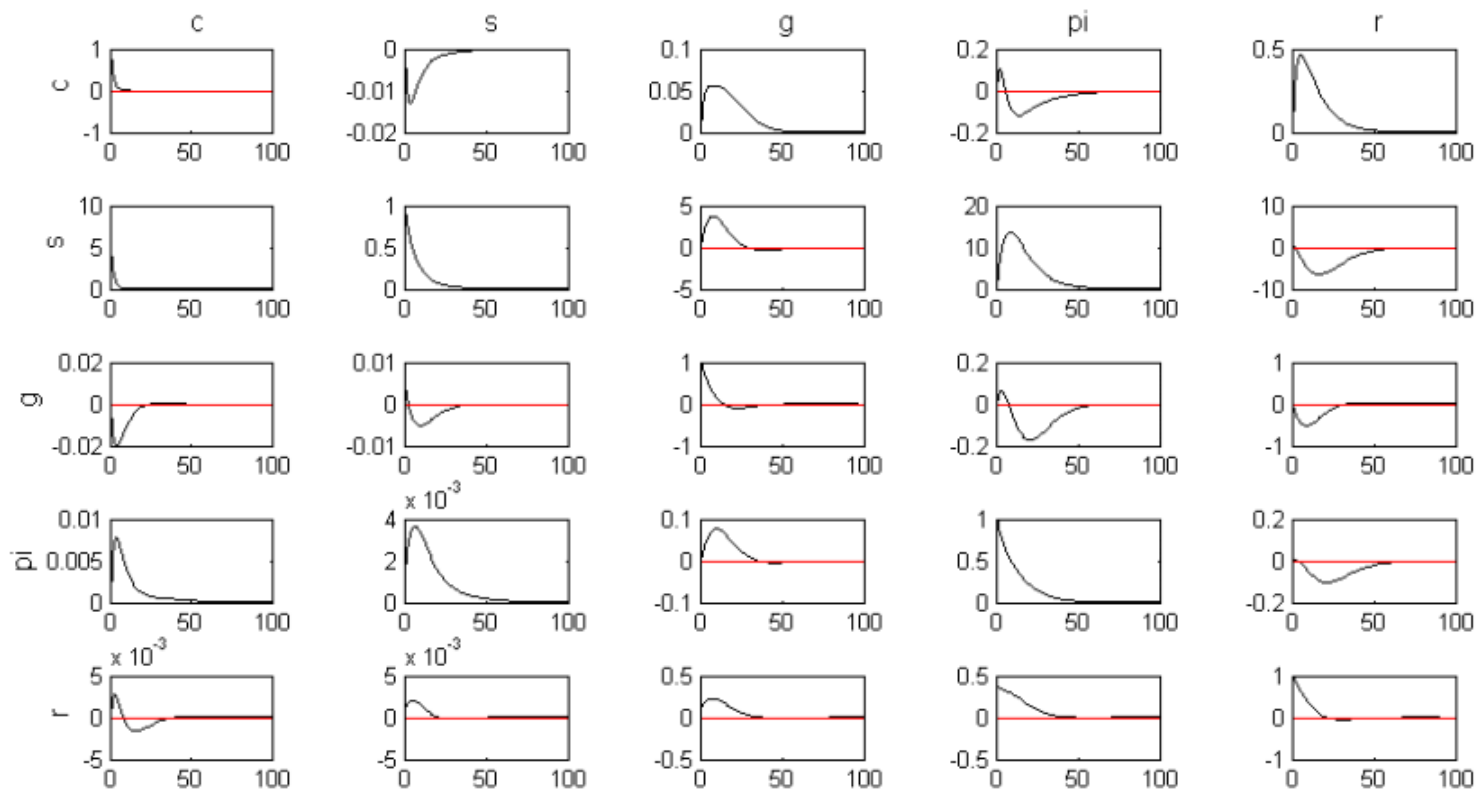


Figure 8: The impulse response functions for observed state variables

Real oil price responses positively to the US output gap and inflation,

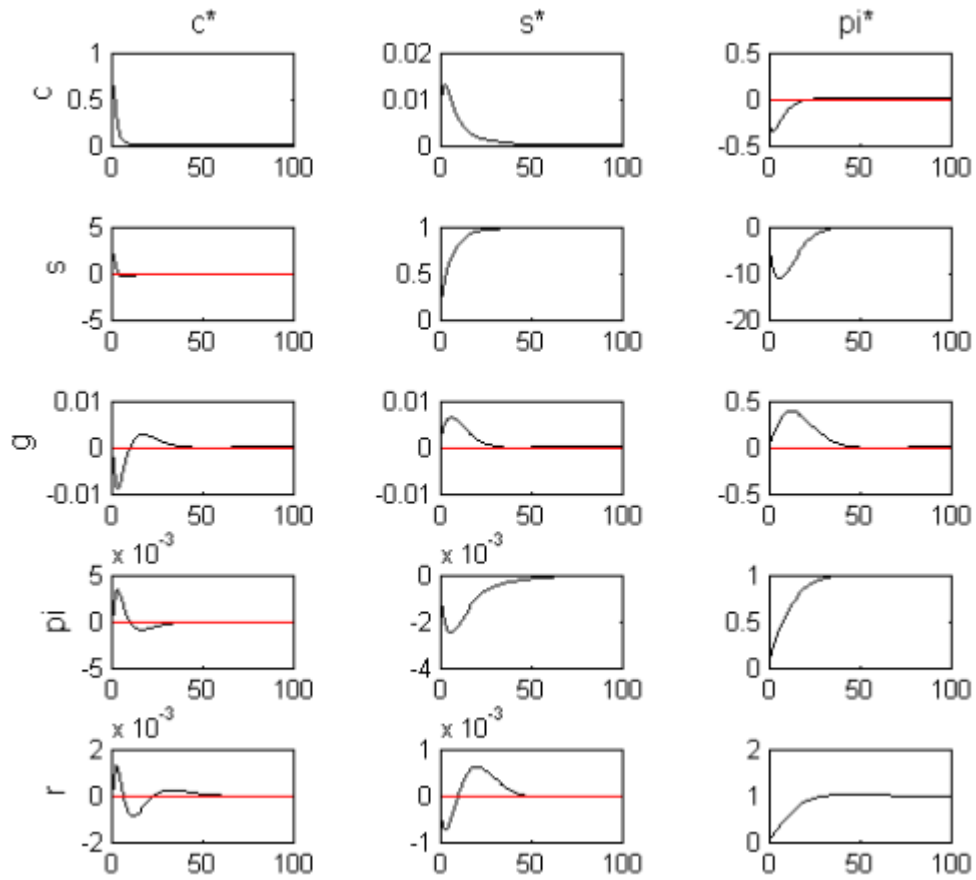


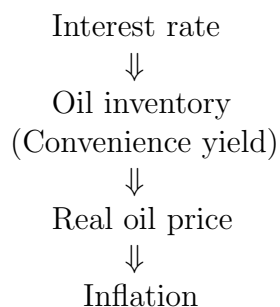
Figure 9: The impulse response functions for latent state variables

negatively to interest rate. Indicating that, economic expansion pushes up real oil price, because higher demand for crude oil is implying. When seeing the growing output with higher inflation rate, the interest rate will increase accordingly in order to slows down the economy and therefore, indirectly reduces crude oil demand, potentially results in a lower real oil price at the end. On the other hand, real oil price has positive impact to inflation and interest rate, however depresses the real output, the story behind this is that, when

the real oil price increases together with the economic expansion, because of higher demand pressure from household consumption and industrial production, burdening output and pushing up inflation. As a reaction, a descending pressure will be created by the policy maker, via the monetary transmission mechanism through inflation targeting method, but theoretically this only happens when the excessive output gap and inflation is observed, as the Taylor rule implies.

Policy makers has been eager to study the subject of monetary transmission mechanism, they would like to figure out how exactly are their interest rate policies transmitted to the inflation and further to the real output. From our results, we realise that the real oil price and the oil inventory in the economy has been playing rather notable role in such mechanism, We believe this role is just like a “bridge” in the middle of the interest rate and the inflation, and is responsible for a part of the monetary shock to be carried through to the real economy. More specifically, from the the last section, we have discussed the statistically significant effect from interest rate to the convenience yield, that is also the oil inventory. The oil inventory passes the signal of oil demand to the real oil price, which at the end place an significant effect on the inflation rate. In another way of description, at the time when the output

The monetary transmission with real oil price



is weak, policy makers will put a hand on the inflation targeting, so that the

the interest rate is cut to a low level. Although, on the surface, the oil supply will pay no attention to one country’s interest rate policy, as indicated by the insignificant $\phi_{s,r}$, the oil supply will certainly observe the shrinking oil inventory in the market, and hence expand the its production, in order to refill the the oil tanks. This creates a demand pressure which pushes up the real oil price, so that the inflation rate will be raising accordingly.

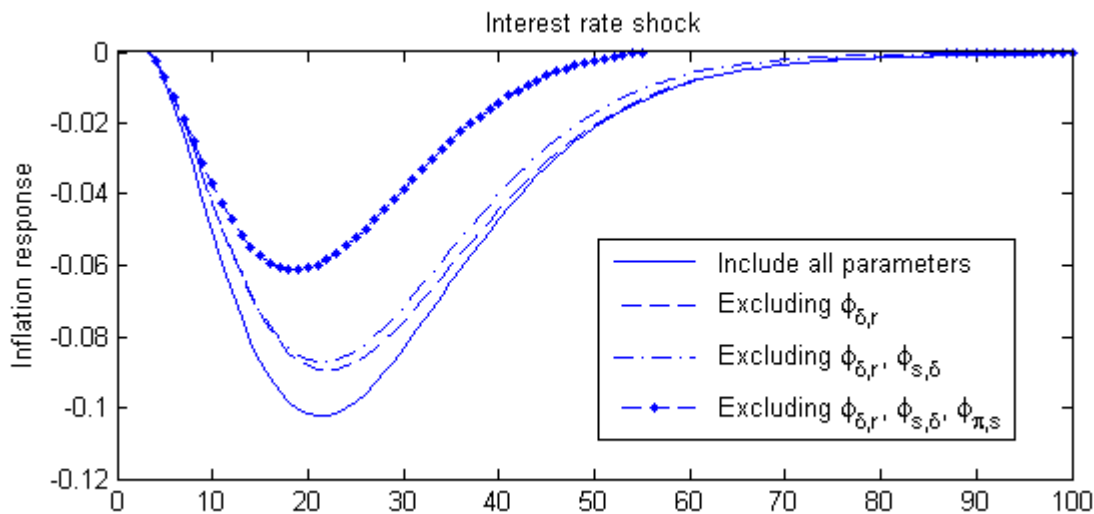


Figure 10: The comparison of impulse response functions for the inflation response to interest rate shock, under different circumstance of parameter absence

To provide more evidence to this argument, we can take a look at how does the impulse response of inflation to shock from the interest rate change, when we break different parts of this monetary transmission “bridge”. Figure (10) shows this impulse response function under circumstances of different parameters absences. As we can see, the negative response of inflation is at its highest if we keep this “bridge” complete, that is the solid line which represent all our model parameters being included. If we remove the passage from interest rate to oil inventory, that is parameter $\phi_{\delta,r}$, this negative response,

as the dashed line points out, obviously shrinks and become less effective. Further removing the passage from oil inventory to real oil price, that is the parameter $\phi_{s,\delta}$, shrinks this response slightly more, as the dash-dotted line, which is not surprising because real oil price has always been very alerting and actively responding to the oil inventory. Finally if we further cut the transmission from real oil price to inflation, that is the parameter $\phi_{\pi,s}$, as the dotted line, we see a even lighter negative response of inflation to the interest rate shock, which is nearly a half to the case when all parameters are included.

6.4 Variance decomposition

We present the variance decomposition of our state variable estimations and the crude oil futures estimations in figure (11) and (12).

In the first row of the figure (11), the latent variable row shows that inflation asymptote (π^*) is very active in explaining variations of output, inflation and interest rate, it also significantly contributes to the variation of convenience yield and spot price, indicating that the policy factor heavily affects classic macro indicators, and it also has some power on oil inventory. Convenience yield trend (δ^*) very significantly affects convenience yield variation, however, it plays very little role in explaining other observed variables, although we can see its tiny effect on the spot price, its effect on output, inflation and interest rate variation are not neglectable. As we have discussed, δ^* can be regarded as a historical shock observer, because it is nicely pointing out the key past events which heavily influence the oil price and the macro economy. Meanwhile, convenience yield as a oil inventory indicator, is clearly playing the role as a supply buffer state cushioning the influences from the past events to the macro economy. When real oil price suddenly increases due to a oil shock, it is the oil inventory to be affected in the first place ahead of the real output, only until it is finally exhausted by the more persistent real oil price increases induced by the progressive raising demand

from the industrial production potentially, the real economy starts to be affected. Extra evidences can be seen if we move back to the impulse responses of other observed variables to the δ^* , we will see that these impulse responses are very sharp and short comparing with others, telling us that the real oil price and real economy might be surprised by the standalone oil shock at the beginning, and react fast and sharply for a rather short period of time, it is however unlikely to be affected by this simple oil shock alone in the long term.

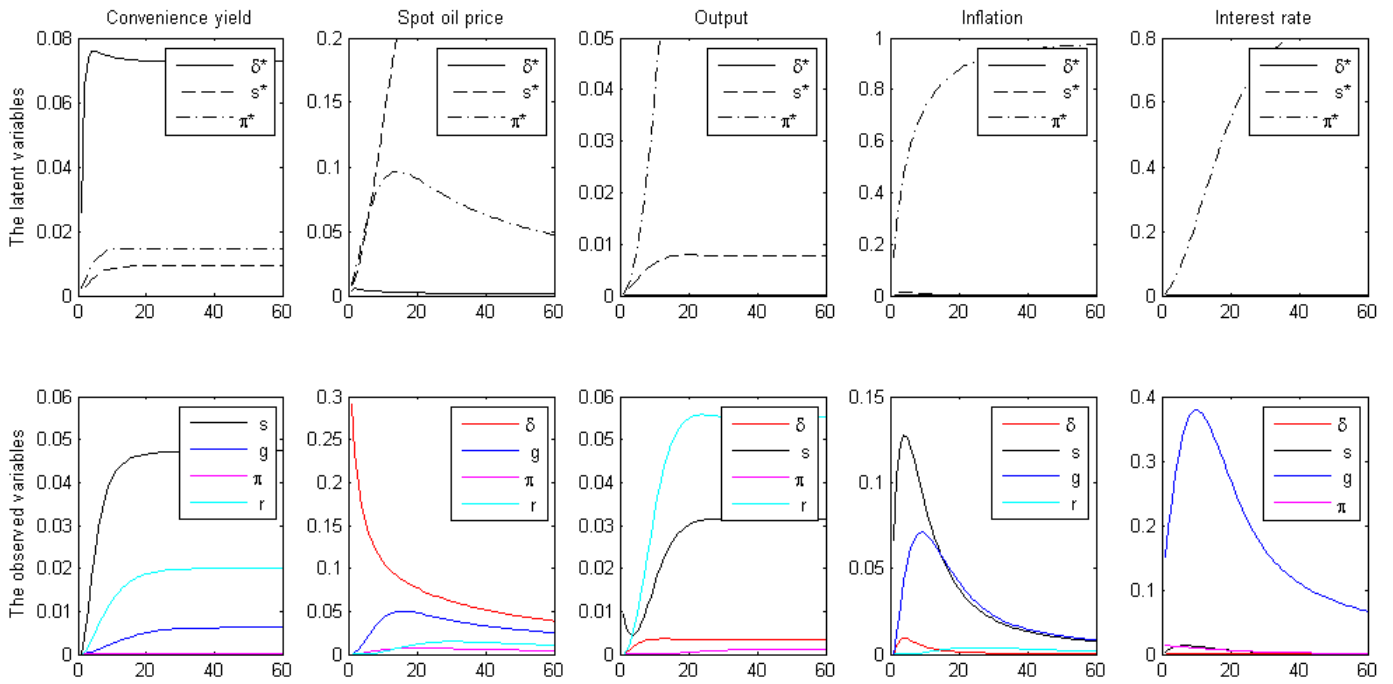


Figure 11: State variables variance decomposition

Moving on to the second row of the figure (11). In general, the variations of output and inflation are heavily depended upon the real oil price. Output

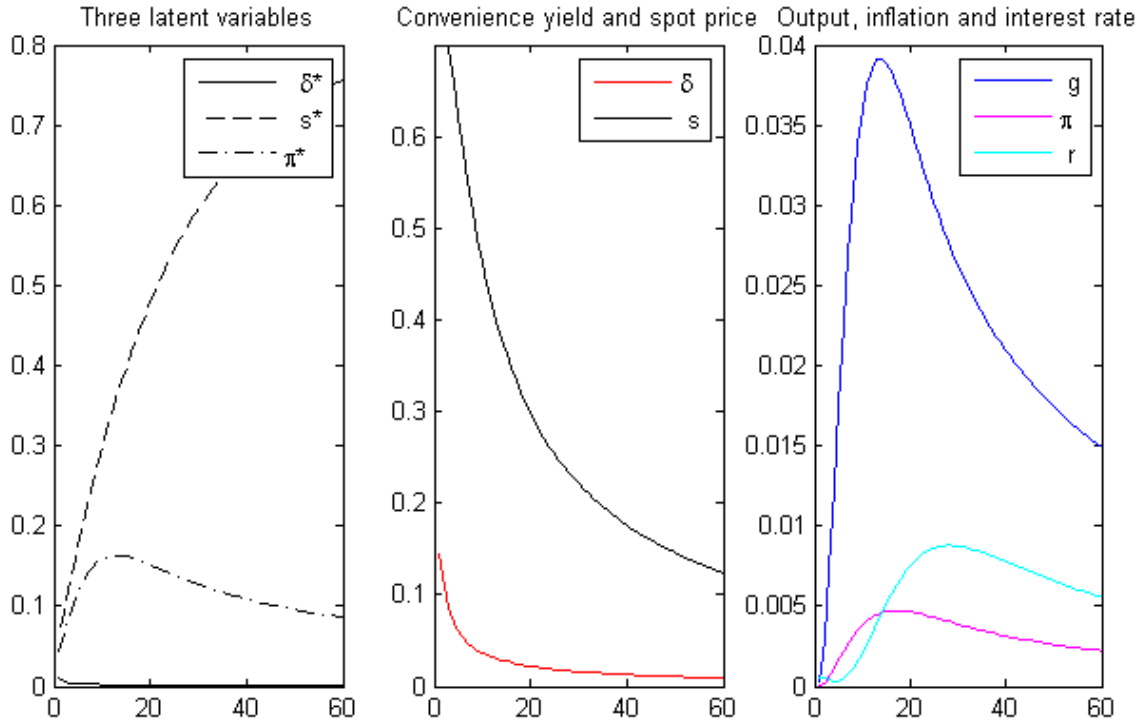


Figure 12: Crude oil futures variance decomposition

gap is rather quiet to the real oil price at the beginning of the period, but as the time moves on, the effect of real oil price start to build up and eventually persists. Inflation, however, is apparently very sensitive to the real oil price at the beginning, it decays along with the time when the policy indicator (π^*) starts to take power gradually. From the first row of the figure (11), we find that it is unlikely for a sudden oil shock to persistently affect the real economy, because oil inventory as a buffer state is going to absorb most of the shock effects until it is finally exhausted. Therefore, it seems obvious to us that, although the oil shock itself has only frail influence on the real economy in the very short term, the persistent real oil price increase induced by the

simple oil shocks, which can hardly digested by the oil inventory, turns out to be the main determinant that is heavily influencing the real economy in a much longer period of time.

On the other hand, the real economy is also explaining the variation of the real oil price evidently, although the underlying oil price (s^*) contributes most of its variation throughout the time as well as the convenience yield, we find that output as a real economic indicator also plays a sizeable role in explaining the real oil price variation, and the policy indicator (π^*) also explains similar level of variation. This means that although the real oil price dynamic is heavily affected by its own determinants, which are unlikely to be interacting with the real economy, such as random events, arbitrary policies, political decisions and diplomatic considerations by the relevant Sovereign states. The real economy and the market supply and demand are still making their limited but undeniable contributions to the real oil price movements, especially under normal economic conditions when its own exogenous factors are not in too much of control.

7 Concluding remarks

In this paper, we have presented a macro-finance model for the crude oil price using standard KVAR setting. This model imposes adjustment for the price level to the log spot oil price, which allows us to look at the real oil price and the real economy. We define the convenience yield trend, underlying spot oil price and inflation asymptote as the latent variables, and include the implied convenience yield, the log real spot WTI oil price, US output gap, US inflation and the Fed fund rate in the observed macro system. We define long term effects from the latent variables to the macro system by imposing equilibrium relationship in the steady state. Assuming the log crude oil futures price is affine to the state variables, we are able to derive a close form solution for the factor model of crude oil futures term structure. We

use the maximum likelihood method based on the Kalman filter algorithm to estimate the model and demonstrate empirical evidences. We also solve the severe missing observation problem using Kalman filter.

We find that underlying oil price points out the potential oil demand in the economy, indicating the level of severity of an oil shock to the real economy. Convenience yield shock observer indicates the major shocks to the real output, including the whichever oil shock has strong impact to the real economy. The convenience yield, as a proxy to the oil inventory plays the role as a two way buffer, not only does it absorb the impact of oil shocks to the real economy, but also it serve as passage between the interest rate and the real oil price, transmitting monetary signals to the real oil price, which affects the inflation rate.

This model allows variations of the oil futures term structure to be explained by its latent variables as well as macroeconomic indicators. It successfully captures the dynamic interaction between the oil futures market and the macroeconomic system. This provides a tractable method for policy makers to evaluate how their monetary policies can influence the crude oil futures market. It also helps us to draw a clearer picture to understand the Central Banks' role in the crude oil futures market, which has been neglected in practice.

8 Appendix

8.1 The companion form under probability measure P and Q

Stacking equation (15) and (21) gives the companion form (24) under the probability measure P , where $W_t = LDV_t$ and:

$$K = \begin{pmatrix} \kappa_{\delta^*} \\ \kappa_{s^*} \\ \kappa_{\pi^*} \\ \kappa_{\delta} + \kappa_{\delta^*}\theta_{\delta,\delta^*} + \kappa_{\delta^*}\theta_{\delta,s^*} + \kappa_{\pi^*}\theta_{\delta,\pi^*} \\ \kappa_s + \kappa_{s^*}\theta_{s,\delta^*} + \kappa_{s^*}\theta_{s,s^*} + \kappa_{\pi^*}\theta_{s,\pi^*} \\ \kappa_g + \kappa_{s^*}\theta_{g,s^*} + \kappa_{\pi^*}\theta_{g,\pi^*} \\ \kappa_{\pi} + \kappa_{s^*}\theta_{\pi,s^*} + \kappa_{\pi^*}\theta_{\pi,\pi^*} \\ \kappa_r + \kappa_{s^*}\theta_{r,s^*} + \kappa_{\pi^*}\theta_{r,\pi^*} \end{pmatrix} \quad (66)$$

$$\Upsilon = \begin{pmatrix} \xi_{\delta^*} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \xi_{\delta^*}\theta_{\delta,\delta^*} & \theta_{\delta,s^*} & \theta_{\delta,\pi^*} & \phi_{\delta,\delta} & \phi_{\delta,s} & 0 & 0 & \phi_{\delta,r} \\ \xi_{\delta^*}\theta_{s,\delta^*} & \theta_{s,s^*} & \theta_{s,\pi^*} & \phi_{s,\delta} & \phi_{s,s} & \phi_{s,g} & \phi_{s,\pi} & \phi_{s,r} \\ 0 & \theta_{g,s^*} & \theta_{g,\pi^*} & 0 & \phi_{g,s} & \phi_{g,g} & \phi_{g,\pi} & \phi_{g,r} \\ 0 & \theta_{\pi,s^*} & \theta_{\pi,\pi^*} & 0 & \phi_{\pi,s} & \phi_{\pi,g} & \phi_{\pi,\pi} & \phi_{\pi,r} \\ 0 & \theta_{r,s^*} & \theta_{r,\pi^*} & 0 & \phi_{r,s} & \phi_{r,g} & \phi_{r,\pi} & \phi_{r,r} \end{pmatrix}. \quad (67)$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \theta_{\delta,\delta^*} & \theta_{\delta,s^*} & \theta_{\delta,\pi^*} & 1 & 0 & 0 & 0 & 0 \\ \theta_{s,\delta^*} & \theta_{s,s^*} & \theta_{s,\pi^*} & c_{s,\delta} & 1 & 0 & 0 & 0 \\ 0 & \theta_{g,s^*} & \theta_{g,\pi^*} & 0 & c_{g,s} & 1 & 0 & 0 \\ 0 & \theta_{\pi,s^*} & \theta_{\pi,\pi^*} & 0 & c_{\pi,s} & c_{\pi,g} & 1 & 0 \\ 0 & \theta_{r,s^*} & \theta_{r,\pi^*} & 0 & c_{r,s} & c_{r,g} & c_{r,\pi} & 1 \end{pmatrix}. \quad (68)$$

$$D = \begin{pmatrix} d_{\delta^*} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_{s^*} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_{\pi^*} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{\delta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_{\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_r \end{pmatrix} \quad V_t = \begin{pmatrix} \nu_{\delta^*,t} \\ \nu_{s^*,t} \\ \nu_{\pi^*,t} \\ \nu_{\delta,t} \\ \nu_{s,t} \\ \nu_{g,t} \\ \nu_{\pi,t} \\ \nu_{r,t} \end{pmatrix} \quad (69)$$

Similarly, under the probability measure Q , K^Q and Υ^Q in equation (33) are defined as:

$$K^Q = \begin{pmatrix} k_{\delta^*}^Q \\ k_{s^*}^Q \\ k_{\pi^*}^Q \\ k_{\delta}^Q \\ k_s^Q \\ k_g^Q \\ k_{\pi}^Q \\ k_r^Q \end{pmatrix} \quad \Upsilon^Q = \begin{pmatrix} \nu_{\delta^*}^Q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu_{s^*}^Q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu_{\pi^*}^Q & 0 & 0 & 0 & 0 & 0 \\ \nu_{\delta,\delta^*}^Q & \nu_{\delta,s^*}^Q & \nu_{\delta,\pi^*}^Q & \nu_{\delta,\delta}^Q & \nu_{\delta,s} & 0 & 0 & \nu_{\delta,r} \\ 0 & -\nu_{\pi,s^*}^Q & -\nu_{\pi,\pi^*}^Q & -1 & 1 - \nu_{\pi,s}^Q & -\nu_{\pi,g}^Q & -\nu_{\pi,\pi}^Q & 1 - \nu_{\pi,r}^Q \\ 0 & \nu_{g,s^*}^Q & \nu_{g,\pi^*}^Q & 0 & \nu_{g,s}^Q & \nu_{g,g}^Q & \nu_{g,\pi}^Q & \nu_{g,r}^Q \\ 0 & \nu_{\pi,s^*}^Q & \nu_{\pi,\pi^*}^Q & 0 & \nu_{\pi,s}^Q & \nu_{\pi,g}^Q & \nu_{\pi,\pi}^Q & \nu_{\pi,r}^Q \\ 0 & \nu_{r,s^*}^Q & \nu_{r,\pi^*}^Q & 0 & \nu_{r,s}^Q & \nu_{r,g}^Q & \nu_{r,\pi}^Q & \nu_{r,r}^Q \end{pmatrix}, \quad (70)$$

8.2 The state space representation

In the measurement equation (54), because observed variables are observed without error, we define j_0 , j_1 and Q as:

$$j_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad j_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (71)$$

$$Q = \begin{pmatrix} q_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_r^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (72)$$

As for the state equation, after incorporating with the long term constrain (29) and (30), in the steady state, A and B in state equation (55) can be respecified from the macro dynamic (24) as:

$$A = \begin{pmatrix} K_z \\ (I - \Upsilon_m)\varphi \end{pmatrix} \quad (73)$$

$$B = \begin{pmatrix} \xi_{\delta^*} & 0 & 0 & 0_{1,5} \\ 0 & 1 & 0 & 0_{1,5} \\ 0 & 0 & 1 & 0_{1,5} \\ (I - \Upsilon_m)R_1 & (I - \Upsilon_m)R_2 & (I - \Upsilon_m)R_3 & \Upsilon_m \end{pmatrix} \quad (74)$$

8.3 Change of probability measure

Following equation (33), the latent dynamics, namely $X_{z,t} = \begin{pmatrix} \delta_t^* & s_t^* & \pi_t^* \end{pmatrix}$ under measure Q , is specified as:

$$X_{z,t} = K_z^Q + \Upsilon_z^Q X_{z,t-1} + W_{z,t}^Q, \quad (75)$$

or in matrix form:

$$X_{z,t} = \begin{pmatrix} \delta_t^* \\ s_t^* \\ \pi_t^* \end{pmatrix} = \begin{pmatrix} \kappa_{\delta^*}^Q \\ \kappa_{s^*}^Q \\ \kappa_{\pi^*}^Q \end{pmatrix} + \begin{pmatrix} v_{\delta^*}^Q & 0 & 0 \\ 0 & v_{s^*}^Q & 0 \\ 0 & 0 & v_{\pi^*}^Q \end{pmatrix} \begin{pmatrix} \delta_{t-1}^* \\ s_{t-1}^* \\ \pi_{t-1}^* \end{pmatrix} + W_{z,t}^Q. \quad (76)$$

Implied by Duffee (2002) essential affine setting, we have the following relationship:

$$W_{z,t}^Q = W_{z,t} + L_z D_z \Lambda_{z,t-1}, \quad (77)$$

and we assume $\Lambda_{z,t}$ is affine to the state variables in the form as:

$$\Lambda_{z,t} = D_z \Lambda_{1,z} + D_z^{-1} \Lambda_{2,z} X_{z,t} + D_z^{-1} \Lambda_{2,m,z} X_{m,t}, \quad (78)$$

where $\Lambda_{1,z}$ is a 3×1 vector, $\Lambda_{2,z}$ is a 3×3 diagonal matrix, and $\Lambda_{2,m,z}$ is a 3×5 matrix. Note that if $\Lambda_{2,m,z} = 0_{3,5}$ then we preserve diagonal/independent dynamics and make the system recursive. Otherwise $\Lambda_{2,m,z}$ is given by the effect of $X_{m,t}$ on Λ_z under Q .

Substituting equation (77) and (78) into equation (75) we have:

$$X_{z,t} = K_z^Q + \Upsilon_z^Q X_{z,t-1} + W_{z,t} + L_z D_z (D_z \Lambda_{1,z} + D_z^{-1} \Lambda_{2,z} X_{z,t-1}), \quad (79)$$

which leads to:

$$K_z^Q = K_z - L_z D_z D_z' \Lambda_{1,z} \quad (80)$$

$$\Upsilon_z^Q = \Upsilon_z - L_z \Lambda_{2,z}, \quad (81)$$

hence:

$$\Lambda_{2,z} = L_z^{-1} (\Upsilon_z - \Upsilon_z^Q). \quad (82)$$

And for the macro system containing $X_{m,t} = \begin{pmatrix} \delta_t & s_t^R & g_t & \pi_t & r_t \end{pmatrix}$ under measure Q in equation (33) with respect to equation (24) under measure P , we have the following expression:

$$X_{m,t} = K_m^Q + \Upsilon_m^Q X_{m,t-1} + \Upsilon_{z,m}^Q X_{z,t-1} + W_{m,t}^Q, \quad (83)$$

and to change the probability measure, we define:

$$W_{m,t}^Q = W_{m,t} + L_m D_m \Lambda_{m,t-1} \quad (84)$$

Following Duffee (2002), we assume that Λ_t is affine in the state variables and has the following form:

$$\Lambda_{m,t} = D_m \Lambda_{1,m} + D_m^{-1} \Lambda_{2,m} X_{m,t} + D_m^{-1} \Lambda_{2,z,m} X_{z,t} \quad (85)$$

where $\Lambda_{1,m}$ is a 5×1 vector, $\Lambda_{2,m}$ is a 5×5 matrix. $\Lambda_{3,m}$ is given by the effect of $X_{z,t}$ on $\Lambda_{m,t}$ and hence $X_{m,t}$ under the measure Q .

Substituting equation (84) and (85) into equation (83) gives:

$$\begin{aligned}
X_{m,t} &= K_m^Q + \Upsilon_m^Q X_{m,t-1} + \Upsilon_{z,m}^Q X_{z,t-1} + W_{m,t} + L_m D_m \Lambda_{m,t-1} \\
&= K_m^Q + \Upsilon_m^Q X_{m,t-1} + \Upsilon_{z,m}^Q X_{z,t-1} + W_{m,t} \\
&\quad + L_m D_m (D_m \Lambda_{1,m} + D_m^{-1} \Lambda_{2,m} X_{m,t-1} + D_m^{-1} \Lambda_{2,z,m} X_{z,t}) \\
&= (K_m^Q + L_m D_m D_m' \Lambda_{1,m}) + (\Upsilon_m^Q + L_m \Lambda_{2,m}) X_{m,t-1} \\
&\quad + (\Upsilon_{z,m}^Q + L_m \Lambda_{2,z,m}) X_{z,t} + W_{m,t}
\end{aligned} \tag{86}$$

and comparing this with (83) gives:

$$K_m^Q = K_m - L_m D_m D_m' \Lambda_{1,m} \tag{87}$$

$$\Upsilon_m^Q = \Upsilon_m - L_m \Lambda_{2,m} \tag{88}$$

$$\Upsilon_{z,m}^Q = \Upsilon_{z,m} - L_m \Lambda_{2,z,m} \tag{89}$$

We can back out $\Lambda_{2,m}$ and $\Lambda_{2,z,m}$ by rearranging equation (89) as:

$$\Lambda_{2,m} = L_m^{-1} (\Upsilon_m - \Upsilon_m^Q) \tag{90}$$

$$\Lambda_{2,z,m} = L_m^{-1} (\Upsilon_{z,m} - \Upsilon_{z,m}^Q) \tag{91}$$

In summary, for K^Q and Υ^Q in equation (33), the change of probability measure from measure Q to measure P can be specified as:

$$K^Q = K - L D D' \Lambda_1 \tag{92}$$

$$\Upsilon^Q = \Upsilon - L \Lambda_2 \tag{93}$$

where Λ_1 is a 8×1 vector, and Λ_2 is a 8×8 matrix, stand for the risk premium parameters, composed by $\Lambda_{1,z}$, $\Lambda_{1,m}$ and $\Lambda_{2,z}$, $\Lambda_{2,m}$, $\Lambda_{2,z,m}$ as:

$$\Lambda_1 = \begin{pmatrix} \Lambda_{1,z} \\ \Lambda_{1,m} \end{pmatrix} \quad \Lambda_2 = \begin{pmatrix} \Lambda_{2,z} & 0_{3,5} \\ \Lambda_{2,z,m} & \Lambda_{2,m} \end{pmatrix} \tag{94}$$

8.4 Analysing risk premium using Return Forecasting Regression

So far we have defined state dynamics under both the risk neutral measure and the real world measure. To follow Cochrane and Piazzesi (2005) and Cochrane and Piazzesi (2008), we estimate Υ^Q directly, and then get Υ^P by adding the risk premium parameters from the return forecasting regression.

We denote the log holding period return from buying an τ -period futures contract at time t and selling it as an $\tau - 1$ period futures contract at time $t + 1$ as:

$$r_{\tau,t+1} = h_{\tau-1,t+1} - h_{\tau,t} \quad (95)$$

the expectation of equation (95) under probability measure P can be written as:

$$E_t^P(r_{\tau,t+1}) = E_t^P(h_{\tau-1,t+1} - h_{\tau,t}) \quad (96)$$

Substituting equation (51) into specification (96), we have:

$$E_t^P((\alpha_{\tau-1} + \Psi_{\tau-1}X_{t+1}) - (\alpha_{\tau} + \Psi_{\tau}X_t)) = E_t^P \left(\underbrace{(\alpha_{\tau-1} - \alpha_{\tau})}_{\textcircled{1}} + \underbrace{(\Psi_{\tau-1}X_{t+1} - \Psi_{\tau}X_t)}_{\textcircled{2}} \right) \quad (97)$$

substituting equation (47) into $\textcircled{1}$, we have:

$$\begin{aligned} \textcircled{1} &= E_t^P(\alpha_{\tau-1} - \alpha_{\tau-1} - \Psi_{\tau-1}K^Q - \frac{1}{2}\Psi_{\tau-1}R\Psi_{\tau-1}) \\ &= E_t^P(-\Psi_{\tau-1}K^Q) \\ &= \Psi_{\tau-1}E_t^P(-K^Q) \end{aligned} \quad (98)$$

substituting equation (45) and (47) into ②, we have:

$$\begin{aligned}
\textcircled{2} &= E_t^P(\Psi_{\tau-1}X_{t+1} - \Psi_{\tau}X_t) \\
&= E_t^P(\Psi_{\tau-1}(K^P + \Upsilon^P X_t) - (\Psi_{\tau-1}\Upsilon^Q)X_t) \\
&= \Psi_{\tau-1}E_t^P((K^P + \Upsilon^P X_t) - \Upsilon^Q X_t)
\end{aligned} \tag{99}$$

using equation (92) and (93) in equation (99):

$$\begin{aligned}
\textcircled{2} &= \Psi_{\tau-1}E_t^P((K^P + \Upsilon^P X_t) - \Upsilon^Q X_t) \\
&= \Psi_{\tau-1}E_t^P((K^Q + L\Lambda_1) + (\Upsilon^Q + L\Lambda_2)X_t - \Upsilon^Q X_t) \\
&= \Psi_{\tau-1}E_t^P(K^Q + L\Lambda_1 + L\Lambda_2 X_t)
\end{aligned} \tag{100}$$

combining ① and ② we have:

$$E_t^P(h_{\tau-1,t+1} - h_{\tau,t}) = \Psi_{\tau-1}E_t^P(L\Lambda_1 + L\Lambda_2 X_t) \tag{101}$$

therefore, we have our return forecasting regression as:

$$r_{\tau,t+1} = \Psi_{\tau-1}L\Lambda_1 + (\Psi_{\tau-1}L\Lambda_2)X_t + \epsilon_{t+1} \quad \epsilon_{t+1} \sim N(0, \sigma^2) \tag{102}$$

Make a plausible assumption that convenience yield and inflation asymptote are not related to the risk premium parameters, the risk premium is driven by the real spot oil price and the macro factors, which means we only need to consider the effects from these factors. In an other word:

$$r_{\tau,t+1} = \Psi_{\tau-1}L\Lambda_1 + (\Psi_{\tau-1}L\Lambda_2 + \Upsilon_{\pi}^Q)X_{m,t} + \epsilon_{t+1} \quad \epsilon_{t+1} \sim N(0, \sigma^2) \tag{103}$$

recall observed vector as:

$$X_{m,t} = \begin{pmatrix} \delta_t \\ s_t^R \\ g_t \\ \pi_t \\ r_t \end{pmatrix} \quad (104)$$

where variables in $X_{m,t}$ for the return forecasting regression are all observable.

8.5 Kalman filter and the maximum likelihood estimation

Representing expectations conditional upon the available information with a ‘hat’ (so that $\hat{z}_t = E_t(z_t)$; $\hat{z}_{s|t} = E_t(z_s)$; $s \geq t$), where m_t is observable but z_t , z_{t+1} and $\epsilon_{z,t+1}$ are unobservable, using (15):

$$\begin{aligned} \hat{z}_{t+1|t} &= E_t(z_{t+1}) = E_t E_{t+1}(z_{t+1}) = E_t(\hat{z}_{t+1}) \\ &= K_z + \Xi \hat{z}_t. \end{aligned} \quad (105)$$

Define the covariance matrices of z_t at time t as:

$$P_{zz} = \hat{V}_t = E_t(z_t - \hat{z}_t)(z_t - \hat{z}_t)'; \quad (106)$$

so that the forecast of \hat{V} at time $t + 1$ is:

$$\begin{aligned} \hat{V}_{t+1|t} &= E_t(z_{t+1} - \hat{z}_{t+1|t})(z_{t+1} - \hat{z}_{t+1|t})' \\ &= \Xi E_t(z_t - \hat{z}_t)(z_t - \hat{z}_t)' \Xi' + \Sigma_{z,t} \\ &= \Xi \hat{V}_t \Xi' + \Sigma_{z,t} \end{aligned} \quad (107)$$

Similarly, using (21): $m_{t+1} = \hat{m}_{t+1|t} + \Theta(z_{t+1} - \hat{z}_{t+1|t}) + \epsilon_{m,t}$ where:

$$\hat{m}_{t+1|t} = K_m + \Theta \hat{z}_{t+1|t} + \Phi m_t, \quad (108)$$

we have:

$$m_{t+1} - \hat{m}_{t+1|t} = \Theta(z_{t+1} - \hat{z}_{t+1|t}) + \epsilon_{m,t} \quad (109)$$

therefore:

$$P_{mm} = E_t(m_{t+1} - \hat{m}_{t+1|t})(m_{t+1} - \hat{m}_{t+1|t})' = \Theta \hat{V}_{t+1|t} \Theta' + \Sigma_{m,t} \quad (110)$$

Similarly, using (50): $f_{t+1} = \hat{f}_{t+1|t} + \Psi'_{z,\tau}(z_{t+1} - \hat{z}_{t+1|t}) + \Psi'_{m,\tau}(m_{t+1} - \hat{m}_{t+1|t}) + i(p_{t+1} - \hat{p}_{t+1|t}) + e_{t+1}$. where:

$$\hat{f}_{t+1|t} = \alpha_\tau + \Psi'_{z,\tau} \hat{z}_{t+1|t} + \Psi'_{m,\tau} \hat{m}_{t+1|t} + i \hat{p}_{t+1|t}. \quad (111)$$

Thus, following our discussion in equation (51):

$$\begin{aligned} h_{t+1} &= f_{t+1} - ip_{t+1} \\ &= \hat{f}_{t+1|t} - i \hat{p}_{t+1|t} + \Psi'_{z,\tau}(z_{t+1} - \hat{z}_{t+1|t}) + \Psi'_{m,\tau}(m_{t+1} - \hat{m}_{t+1|t}) + e_{t+1} \\ &= \hat{h}_{t+1|t} + \Psi'_{z,\tau}(z_{t+1} - \hat{z}_{t+1|t}) + \Psi'_{m,\tau}(m_{t+1} - \hat{m}_{t+1|t}) + e_{t+1}, \end{aligned} \quad (112)$$

where:

$$\hat{h}_{t+1|t} = \alpha_\tau + \Psi_{z,\tau} \hat{z}_{t+1|t} + \Psi_{m,\tau} \hat{m}_{t+1|t}, \quad (113)$$

therefore:

$$h_{t+1|t} - \hat{h}_{t+1|t} = \Psi_{z,\tau}(z_{t+1|t} - \hat{z}_{t+1|t}) + \Psi_{m,\tau}(m_{t+1|t} - \hat{m}_{t+1|t}) + e_{t+1}, \quad (114)$$

hence:

$$\begin{aligned} P_{hh} &= E_t(h_{t+1|t} - \hat{h}_{t+1|t})(h_{t+1|t} - \hat{h}_{t+1|t})' \\ &= (\Psi_{z,\tau} + \Psi_{m,\tau}\Theta)\hat{V}_{t+1|t}(\Psi'_{z,\tau} + \Psi'_{m,\tau}\Theta') + \Psi_{m,\tau}\Sigma_{m,t}\Psi_{m,\tau} + Q_t. \end{aligned} \quad (115)$$

Having found P_{zz} , P_{mm} and P_{hh} , proceeding this way, we can further work out other cross products:

$$P_{zh} = E_t(z_{t+1} - \hat{z}_{t+1|t})(h_{t+1} - \hat{h}_{t+1|t}) = (\Psi_{m,\tau}\Theta + \Psi_{z,\tau})\hat{V}_{t+1|t}, \quad (116)$$

$$P_{hm} = E_t(h_{t+1} - \hat{h}_{t+1|t})(m_{t+1} - \hat{m}_{t+1|t}) = \Psi_{2,\tau}\Sigma_{m,t} + (\Psi_{2,\tau}\Theta + \Psi_{1,\tau})\hat{V}_{t+1|t}\Theta', \quad (117)$$

$$P_{mz} = E_t(m_{t+1} - \hat{m}_{t+1|t})(z_{t+1} - \hat{z}_{t+1|t}) = \Theta\hat{V}_{t+1|t}, \quad (118)$$

The t -conditional covariance matrix for this $t + 1$ dated system is:

$$\begin{pmatrix} P_{hh} & P_{hm} & P_{hz} \\ P'_{hm} & P_{mm} & P_{mz} \\ P'_{hz} & P'_{mz} & P_{zz} \end{pmatrix} = E_t \begin{pmatrix} h_{t+1} - \hat{h}_{t+1|t} \\ m_{t+1} - \hat{m}_{t+1|t} \\ z_{t+1} - \hat{z}_{t+1|t} \end{pmatrix} \begin{pmatrix} h_{t+1} - \hat{h}_{t+1|t} & m_{t+1} - \hat{m}_{t+1|t} & z_{t+1} - \hat{z}_{t+1|t} \end{pmatrix}, \quad (119)$$

This allows the expectations to be updated as:

$$\hat{z}_{t+1} = \hat{z}_{t+1|t} + \begin{pmatrix} P_{zh} & P_{zm} \end{pmatrix} \begin{pmatrix} P_{hh} & P_{hm} \\ P_{mh} & P_{mm} \end{pmatrix}^{-1} \begin{pmatrix} h_{t+1} - \hat{h}_{t+1|t} \\ m_{t+1} - \hat{m}_{t+1|t} \end{pmatrix}, \quad (120)$$

$$\hat{V}_{t+1} = \hat{V}_{t+1|t} - \begin{pmatrix} P_{zh} & P_{zm} \end{pmatrix} \begin{pmatrix} P_{hh} & P_{hm} \\ P_{mh} & P_{mm} \end{pmatrix}^{-1} \begin{pmatrix} P_{zh} \\ P_{zm} \end{pmatrix}. \quad (121)$$

Let H_{t-1} refer to the hyper-parameter at the last time stage $t - 1$, and

our target is to maximize the log likelihood function with respect to this hyper-parameter H_{t-1} :

$$\begin{aligned} \ln \mathcal{L}(H_{t+1}) = & -\frac{nT}{2} \ln(2\pi) - \frac{1}{2} \ln \left(\text{Det} \begin{pmatrix} P_{hh} & P_{hm} \\ P_{mh} & P_{mm} \end{pmatrix} \right) \\ & - \frac{1}{2} \begin{pmatrix} h_{t+1} - h_{t+1|t} & m_{t+1} - m_{t+1|t} \end{pmatrix} \begin{pmatrix} P_{hh} & P_{hm} \\ P_{mh} & P_{mm} \end{pmatrix}^{-1} \begin{pmatrix} h_{t+1} - \hat{h}_{t+1|t} \\ m_{t+1} - \hat{m}_{t+1|t} \end{pmatrix} \end{aligned} \quad (122)$$

8.6 Missing observations

Constrained by the availability of our data, different series in our data set is not aligned to each another. In another word, we have encountered the missing observation problem. Specifically, during the period between Q1 1964 to Q1 1984, we only have the spliced real log spot oil price and the three macro variables available. After Q1 1984, for WTI oil futures, we are missing data from Q1 1984 to Q3 1989 for oil futures contract with 18 months of maturity, and for 24 months oil futures data we are missing from Q1 1984 to Q3 1995.

We solve this problem by introducing indicator matrices to identify the observed data and exclude missing values in the data set, suggested by Tsay (2005), Durbin and Koopman (2012), recall measurement equation (52):

$$y_t = D + HX_t + e_t \quad e_t \sim N(0, Q) \quad (123)$$

let y_t be the full data set with no missing observations, and y_t^- to be the observed data set that we obtain from our data source with missing observations, furthermore, let matrix J_t be the $n * n$ indicator matrix at time t sharing the same number of dimension as H , its rows are a subset of the rows of the $n * n$ identity matrix. which means when there is no missing observations in y_t^- , J_t is to be a identity matrix, whereas when observation

is missing at maturity τ , therefore, for every time t , we have $y_t^- = J_t y_t$, and accordingly, $D^- = J_t D$, $H^- = J_t H$, $e_t^- = J_t e_t$ and $Q^- = J_t Q J_t'$, hence measurement equation (123), can be rewritten in a reduced form as:

$$y_t^- = D^- + H^- X_t + e_t^- \quad e_t^- \sim N(0, Q^-) \quad (124)$$

by doing this we can continue estimating our model using maximum likelihood estimation based on Kalman filter in the same way as data set is complete, as long as we carry out modified measurement equation at time t , if there is missing value at that point in time.

As we can see, the J_t matrix is here to indicate missing observations, and exclude them from the estimation by setting its relevant diagonal elements to zero. This means that in practice, we do not need to specify J_t directly, as long as we follow its implication to adjust the measurement equation accordingly.

For example, during the period between Q1 1964 to Q1 1984, we can only observe the spliced real log spot oil price and the three macro variables. Considering oil futures prices for all maturities are missing observations, in summary, the state space representation for this time period is:

$$y_t^- = D^- + H^- X_t + e_t^- \quad e_t^- \sim N(0, Q^-) \quad (125)$$

$$X_t = A + B X_{t-1} + W_t \quad W_t \sim N(0, \Sigma_t), \quad (126)$$

to see it more clearly, the reduced form of measurement equation (125) in

matrices can be specified as:

$$\begin{pmatrix} \delta_t^o \\ s_t^{R,o} \\ g_t^o \\ \pi_t^o \\ r_t^o \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta_t^* \\ s_t^* \\ \pi_t^* \\ r_t^* \\ \delta_t \\ s_t^R \\ g_t \\ \pi_t \\ r_t \end{pmatrix} + e_t^- \quad e_t^- \sim N(0, Q^-),$$

(127)

where the measurement error is a 4×4 matrix with all elements equal to zero, namely: $Q^- = 0_{4,4}$. This is because as we have specified, the more macro variables are observed without any measurement error. The state equation (126) under measure P remains unchanged as what we have previously specified in equation (55). Similarly, we readjust specification of the measurement equation in this way accordingly for the other two period: Q1 1984 to Q3 1989 and Q1 1984 to Q3 1995, that we are missing 18 months and 24 months oil futures prices.