

A Generalisation of Closed Unbounded and Stationary Sets

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uncountable cofinality \rightarrow clubs generate a filter (measure)
- ▶ a stationary set
defined at ordinals of uncountable cofinality only

A sketch

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C is *1-club* in κ iff C is stationary in κ and stationary-closed.

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- 3 C is γ -club in κ if C is γ -stationary closed and γ -stationary in κ .
- 4 κ is γ -s-reflecting if for any γ -stationary $S, T \subseteq \kappa$ there is $\alpha < \kappa$ with S and T both γ -stationary below α .

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Notation

$d_\gamma(A) := \{\alpha : A \text{ is } \gamma\text{-stationary below } \alpha\}$

Restating the Definitions in Terms of d_γ

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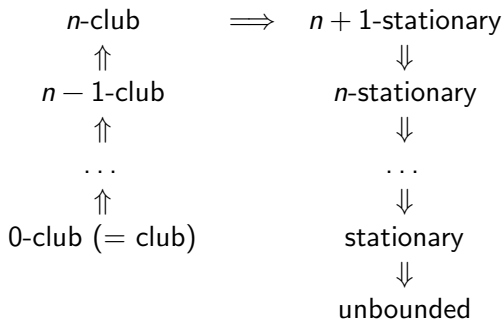
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Definition (restated)

- 1 $S \subseteq On$ is 0-stationary in κ if it is unbounded in κ .
- 2 $C \subseteq On$ is γ -stationary closed if $d_\gamma(C) \subseteq C$.
- 3 C is γ -club in κ if C is γ -stationary closed and γ -stationary below κ .
- 4 κ is γ -reflecting if for any γ -stationary $S, T \subseteq \kappa$,
 $d_\gamma(S) \cap d_\gamma(T) \cap \kappa \neq \emptyset$.
- 5 $S \subseteq \kappa$ is $n+1$ -stationary if κ is n -reflecting and $S \cap C \neq \emptyset$ for every C n -club in κ

how large is a subset of κ ?

If κ is n -reflecting, then for a subset of κ we have these implications:



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- ▶ Hellsten, A. (2003). *Diamonds on large cardinals* (Vol. 134). Suomalainen Tiedeakatemia.
- ▶ L. Beklemishev, D. Gabelaia, (2014) *Topological interpretations of provability logic*, Leo Esakia on duality in modal and intuitionistic logics, Outstanding Contributions to Logic, 4, eds. G. Bezhanishvili, Springer, 257290
- ▶ Bagaria, J., Magidor, M., and Sakai, H. (2015) *Reflection and indescribability in the constructible universe*. Israel Journal of Mathematics 208.1: 1-11.
- ▶ Bagaria, J. (2016). *Derived topologies on ordinals and stationary reflection*. <https://www.newton.ac.uk/files/preprints/ni16031.pdf>

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- ▶ Defining γ -stationary sets in this way is equivalent to defining them in terms of generalised clubs.
 - ▶ This is easy to show by induction: the key is that if κ is γ -stationary and $T \subseteq \kappa$ is η -stationary for $\eta < \gamma$, then $d_\eta(T)$ is η -club.

Definition 3: Topologies

Let Ω be an ordinal and \mathcal{T} a topology on Ω . For $A \subseteq \Omega$ we set

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- ▶ These topologies are closely related to γ -stationarity.

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$$\alpha \in d_{\mathcal{T}_1}(A)$$

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$$\Leftrightarrow \forall C, D \text{ club in } \alpha \text{ we have } C \cap D \cap A \neq \emptyset$$

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In fact we can show that for any γ , $d_{\mathcal{T}_\gamma} = d_\gamma$. Thus a point α is *non-isolated* in \mathcal{T}_γ iff for every $\gamma' < \gamma$, α is γ' -s-reflecting (i.e. α is γ -stationary), and \mathcal{T}_γ is non-discrete iff there is an ordinal $\alpha < \Omega$ that is γ stationary.

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Theorem (B., Bagaria)

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A regular cardinal that is 1-s-reflecting is Π_1^1 -indescribable in L . Thus the existence of a 1-s-reflecting cardinal is equiconsistent with the existence of a Π_1^1 -indescribable.

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Theorem (B.)

Let κ be a regular cardinal that is γ -s-reflecting such that the γ -club filter on κ is normal, and for “many” cardinals λ below κ we have λ is η -s-reflecting implies the η -club filter on λ is normal. Then κ is Π_γ^1 -indescribable in L .

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Conjecture:

For $\gamma > 1$ the consistency strength of a γ -s-reflecting cardinal is below that of a Π_γ^1 -indescribable.

Generalised \square sequences

Definition

A \square^γ sequence on κ is a sequence $\langle C_\alpha : \alpha \in d_\gamma(\kappa) \rangle$ such that for each α :

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Theorem (B.)($V = L$)

If κ is Π^1_γ - but not $\Pi^1_{\gamma+1}$ -indescribable then there is an $\gamma + 1$ -stationary set $E \subseteq \kappa$ and \square^γ sequence avoiding E . Thus κ is not $\gamma + 1$ -reflecting.