A Generalisation of Closed Unbounded and Stationary Sets

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C is *1-club* in κ iff C is stationary in κ and stationary-closed.

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5. $S \subseteq \kappa$ is γ-stationary if for every $\gamma' < \gamma$ we have κ is γ'-s-reflecting and for any C γ'-club in κ we have $S \cap C \neq \emptyset$

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Notation

$d_\gamma(A) := \{\alpha : A \text{ is } \gamma\text{-stationary below } \alpha\}$

Restating the Definitions in Terms of d_γ

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Definition (restated)

1. $S \subseteq On$ is 0-stationary in κ if it is unbounded in κ.
2. $C \subseteq On$ is γ-*stationary closed* if $d_\gamma(C) \subseteq C$.
3. C is γ-*club* in κ if C is γ-stationary closed and γ-stationary below κ.
4. κ is γ-*reflecting* if for any γ-stationary $S, T \subseteq \kappa$, $d_\gamma(S) \cap d_\gamma(T) \cap \kappa \neq \emptyset$.
5. $S \subseteq \kappa$ is $n+1$-stationary if κ is n-reflecting and $S \cap C \neq \emptyset$ for every C n-club in κ

how large is a subset of κ?

If κ is *n*-reflecting, then for a subset of κ we have these implications:

$$
\begin{array}{ccc}
n\text{-club} & \Longrightarrow & n+1\text{-stationary} \\
\Uparrow & & \Downarrow \\
n-1\text{-club} & & n\text{-stationary} \\
\Uparrow & & \Downarrow \\
\dots & & \dots \\
\Uparrow & & \Downarrow \\
0\text{-club } (= \text{club}) & & \text{stationary} \\
 & & \Downarrow \\
 & & \text{unbounded}
\end{array}
$$

Origins

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- Defining γ-stationary sets in this way is equivalent to defining them in terms of generalised clubs.
- This is easy to show by induction: the key is that if κ is γ-stationary and $T \subseteq \kappa$ is η-stationary for $\eta < \gamma$, then $d_\eta(T)$ is η-club.

Definition 3: Topologies

Let Ω be an ordinal and $\mathcal{T}$ a topology on Ω. For $A \subseteq \Omega$ we set

$$d_{\mathcal{T}}(A) = \text{ the set of limit points of } A \text{ in the topology } \mathcal{T}.$$

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- These topologies are closely related to γ-stationarity.

$\mathcal{T}_\gamma$ and γ-stationarity

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We have the following equivalences:

$$\begin{aligned} & \alpha \in d_{\mathcal{T}_1}(A) \\ \Leftrightarrow\ & \forall X, Y \subseteq \Omega\ \alpha \in d_0(X) \cap d_0(Y) \to d_0(X) \cap d_0(Y) \cap \alpha \cap A \neq \emptyset \\ \Leftrightarrow\ & \forall C, D \text{ club in } \alpha \text{ we have } C \cap D \cap A \neq \emptyset \\ \Leftrightarrow\ & A \text{ is stationary in } \alpha\text{, i.e. } \alpha \in d_1(A) \end{aligned}$$

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Thus:

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In fact we can show that for any γ, $d_{T_\gamma} = d_\gamma$. Thus a point α is *non-isolated* in $\mathcal{T}_\gamma$ iff for every $\gamma' < \gamma$, α is γ'-s-reflecting (i.e. α is γ-stationary), and $\mathcal{T}_\gamma$ is non-discrete iff there is an ordinal $\alpha < \Omega$ that is γ stationary.

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Upper bound:

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Theorem (B., Bagaria)
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Lower bound?

Theorem (Magidor)
A regular cardinal that is 1-s-reflecting is Π^1_1-indescribable in L. Thus the existence of a 1-s-reflecting cardinal is equiconsistent with the existence of a Π^1_1-indescribable.

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Theorem (B.)
Let κ be a regular cardinal that is γ-s-reflecting such that the γ-club filter on κ is normal, and for "many" cardinals λ below κ we have λ is η-s-reflecting implies the η-club filter on λ is normal. Then κ is Π^1_γ-indescribable in L.

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Conjecture:
For $\gamma > 1$ the consistency strength of a γ-s-reflecting cardinal is below that of a Π^1_γ-indescribable.

Generalised $\square$ sequences

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A $\square^\gamma$ *sequence* on κ is a sequence $\langle C_\alpha : \alpha \in d_\gamma(\kappa)\rangle$ such that for each α:

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2. (Coherence) for every $\beta \in d_\gamma(C_\alpha)$ we have $C_\beta = C_\alpha \cap \beta$

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Theorem (B.)($V = L$)
If κ is Π^1_γ- but not $\Pi^1_{\gamma+1}$-indescribable then there is an $\gamma + 1$-stationary set $E \subseteq \kappa$ and $\square^\gamma$ sequence avoiding E. Thus κ is not $\gamma + 1$-reflecting.