Strong Reflection Principles and the Hierarchy of Large Cardinals

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Large cardinals

Large cardinals

Usually, by a large cardinal axiom one means a set-theoretic assertion that implies the existence of a cardinal having some properties that make it very large, and whose existence cannot be proved in ZFC (because it implies the consistency of ZFC).

Empirical fact

There is strong evidence that every statement of the language of set theory (and therefore every mathematical statement) that is independent of ZFC is equiconsistent with either ZFC or ZFC plus a large cardinal axiom.

The consistency strength of a set-theoretic statement φ that is consistent with ZFC but not equiconsistent with it can be measured by large cardinal axioms. That is, there are large cardinal axioms A_1 and A_2 such that

$$CON(ZFC + \varphi) \Rightarrow CON(ZFC + A_1)$$

and

$$CON(ZFC + A_2) \Rightarrow CON(ZFC + \varphi)$$

We refer to A_1 as a lower bound for the consistency of φ and A_2 as an upper bound.

In the fortunate cases when the lower and upper bound coincide, we obtain an exact measure of the consistency strength of φ.

Some examples of large cardinals (in chronological order)

- κ is weakly inaccessible if it is regular, uncountable, and a limit cardinal. (Hausdorff 1908)
- κ is inaccessible if it is regular, uncountable, and a strong limit, i.e., $2^\lambda < \kappa$ for all $\lambda < \kappa$. (Sierpiński-Tarski, Zermelo 1930)
- κ is measurable if it is uncountable and there is a κ-additive non-trivial two-valued measure on $\mathcal{P}(\kappa)$. (Ulam 1930)
- κ is weakly compact if $\kappa \to (\kappa)^2_2$. (Erdös-Tarski 1961)
- Vopěnka's Principle asserts that there is no rigid proper class of graphs. (Vopěnka 1960's)

Some equiconsistencies

The following are pairs of equiconsistent statements:

1. - Every co-analytic uncountable set of real numbers contains a perfect set.
   - There exists an inaccessible cardinal.

2. - There is a countably-additive measure that extends the Lebesgue measure and measures all sets of reals.
   - There exists a measurable cardinal.

3. - The GCH first fails at $\aleph_\omega$.
   - There exists a measurable cardinal κ of Mitchell order κ^{++}.

4. - A category is bounded iff it has a colimit-dense subcategory.
   - Vopěnka's principle.
     (They are actually equivalent!)

Three questions

Question

Are large cardinal axioms true axioms of set theory?

Question

Why do large cardinals form a linear hierarchy measuring the consistency strength of set-theoretic statements?

Question

Is there a uniform and complete hierarchy of large cardinals?

Reflection Phenomena

Reflection Phenomena I: Reflecting the theory of V in some V_α

The Reflection Theorem (Levy 1960, Montague 1961)

Every formula of the first-order language of set theory is reflected in some V_α. More precisely, for every formula $\varphi(x_1, \ldots, x_n)$, ZF proves that for every $a_1, \ldots, a_n$ there is an ordinal α such that

$$\varphi(a_1, \ldots, a_n) \quad \text{if and only if} \quad V_\alpha \models \varphi(a_1, \ldots, a_n).$$

In fact, for every n, ZF proves that there exists a closed and unbounded proper class $C^{(n)}$ of ordinals such that $V_\alpha \preceq_{\Sigma_n} V$, for every $\alpha \in C^{(n)}$.

Levy observed that the Reflection Theorem is equivalent to the axioms of Infinity and Replacement (modulo the other ZF axioms), which shows that the reflection phenomenon is not only deeply ingrained in the ZF axioms, but it captures the essence of set theory.

Thus, it is not surprising that many people have suggested that any intrinsic[1] justification of new set-theoretic axioms, beyond ZFC, and in particular the axioms of large cardinals, should be based on stronger forms of the Reflection Theorem, e.g., second-order, or even higher-order, reflection.

However, Koellner[2] has shown that any (Levy-Montague type) Reflection principle for reasonable classes of higher order formulas either follows from the existence of $\kappa(\omega)$ or is outright inconsistent.

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[1] I.e., based on the iterative conception of the set-theoretic universe.

[2] On Reflection Principles, Annals of Pure and Applied Logic, Vol. 157, Nos. 2-3, 2009, pp. 206-219.

Reflection Phenomena II: Reflecting the theory of a structure in a smaller one

The reflection phenomenon is ubiquitous. It occurs both globally and locally.

The Downwards Löwenheim-Skolem-Tarski Theorem

For every sentence φ of a first-order language $\mathcal{L}$, and every $M \models \varphi$, there exists $N \subseteq M$ such that $N \models \varphi$ and $|N| \leqslant |\mathcal{L}|$.

In fact, for every structure M in a first-order language $\mathcal{L}$, there exists $N \preceq M$ such that $|N| \leqslant |\mathcal{L}|$.

Note: N need not be of the same kind as M. E.g., if M is transitive, then N need not be so.

Large cardinals as second-order local reflection principles. Early results

- κ is inaccessible iff for every $A \subseteq V_\kappa$ there is $\lambda < \kappa$ such that

  $$\langle V_\lambda, \in, A \cap V_\lambda \rangle \preceq \langle V_\kappa, \in, A \rangle.$$

  Equivalently, iff it is Σ^1_1-indescribable. (Levy 1960)

- κ is weakly compact iff it is Π^1_1-indescribable. (Hanf-Scott 1961, Keisler 1962)

Large cardinals as resemblance principles. Early results

- κ is measurable iff it is the critical point of an elementary embedding $j : V \to M$, with M transitive. (Scott 1961, Keisler)
- $0^\sharp$ exists iff there is a non-trivial elementary embedding $j : L \to L$. Equivalently, iff there is an elementary embedding $j : L_\alpha \to L_\beta$ with critical point less than $|\alpha|$. (Kunen 1971)
- Vopěnka's Principle holds iff for every proper class of structures $\mathcal{C}$ of the same type, there exist distinct M and N in $\mathcal{C}$ and an elementary embedding $j : N \to M$. (Vopěnka 1970s)

Elementary embeddings and larger cardinals

- κ is supercompact if, for every $\lambda > \kappa$ there exists an elementary embedding $j : V \to M$, M transitive, with critical point κ, and such that $j(\kappa) > \lambda$ and M is closed under λ-sequences. (Solovay-Reinhardt, late 1960's)
- κ is Reinhardt if there exists an elementary embedding $j : V \to V$ with $crit(j) = \kappa$. (Reinhardt 1965)

Kunen's Theorem

Theorem (Kunen 1971)

Reinhardt cardinals don't exist.
In fact, there is no non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$, for any λ.

- κ is extendible if for every λ greater than κ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with $crit(j) = \kappa$, and $j(\kappa) > \lambda$. (Reinhardt 1974)

Reflection Phenomena III:
Reflecting a structure in a smaller one **of the same kind**

A class of structures $\mathcal{C}$ (of the same kind) is given by some formula $\varphi(x)$, possibly with parameters, so that

$$\mathcal{C} = \{A = \langle X, \in, \langle R_i \rangle_{i \in I} \rangle : \varphi(A)\}.$$

The notion of reflecting the class $\mathcal{C}$ can be naturally construed in the sense that some V_α reflects φ. One natural way to make this precise is to say that for every structure $A \in \mathcal{C}$ there exists $B \in \mathcal{C}$ which belongs to V_α and is *like* A. Since, in general, A may be much larger than any B in V_α, the closest resemblance of B to A will be attained in the case B is isomorphic to an elementary substructure of A, i.e., B can be elementarily embedded into A.

We emphasize that what is reflected is not the formula φ, but the structural property defined by φ. This is the crucial difference with the Levy-Montague type of reflection.

Reflection Phenomena III: Reflecting a structure in a smaller one of the same kind

Structural Reflection

$SR_\kappa(\mathcal{C})$: κ reflects $\mathcal{C}$, i.e., for every A in $\mathcal{C}$ there exist B in $\mathcal{C} \cap V_\kappa$ and an elementary embedding from B into A.

Notation: If Γ is Σ_n or Π_n, then $SR_\kappa(\Gamma)$ means that $SR_\kappa(\mathcal{C})$ holds for every Γ-definable class $\mathcal{C}$ of structures.

$SR(\Gamma)$ means that there exists κ such that $SR_\kappa(\Gamma)$ holds.

Theorem

$SR_\kappa(\Sigma_1)$ *holds for every* $\kappa \in C^{(1)}$.

Large cardinals as principles of Structural Reflection (SR)

Supercompact and extendible cardinals as SR principles

Recall:

A cardinal κ is supercompact if, for every $\lambda > \kappa$ there exists an elementary embedding $j : V \to M$, M transitive, with critical point κ, and such that $j(\kappa) > \lambda$ and M is closed under λ-sequences.

Theorem

If κ is supercompact, then $SR_\kappa(\Sigma_2)$ holds.

This, together with the following theorem of Magidor's yields a characterization of supercompactness in terms of SR.

Theorem (Magidor 1971[3])

If κ is the least cardinal that reflects the Π_1-definable proper class $\mathcal{C}$ of structures of the form $\langle V_\lambda, \in \rangle$, then κ is supercompact.

[3]Magidor, M. (1971) On the role of supercompact and extendible cardinals in logic. *Israel Journal of Mathematics* **10**, 147 – 157

Corollary

The following are equivalent for a cardinal κ*:*

1. κ *is the first supercompact cardinal.*
2. κ *is the least cardinal such that* $SR_\kappa(\Pi_1)$ *holds.*
3. κ *is the least cardinal such that* $SR_\kappa(\Sigma_2)$ *holds.*

Corollary

The following are equivalent:

1. $SR(\Pi_1)$
2. $SR(\Sigma_2)$
3. *There exists a supercompact cardinal.*

Beyond supercompactness

Recall: κ is extendible if for every $\lambda > \kappa$ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with $crit(j) = \kappa$, and $j(\kappa) > \lambda$.

Theorem

The following are equivalent for a cardinal κ:

1. *κ is the first extendible cardinal.*
2. *κ is the least cardinal such that $SR_\kappa(\Pi_2)$ holds.*
3. *κ is the least cardinal such that $SR_\kappa(\Sigma_3)$ holds.*

Corollary

The following are equivalent:

1. $SR(\Pi_2)$
2. $SR(\Sigma_3)$
3. *There exists an extendible cardinal.*

For the higher levels ($n > 3$) we need the notion of $C^{(n)}$-extendible cardinal.

Definition

κ is $C^{(n)}$-extendible if for every λ greater than κ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with $crit(j) = \kappa$, $j(\kappa) > \lambda$, and $j(\kappa) \in C^{(n)}$, i.e., $V_{j(\kappa)} \preceq_{\Sigma_n} V$.

Note: κ is extendible if and only if it is $C^{(1)}$-extendible.

Theorem

The following are equivalent for a cardinal κ, and $n \geqslant 1$:

1. κ *is the first* $C^{(n)}$*-extendible cardinal.*
2. κ *is the least cardinal such that* $SR_\kappa(\Pi_{n+1})$ *holds.*
3. κ *is the least cardinal such that* $SR_\kappa(\Sigma_{n+2})$ *holds.*

Corollary

The following are equivalent for $n \geqslant 1$:

1. $SR(\Pi_{n+1})$
2. $SR(\Sigma_{n+2})$
3. *There exists a* $C^{(n)}$*-extendible cardinal.*

Remark

Suppose $n \geqslant 1$. Given a Σ_{n+1}-definable class of structures $\mathcal{C}$, say via the Σ_{n+1} formula $\varphi(x)$, let $\mathcal{C}^*$ be the class of structures of the form

$$A^* = \langle V_\alpha, \in, \alpha, A\rangle$$

where α is the least ordinal in $C^{(n)}$ such that $V_\alpha \models \varphi(A)$. If $A \in \mathcal{C}$, then such an α exists, because the set of ordinals α such that $V_\alpha \models \varphi(A)$ is club. Conversely, if $\langle V_\alpha, \in, \alpha, A\rangle \in \mathcal{C}^*$, then $V_\alpha \models \varphi(A)$ and since $\alpha \in C^{(n)}$, $A \in \mathcal{C}$. Thus, we have

$$A \in \mathcal{C} \text{ if and only if } A^* \in \mathcal{C}^*.$$

Now notice that $\mathcal{C}^*$ is Π_n definable. This explains why a cardinal reflects Π_n classes if and only if it reflects Σ_{n+1} classes.

$C^{(n)}$-extendible cardinals from Vopěnka's Principle

Recall that Vopěnka's Principle holds (schematically) if for every definable proper class $\mathcal{C}$ of structures of the same type there exist $A \neq B$ in $\mathcal{C}$ and an elementary embedding $j : A \to B$.

Theorem

Suppose $n \geqslant 1$. If Vopěnka's Principle holds for Π_{n+1}-definable classes, then there exists a $C^{(n)}$-extendible cardinal.

Corollary

The following are equivalent:

1. $SR(\mathcal{C})$ *holds for every definable (with parameters) class* $\mathcal{C}$.
2. *There exists a* $C^{(n)}$*-extendible cardinal, for every* n.
3. *There exists a proper class of* $C^{(n)}$*-extendible cardinals, for every* n.
4. *Vopěnka's Principle.*

SR relative to inner models

SR for classes of structures in inner models

Example

Let $\mathcal{C}$ be the class of structures of the form $\langle L_\beta, \in, \gamma \rangle$, where γ and β are cardinals (in V) and $\gamma < \beta$. Note that $\mathcal{C}$ is Π_1 definable (without parameters).

Theorem

The following are equivalent:

1. $SR(\mathcal{C})$
2. $0^\sharp$ *exists.*

The *shadow* of supercompactness on inner models

A similar equivalence holds relative to any set of ordinals X:

let $\mathcal{C}_X$ be the class of structures of the form $\langle L_\beta[X], \in, \gamma, X\rangle$, where γ and β are cardinals and $sup(X) < \gamma < \beta$.
Note that $\mathcal{C}$ is Π_1 definable with X as a parameter.

Theorem

The following are equivalent:

1. $SR(\mathcal{C}_X)$
2. $X^\sharp$ *exists.*

Similar results hold for larger inner models, e.g., $L[\mu]$. Letting $\mathcal{C}$ be the class of structures of the form $\langle L_\beta[\mu], \in, \gamma, \mu \rangle$, where γ and β are cardinals (in V) and $\gamma < \beta$, we have:

Theorem

The following are equivalent:

1. *There exists* κ *such that* $SR_\kappa(\mathcal{C})$ *holds.*
2. $0^\dagger$ *exists.*

SR below a supercompact

Between $SR(\Sigma_1)$ and $SR(\Pi_1)$

For $\mathcal{R}$ a set of Π_1 predicates, and κ an infinite cardinal, let's define:

Definition

$SR_\kappa(\mathcal{R})$ iff κ reflects every $\Sigma_1(\mathcal{R})$ definable class of structures $\mathcal{C}$ of the same type and closed under isomorphisms.

We write $SR(\mathcal{R}) = \kappa$ to indicate that κ is the least cardinal for which $SR_\kappa(\mathcal{R})$ holds.

We have that $SR(\emptyset) = \aleph_1$. However, if R is the Π_1 relation

$$\text{“}x \text{ is an ordinal and } y = V_x\text{”}$$

then $SR(R) = \kappa$ if and only if κ is the first supercompact cardinal. Moreover, if κ is supercompact, then $SR_\kappa(\mathcal{R})$ holds for every $\mathcal{R}$.

The Löwenheim-Skolem-Tarski property for a logic $\mathcal{L}^*$

By a *logic* $\mathcal{L}^*$ we mean one of the following,

- First-order logic ($\mathcal{L}_{\omega\omega}$).
- Infinitary logic ($\mathcal{L}_{\kappa\lambda}$).
- Higher-order logic ($\mathcal{L}^n$, $n \geqslant 2$).

possibly extended with generalized quantifiers.

Definition

$LST(\mathcal{L}^*)(\kappa)$: for every $\mathcal{L}^*$-sentence φ and every $M \models \varphi$, there is $N \subseteq M$ such that $N \models \varphi$ and $|N| < \kappa$.

Notice that if $LST(\mathcal{L}^*)(\kappa)$ holds, then it also holds for any larger cardinal. We call the least cardinal κ for which $LST(\mathcal{L}^*)(\kappa)$ holds, provided it exists, the $LST(\mathcal{L}^*)$*-number*, and we write $LST(\mathcal{L}^*) = \kappa$ to indicate this.

Examples

- $LST(\mathcal{L}_{\omega\omega}) = LST(\mathcal{L}_{\omega_1\omega}) = \aleph_1$.
- $LST(\mathcal{L}_{\omega\omega}(MM_{\aleph_1})) = \aleph_2$, where $MM_{\aleph_1}$ is the *Magidor-Malitz quantifier*. Namely,

$$MM_{\aleph_1} x \varphi(x, \vec{y})$$

if and only if there exists X such that $|X| \geqslant \aleph_1$ and $\varphi(a, \vec{y})$ holds for all $a \in X$.

The close relationship between $SR(\mathcal{R})$ and $LST(\mathcal{L}^*)$

If $\mathcal{C}$ is a class of structures with vocabulary L, and $L' \subseteq L$, then we can take the *projection* of $\mathcal{C}$ to L', that is

$$\mathcal{C} \upharpoonright L' := \{M \upharpoonright L' : M \in \mathcal{C}\}.$$

A class $\mathcal{C}$ of structures in some fixed vocabulary is said to be $\mathcal{L}^*$*-definable* if there is a sentence $\varphi \in \mathcal{L}^*$ such that $\mathcal{C} = \{M : M \models \varphi\}$.

Sometimes, a class $\mathcal{C}$ is a projection of an $\mathcal{L}^*$-definable class, and the complement is also a projection of an $\mathcal{L}^*$-definable class. Then we say that $\mathcal{C}$ is $\Delta(\mathcal{L}^*)$*-definable*.

Example (A paradigm example)

The class $\mathcal{W}$ of structures $(M, <)$, where $<$ well-orders M is $\Delta(\mathcal{L}^*)$-definable, where $\mathcal{L}^*$ is $\mathcal{L}_{\omega\omega}(I)$, i.e., first-order logic with the additional quantifier I, known as the *Härtig quantifier*, given by

$$Ixy\varphi(x)\psi(y) \leftrightarrow |\varphi(\cdot)| = |\psi(\cdot)|.$$

Symbiosis

Symbiosis

Definition (Väänänen 1979)

A finite set $\mathcal{R}$ of predicates and a logic $\mathcal{L}^*$ are *symbiotic* if the following conditions are satisfied:

1. Every $\mathcal{L}^*$-definable model class is $\Delta_1(\mathcal{R})$-definable.
2. Every $\Delta_1(\mathcal{R})$-definable model class closed under isomorphisms is $\Delta(\mathcal{L}^*)$-definable.

Examples

The following pairs $(\mathcal{R}, \mathcal{L}^*)$ are symbiotic.

1. $\mathcal{R}$: Cd, where Cd is the predicate "x is a cardinal".
   $\mathcal{L}^*$: $\mathcal{L}_{\omega\omega}(I)$, where $Ixy\varphi(x)\psi(y) \leftrightarrow |\varphi| = |\psi|$ is the Härtig quantifier.
2. $\mathcal{R}$: Rg, where Rg is the predicate "x is a regular cardinal".
   $\mathcal{L}^*$: $\mathcal{L}_{\omega\omega}(I, W^{Rg})$, where $W^{Rg}xy\varphi(x,y) \leftrightarrow \varphi(\cdot,\cdot)$ has the order-type of a regular cardinal.
3. $\mathcal{R}$: Cd, WC, where $WC(x,\alpha)$ is the relation "α is a limit ordinal and x is a partial ordering with no chain of order-type α".
   $\mathcal{L}^*$: $\mathcal{L}_{\omega\omega}(I, Q_{Br})$, where $Q_{Br}xy\varphi(x,y) \leftrightarrow \varphi(\cdot,\cdot)$ is a tree order of height some α and has no branch of length α.
4. $\mathcal{R}$: $PwSet$, where $PwSet$ be the relation $\{(x,y) : y = \mathcal{P}(x)\}$.
   $\mathcal{L}^*$: $\mathcal{L}^2$.

Recall:

$SR_\kappa(\mathcal{R})$: κ reflects every $\Sigma_1(\mathcal{R})$ definable class of structures $\mathcal{C}$ of the same type and closed under isomorphisms.

$LST(\mathcal{L}^*)(\kappa)$: for every $\mathcal{L}^*$-sentence φ and every $M \models \varphi$, there is $N \subseteq M$ such that $N \models \varphi$ and $|N| < \kappa$.

Theorem (B. and Väänänen 2014)

Suppose $\mathcal{L}^*$ and $\mathcal{R}$ are symbiotic. Then the following are equivalent, for every cardinal κ:

(i) $SR_\kappa(\mathcal{R})$

(ii) $LST(\mathcal{L}^*)(\kappa)$.

It follows that for symbiotic $\mathcal{L}^*$ and $\mathcal{R}$,

$$LST(\mathcal{L}^*) = \kappa \text{ if and only if } SR(\mathcal{R}) = \kappa.$$

Thus, writing $\equiv$ to indicate that the corresponding cardinals are the same, assuming they exist, we have the following:

Corollary

1. $SR(Cd) \equiv LST(\mathcal{L}_{\omega\omega}(I))$.
2. $SR(Rg) \equiv LST(I, \mathcal{L}_{\omega\omega}(W^{Rg}))$.
3. $SR(Cd, WC) \equiv LST(\mathcal{L}_{\omega\omega}(I, Q_{Br}))$.
4. $SR(PwSet) \equiv LST(\mathcal{L}^2)$.

Theorem (Magidor and Väänänen 2011)

If $LST(\mathcal{L}_{\omega\omega}(I)) = \kappa$, then PD holds, the SCH holds above κ, etc. Moreover, it is consistent (modulo a supercompact cardinal) that the $LST(\mathcal{L}_{\omega\omega}(I))$ number is the first supercompact cardinal. But it is also consistent (modulo a supercompact cardinal) that the $LST(\mathcal{L}_{\omega\omega}(I))$ number is the first inaccessible cardinal.

Theorem (Magidor 1971)

$LST(\mathcal{L}^2) = \kappa$ if and only if κ is the first supercompact cardinal.

So, in order to characterize smaller large cardinals we need to consider different *SR* principles.

SR for structures of limited size

Weak Structural Reflection

$SR^{-}_{\kappa}(\mathcal{R})$: If $\mathcal{C}$ is a $\Sigma_1(\mathcal{R})$ class of structures closed under isomorphisms and $M \in \mathcal{C}$ has cardinality κ, then there exists $N \in \mathcal{C}$ of cardinality less than κ and an elementary embedding $e : N \to M$.

Definition (The strict Löwenheim-Skolem-Tarski property)

$SLST(\mathcal{L}^*)(\kappa)$: for every $\mathcal{L}^*$-sentence φ and every $M \models \varphi$, if $|M| = \kappa$, then there is $N \subseteq M$ such that $N \models \varphi$ and $|N| < \kappa$.

Theorem (B. and Väänänen 2014)

If $\mathcal{R}$ and $\mathcal{L}^*$ are symbiotic, then the following are equivalent for every cardinal κ:

(i) $SR^-_\kappa(\mathcal{R})$

(ii) $SLST(\mathcal{L}^*)(\kappa)$.

Corollary

1. $SR^-(Cd) \equiv SLST(\mathcal{L}_{\omega\omega}(I))$
2. $SR^-(Rg) \equiv SLST(I, \mathcal{L}_{\omega\omega}(W^{Rg}))$
3. $SR^-(Cd, WC) \equiv SLST(\mathcal{L}_{\omega\omega}(I, Q_{Br}))$.

Theorem (B. and Väänänen 2014)

1. $SR^{-}(Cd) = \kappa$ *if and only if* κ *is the least weakly inaccessible cardinal.*
2. $SR^{-}(Rg) = \kappa$ *if and only if* κ *is the first weakly Mahlo cardinal.*
3. $SR^{-}(Cd, WC) = \kappa$ *if and only if* κ *is the first weakly compact cardinal.*

Generic SR

Generic Structural Reflection

Generic Structural Reflection

$GSR_\kappa(\Gamma)$: For every Γ-definable (in the first-order language of set theory, possibly with parameters) class of structures $\mathcal{C}$, κ generically reflects $\mathcal{C}$, i.e.,

for every A in $\mathcal{C}$ there exists B in $\mathcal{C} \cap V_\kappa$ such that in some generic extension of V there is an elementary embedding from B into A.

Thus, $GSR_\kappa(\mathcal{C})$ says: $SR_\kappa(\mathcal{C})$ holds, but the witnessing elementary embeddings may only exist in some generic extension of V.

Proposition

The following are equivalent for structures B and A in the same language.

1. *In $V^{\mathrm{Coll}(\omega,B)}$, there is an elementary embedding $j : B \to A$.*
2. *There is a complete Boolean algebra $\mathbb{B}$ such that*

$$V^{\mathbb{B}} \models \text{“There exists an elementary embedding } j : B \to A.\text{”}$$

So, we may reformulate *GSR* as follows:

Generic Structural Reflection

$GSR_\kappa(\Gamma)$: For every Γ-definable (in the first-order language of set theory, possibly with parameters) class of structures $\mathcal{C}$, κ generically reflects $\mathcal{C}$, i.e.,
for every A in $\mathcal{C}$ there exists B in $\mathcal{C} \cap V_\kappa$ such that in $V^{\mathrm{Coll}(\omega,B)}$ there is an elementary embedding from B into A.

It turns out that some large cardinals, e.g., remarkable cardinals, can be characterized in terms of *GSR*:

Remarkable cardinals

A cardinal κ is remarkable (Schindler 2000) if and only if for all regular cardinals $\theta > \kappa$ and for every $a \in H(\theta)$ there are M, N, π, $\bar{\kappa}$, σ, and $\bar{\theta}$ such that the following hold:

1. M and N are countable and transitive,
2. $\pi : M \to H(\theta)$ is an elementary embedding,
3. a is in the range of π,
4. $\pi(\bar{\kappa}) = \kappa$,
5. $\sigma : M \to N$ is an elementary embedding with $crit(\sigma) = \bar{\kappa}$,
6. $\bar{\theta} = M \cap OR$ is a regular cardinal in N,
7. $\sigma(\bar{\kappa}) > \bar{\theta}$, and
8. $M = H(\bar{\theta})^N$, i.e., $M \in N$ and $N \models$ "$M = \{x : |TC(x)| < \bar{\theta}\}$".

Equivalently (Schindler 2014): if for every regular cardinal $\lambda > \kappa$, there is a regular cardinal $\bar{\lambda} < \kappa$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j : H^V_{\bar{\lambda}} \to H^V_{\lambda}$ with $j(\mathrm{crit}(j)) = \kappa$.

By a result of Magidor (1971), remarkable cardinals are virtually supercompact.

However, remarkable cardinals are much weaker than supercompact cardinals. They are downward absolute to L and their consistency consistency strength is below an ω-Erdös cardinal.
Remarkable cardinals are in $C^{(2)}$. Moreover, they are totally indescribable and limits of totally indescribable cardinals.

GSR and remarkable cardinals

All the following results are from B.-Gitman-Schindler 2016[4]

Theorem

1. *If κ is a remarkable cardinal, then $GSR_\kappa(\Sigma_2)$ holds.*
2. *If $GSR_\kappa(\Pi_1)$ holds, then either there is a remarkable cardinal, or there is a transitive model of ZFC with a proper class of remarkable cardinals.*

Corollary

The following are equiconsistent:

1. $GSR(\Pi_1)$
2. $GSR(\Sigma_2)$
3. *There exists a remarkable cardinal.*

[4] Bagaria, J., Gitman, V., Schindler, R., Generic Vopěnka's Principle, Remarkable Cardinals, and the Weak Proper Forcing Axiom. Arch. Math. Logic (2016).

Definition

A cardinal κ is virtually extendible if for every $\alpha > \kappa$, in some set-forcing extension (equivalently in $V^{\mathrm{Coll}(\omega, V_\alpha)}$) there is $j : V_\alpha \to V_\beta$ such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.
A cardinal κ is virtually $C^{(n)}$-extendible if additionally $j(\kappa) \in C^{(n)}$.

Note that virtually extendible cardinals are virtually $C^{(1)}$-extendible because $j(\kappa)$ must be inaccessible in V.

Theorem

If $0^\sharp$ exists, then in L every Silver indiscernible is virtually $C^{(n)}$-extendible, for all n.

Theorem

If κ is virtually $C^{(n)}$-extendible, then $GSR(\Sigma_{n+2})$ holds.

Theorem

If $GSR(\Pi_{n+1})$ holds, then either there is a virtually $C^{(n)}$-extendible cardinal or there is a transitive model of ZFC with a proper class of virtually $C^{(n)}n$-extendible cardinals.

Theorem

The following are equiconsistent for $n \geqslant 1$:

1. $GSR(\Pi_{n+1})$
2. $GSR(\Sigma_{n+2})$
3. *There is a virtually $C^{(n)}$-extendible cardinal.*

Question

Can we replace "equiconsistent" by "equivalent"?

If in the definition of GSR we require the embeddings to be overspilling, i.e., $j(crit(j)) > rank(B)$, then YES.

But without this requirement the answer is NO, by a recent result of Gitman-Hamkins (2017).

Theorem

The following are equivalent for every $n \geqslant 1$:

1. *GSR for Σ_{n+2} classes of structures of the form $\langle V_\alpha, \in \rangle$ (and we require overspilling).*
2. *There is a virtually $C^{(n)}$-extendible cardinal.*

Theorem

The following are equivalent:

1. *GSR for every definable class of structures of the form $\langle V_\alpha, \in \rangle$ (and we require overspilling).*
2. *For every n there is a virtually $C^{(n)}$-extendible cardinal.*
3. *The Generic Vopěnka Principle holds.*

Conclusions

We have the following equivalences (modulo ZFC):

- $SR(\Sigma_1) \equiv$ True
- $SR(\Pi_1) \equiv SR(\Sigma_2) \equiv$ There exists a supercompact cardinal
- $SR(\Pi_2) \equiv SR(\Sigma_3) \equiv$ There exists an extendible cardinal
- $SR(\Pi_n) \equiv SR(\Sigma_{n+1}) \equiv$ There exists a $C^{(n)}$-extendible card.
- $SR \equiv$ Vopěnka's Principle
- $SR(\Pi_1 \cap L) \Rightarrow 0^\sharp$ exists
- $SR(\Pi_1 \cap L[X]) \Rightarrow X^\sharp$ exists
- $SR(\Pi_1 \cap L[\mu]) \Rightarrow 0^\dagger$ exists
- $SR^-(Cd) \equiv$ There exists a weakly inaccessible card.
- $SR^-(Rg) \equiv$ There exists a weakly Mahlo card.
- $SR^-(Cd, WC) \equiv$ There exists a weakly-compact card.
- $GSR(\Sigma_2) \equiv$ There exists a remarkable card.
- $GSR(\Sigma_{n+2}) \equiv$ There exists a virtually $C^{(n)}$-extendible card.
- $GSR \equiv$ Generic Vopěnka's principle

A Conjecture (aspiring to become a Definition)

Conjecture

Every large cardinal axiom is equivalent to some natural form of SR.