Maths for Computing

Lecture 2

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Please note

- Last retrieval date on Friday after coursework is due (only 1 day lateness allowed)
- This is due to University regulations

Last time …

Read:
“is contained in”
“is subset of”

- … we learned about

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

- … and how asking for **closure** with respect to certain operations leads from one set to the next.
- We ended learning that $\sqrt{2}$ is not a rational number (proof today), the set including those is called **real numbers**, or $\mathbb{R}$.

Last time …

- I wrote things like

$$2 = \frac{a^2}{b^2} \quad \Rightarrow \quad 2b^2 = a^2$$

= sign denotes equality. Do not use in any other sense!

$\Rightarrow$ denotes "implies" i.e. if LHS is true then RHS is true. Does not imply the inverse.

I.e.: Symbols of logic

- I am using symbols of logic:

$\Longrightarrow$ means "implies", for example $a > 5 \Rightarrow a > 3$

In general, the opposite need not be true:

$a > 3 \not\Rightarrow a > 5$

If both directions are true, it's called equivalent:

$a = 3 \;\Leftrightarrow\; 2a = 6$

$\Longleftrightarrow$ means "equivalent"

$\sqrt{2}$ is not a rational number

- I will show a proof on the black board
- It is called “proof by contradiction”
- In a nutshell, it works like this:
  - Assume **Fact A** (we do not know whether it is true)
  - Show **Fact A** $\Rightarrow$ **Fact B** and we know B is false.
  - This is a **contradiction** and proves **Fact A** was false to begin with.

BB: $\sqrt{2} \notin \mathbb{Q}$: Proof by contradiction

- Assume $\sqrt{2}$ is rational, i.e. $\sqrt{2} = a/b$, where *a* and *b* are integers, expressed in lowest terms, i.e. irreducible form.
- Then, $2 = a^2/b^2$, and $a^2 = 2b^2$
- Since b^2 is an integer, a^2 is even.
- Since the square of any odd number is odd, and a^2 is even, then a must be even.
- So we can write $a = 2c$, then $b^2 = 2c^2$, hence b^2 is even and, as before b is even.

BB: Proof continued …

- If both a and b are even, a/b wasn't expressed in lowest terms!
- This is a contradiction.

$\sqrt{2}$ **is not a rational number!**

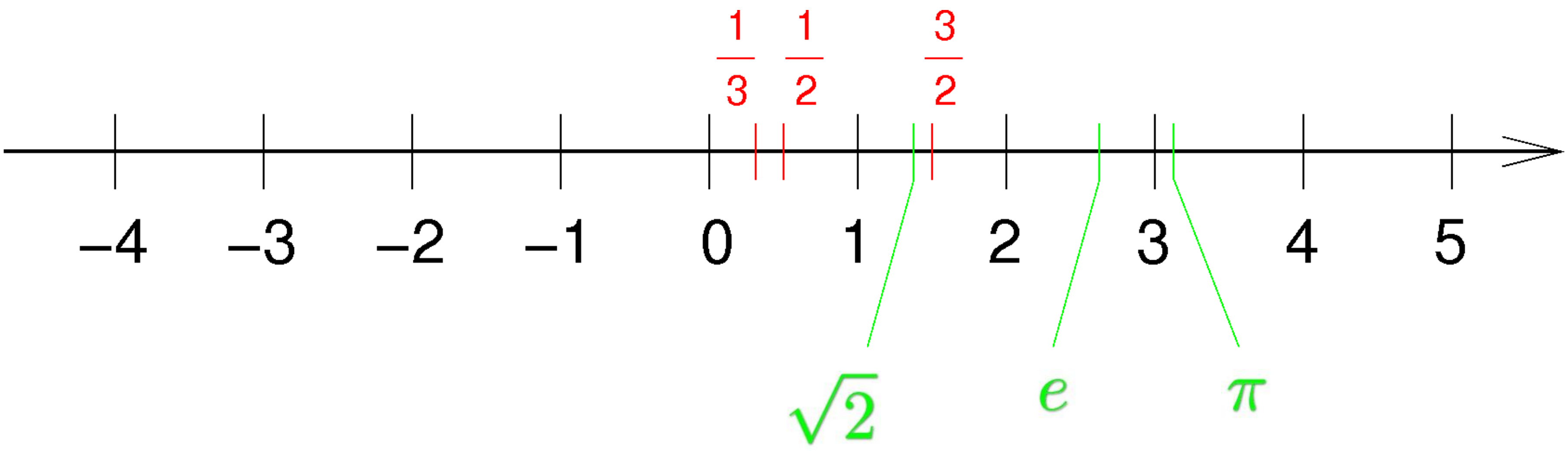
$\mathbb{R}$: More about real numbers

- We can think of $\mathbb{R}$ as those numbers consisting of unending decimal places, e.g., 512.37988032274658 … (no end)
- The real numbers that are not rational are sometimes called “irrational numbers”, $\mathbb{R}\backslash\mathbb{Q}$
- True or false:
  - If the decimal representation is finite, the number is rational? **True**
  - If the decimal representation does not end, the number is irrational? **False**
  - If the number is irrational, the decimal representation does not end? **True**

More about real numbers

- In computers, the distinction between $\mathbb{Q}$ and $\mathbb{R}$ doesn't matter so much. Why not?
- We can visualise $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ on a line BB
- Notation: $\mathbb{R}_+$ are all $x \in \mathbb{R}$ with $x > 0$
- Similar: $\mathbb{Q}_+$
- Why did mathematicians not stop with $\mathbb{R}$?

BB Number line



Putting numbers on the number line is possible because we have the ordering “>” on all numbers in $\mathbb{R}$.

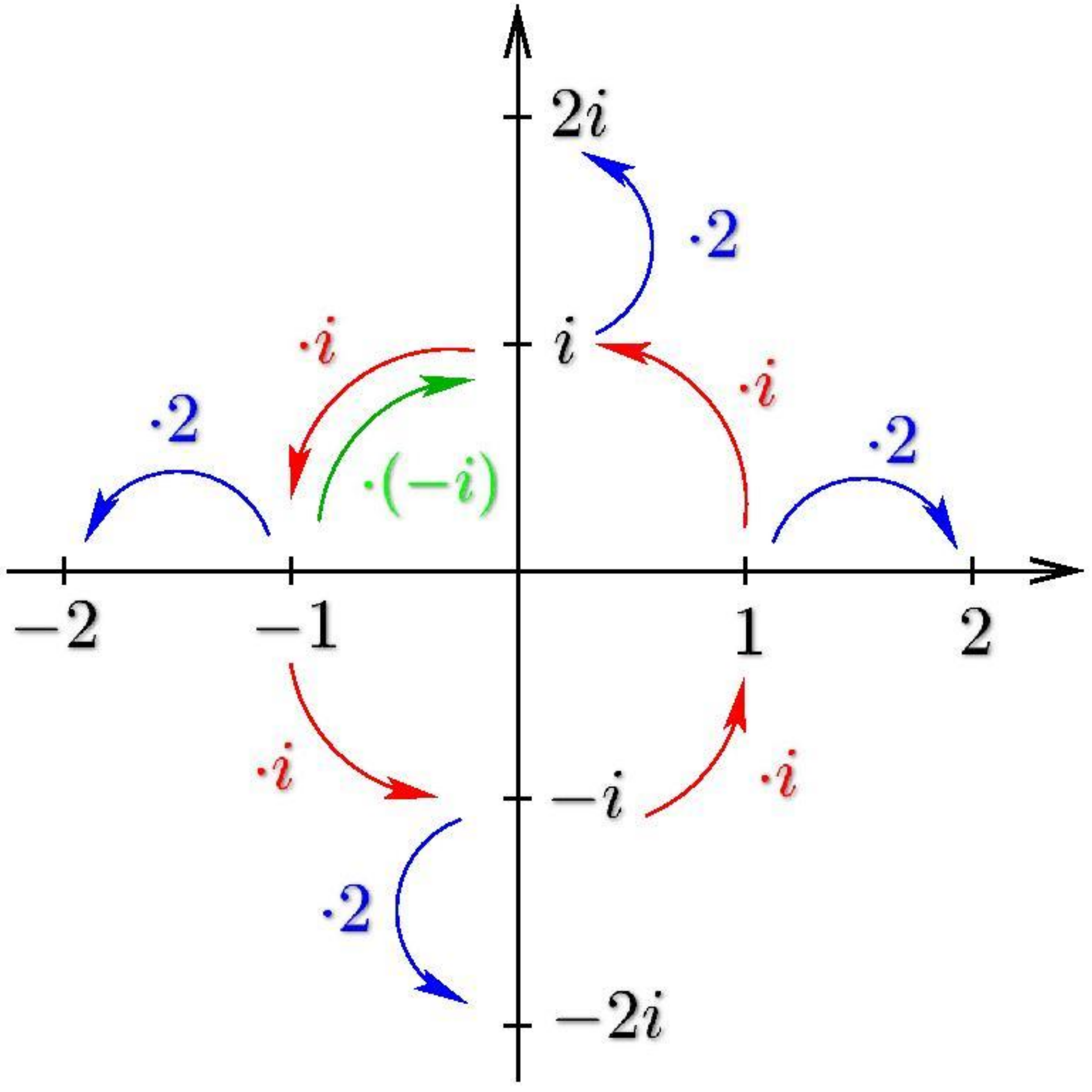
$\mathbb{C}$: The complex numbers

- Squares of real numbers are positive.
  I.e. there is no solution to $x^2 = -1$
- Introduce the "imaginary number" i with:

$$i^2 = -1 \quad \Leftrightarrow \quad i = \sqrt{-1}$$

- Complex numbers cannot be ordered along a line, they require a plane. **BB**
- (Complex numbers are **algebraically** closed. **BG**)

BB Complex plane



(insert your own multiplications and additions etc)

BG Algebraic closure

$\mathbb{C}$ is called **algebraically closed** because all polynomial equations have a solution within $\mathbb{C}$:

$$x^n = y$$

Has a solution $x \in \mathbb{C}$ for all $n \in \mathbb{N}$ and $y \in \mathbb{C}$

(actually, it always has n solutions)

$\mathbb{C}$: The complex numbers

- Every complex number can be written as

$$z = x + iy \qquad x \in \mathbb{R}, \quad y \in \mathbb{R}$$

Real part (x) — **Imaginary part** (y)

- We can operate on complex numbers like usual (BODMAS), with the additional rule $i^2 = -1$, Examples: **BB**

BB Examples of calculating with complex numbers

$$5 + 3i - (3 + i) = 5 - 3 + 3i - i = 2 + 2i$$

$$\begin{aligned}(5 + 3i) \cdot (3 + i) &= 5 \cdot 3 + 5 \cdot i + 3i \cdot 3 + 3i \cdot i \\ &= 15 + 5i + 9i + 3 \cdot (-1) = 12 + 14i\end{aligned}$$

$$\begin{aligned}(x + yi)^2 &= x^2 + x \cdot yi + yi \cdot x + yi \cdot yi \\ &= x^2 + 2xyi - y^2\end{aligned}$$

Intuition test

- How many? $\mathbb{N}_0$
- How much? $\mathbb{R}$
- How long? $\mathbb{R}_+$
- Which percentage? $\mathbb{Q}_+$
- How far? $\mathbb{R}_+$
- How expensive **in pounds**? $\mathbb{Q}_+$
- Account balance **in pennies**? $\mathbb{Z}$

Summary

- There are several **number systems**:

  $\mathbb{N}$ – the **natural** numbers

  $\mathbb{Z}$ – the **integers**

  $\mathbb{Q}$ – the **rational** numbers

  $\mathbb{R}$ – the **real** numbers

  $\mathbb{C}$ – the **complex** numbers
- They contain each other

Read:
"is contained in"
"is subset of"

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Summary

- The different number systems are closed under different operations
- Symbols introduced on the way:

$\in$	is an element of	$\notin$	is not an element of
$\subset$	is a subset of	$\not\subset$	is not a subset of

number systems: $\mathbb{N}$ $\mathbb{Z}$ $\mathbb{Q}$ $\mathbb{R}$ $\mathbb{C}$ $\mathbb{Q}_+$ $\mathbb{R}_+$

i “imaginary unit” defined by $i^2 = -1$

Further reading

- If you are interested how things are represented in the computer, try:

  Truss, J. (1999) Discrete mathematics for computer scientists (2nd edition). Addison-Wesley. Chapter 1.

SUMS AND PRODUCTS

Notations

infix notation: $a + b$ (used in maths)

prefix notation: $+(a, b)$

postfix notation: $(a, b)+$

(sometimes used in computer science)

For sums of many operands (i.e. numbers, variables), there is an additional notation:

Summation notation ($\sum$ notation)

upper limit

Definition: $$\sum_{j=1}^{3} x_j := x_1 + x_2 + x_3$$

summation index

lower limit

Note: Increment always by 1!

It is like a "for" loop:

```
a= 0;
 for ( j=1; j <= 3; j= j + 1) {
    a= a+x_j
 }
```

Some alternative notations **BB**

BB Some alternative notations

$$\sum_{i \in \{1,\dots,N\}} x_i = \sum_{i=1}^{N} x_i$$

$$\sum_{i \in \{1,\dots,N\}, i \neq 2} x_i = \left(\sum_{i=1}^{N} x_i \right) - x_2$$

$$\sum_{\substack{i \in \{1,\dots,N\} \\ j \in \{1,\dots,M\}}} x_i x_j = \sum_{i=1}^{N} \sum_{j=1}^{M} x_i x_j$$

Summation notation

Summation notation is extremely convenient and is used everywhere.

Note: Empty sums are zero:

$$\sum_{k=2}^{1} k^2 = 0$$

Examples: **BB**

BB "Normal" examples

$$\sum_{i=1}^{4} 3 \cdot i = 3 + 6 + 9 + 12$$

$$\sum_{j=3}^{5} \sum_{k=1}^{2} j \cdot k = \sum_{j=3}^{5} (j \cdot 1 + j \cdot 2)$$

$$= 3 \cdot 1 + 3 \cdot 2 + 4 \cdot 1 + 4 \cdot 2 + 5 \cdot 1 + 5 \cdot 2$$

$$= 3 + 6 + 4 + 8 + 5 + 10$$

BB More examples

$$\sum_{j=3}^{5}\sum_{k=1}^{2} j \cdot k = \sum_{k=1}^{2} 3 \cdot k + \sum_{k=1}^{2} 4 \cdot k + \sum_{k=1}^{2} 5 \cdot k$$

$$= 3 \cdot 1 + 3 \cdot 2 + 4 \cdot 1 + 4 \cdot 2 + 5 \cdot 1 + 5 \cdot 2$$

$$= 3 + 6 + 4 + 8 + 5 + 10$$

Note how the order of expansion does not matter!
Reason:

$a + (b + c) = (a + b) + c$ (Associativity)

$a + b = b + a$ (Commutativity)

Product notation

Definition: $$\prod_{j=1}^{5} a_j := a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5$$

Think about a for loop, but multiplication inside, instead of summation.

Example: $$\prod_{j=1}^{5} j = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

Product notation

Product notation is somewhat less common but also still used frequently (and useful)

Note: Empty products are 1:

$$\prod_{i=5}^{1} x_i = 1$$

This makes a lot of sense:

Empty sum

```
a= 0;
for ( j=1; j <= 3; j= j + 1) {
   a= a+x_j
}
```

If the loop is never executed -> sum is 0.

Empty product

```
a= 1;
for ( j=1; j <= 3; j= j + 1) {
   a= a*x_j
}
```

If the loop is never executed -> product is 1.

Why one implies the other: **BB**

BB … and why one implies the other

For $a, b \in \mathbb{R}$

$$a \cdot b = \exp(\log(a \cdot b)) = \exp(\log(a) + \log(b))$$

$$\prod_{i=1}^{n} x_i = \exp\left(\sum_{i=1}^{n} \log(x_i)\right)$$

If the product is empty, then the sum is empty and

$$\prod_{i=1}^{n} x_i = \exp(0) = 1$$

Summary

Sum: $\sum_{j=1}^{3} x_j := x_1 + x_2 + x_3$, empty sum is 0.

Product: $\prod_{j=1}^{5} a_j := a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5$, empty product is 1

Multiple sums or products:

$$\sum_{j=3}^{5} \sum_{k=1}^{2} j \cdot k$$

… order of evaluation does not matter