Mathematical Concepts (G6012)

Lecture 15

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LIMITS

The idea

- Problem: Sequence of numbers, e.g.

$$a_n = \frac{1}{n} \quad \text{and} \quad n = 1, 2, \ldots$$

What happens to $\frac{1}{n}$ if we make *n* ever larger?

Mathematicians write $\lim_{n \to \infty} \frac{1}{n} = 0$

Two *typical* cases:
The numbers go ever larger (the series **diverges**), or (more interesting) it **converges** to a number (here 0).

Formal Definition

A sequence a_n converges to a number $x \in \mathbb{R}$

if for all (small) $\epsilon > 0$ there is a number $N \in \mathbb{N}$

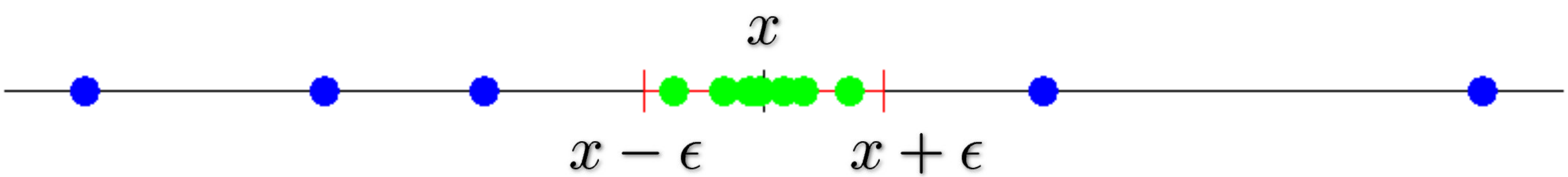
such that $|a_n - x| < \epsilon$ for all $n \geq N$.

Intuition: **BB**

BB Intuition: Convergence

Blue dots: a_n up to some $N \in \mathbb{N}$

Green dots: a_n for all larger n



Examples

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

BB

The limit for the sequence $a_n = n$ for n to infinity does not exist, it grows without bound, people write:

$$\lim_{n\to\infty} n = \infty$$

BB

Claim: $\lim_{n\to\infty} \frac{1}{n} = 0$

Proof: Let $\epsilon > 0$.

We choose $N = \text{floor}\left(\frac{1}{\epsilon}\right) + 1$.

Then, for all $n \geq N$, we know

$$\left|\frac{1}{n} - 0\right| \leq \left|\frac{1}{N}\right| < \left|\frac{1}{\frac{1}{\epsilon}}\right| = \epsilon$$

Practical rules for limits

- If all limits involved exist,

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(k \cdot a_n) = k \cdot \lim_{n\to\infty} a_n$$

“Thumb” rules for calculating limits

(Please don’t show this slide to proper mathematicians)

If $\lim_{x \to \infty} x = \infty$

then $\lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0$

Example

$$f(x) = \exp(-x) = e^{-x}$$

$$\lim_{x \to \infty} \exp(-x) = \lim_{x \to \infty} \frac{1}{\underbrace{\exp(x)}_{\to \infty}} = 0$$

Some more

(Please don't show this slide to proper mathematicians)

$$k \cdot \infty = \begin{cases} \infty & \text{if } k > 0 \\ -\infty & \text{if } k < 0 \end{cases}$$

$$\frac{k}{0} = \begin{cases} \infty & \text{if } k > 0 \\ -\infty & \text{if } k < 0 \end{cases}$$

Another example

(Please don't show this slide to proper mathematicians)

$$\lim_{x \to 0} \exp\left(-\frac{1}{x}\right) =$$

$$\exp\left(-\frac{1}{0}\right) = \exp(-\infty) = \frac{1}{\exp(\infty)} = 0$$

Warning: Not everything converges or diverges (goes to ∞)

What about $\lim_{n \to \infty} \sin(n)$? **It does not exist at all!**

Big O notation: Speed of divergence

- Is used especially in Computer Science
- A function f is $\mathcal{O}(g)$, where g is another function, if $|f(x)| \leq k \cdot |g(x)|$ for $x > N$, $N \in \mathbb{N}$ and some $k \in \mathbb{R}$.
- In other words, if f is **on the order of** g for large x (or less than g …)

Examples

$f(x) = x^2 + 5x + 10$ is $\mathcal{O}(x^2)$

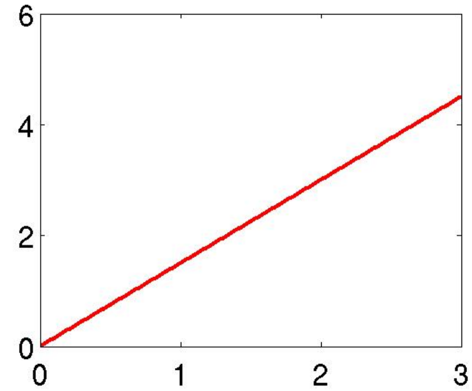
It is also $\mathcal{O}(x^3)$

$x\sqrt{x}$ is $\mathcal{O}(x^2)$

Note: if f is $\mathcal{O}(g)$ and g converges to 0, then f converges to 0 as well!

DERIVATIVES

Slope, Derivative



First for **linear functions**:

$$f(x) = a \cdot x$$

The slope or derivative is the ratio of the change of f(x) and the change of x.

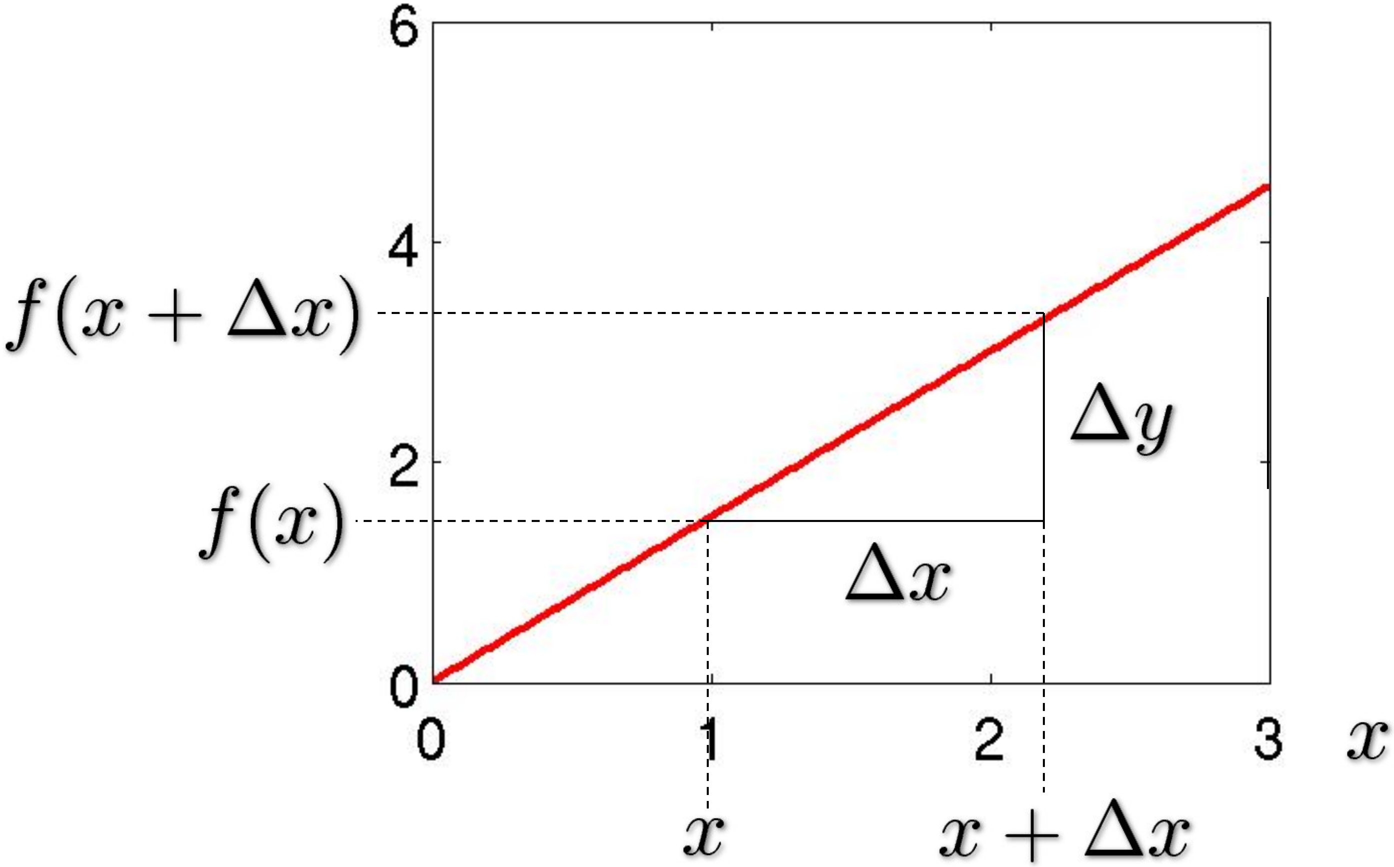
$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

BB

BB

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f(x) = 1.5 \cdot x$$



BB Calculating the linear example

$$f(x) = 1.5 \cdot x$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

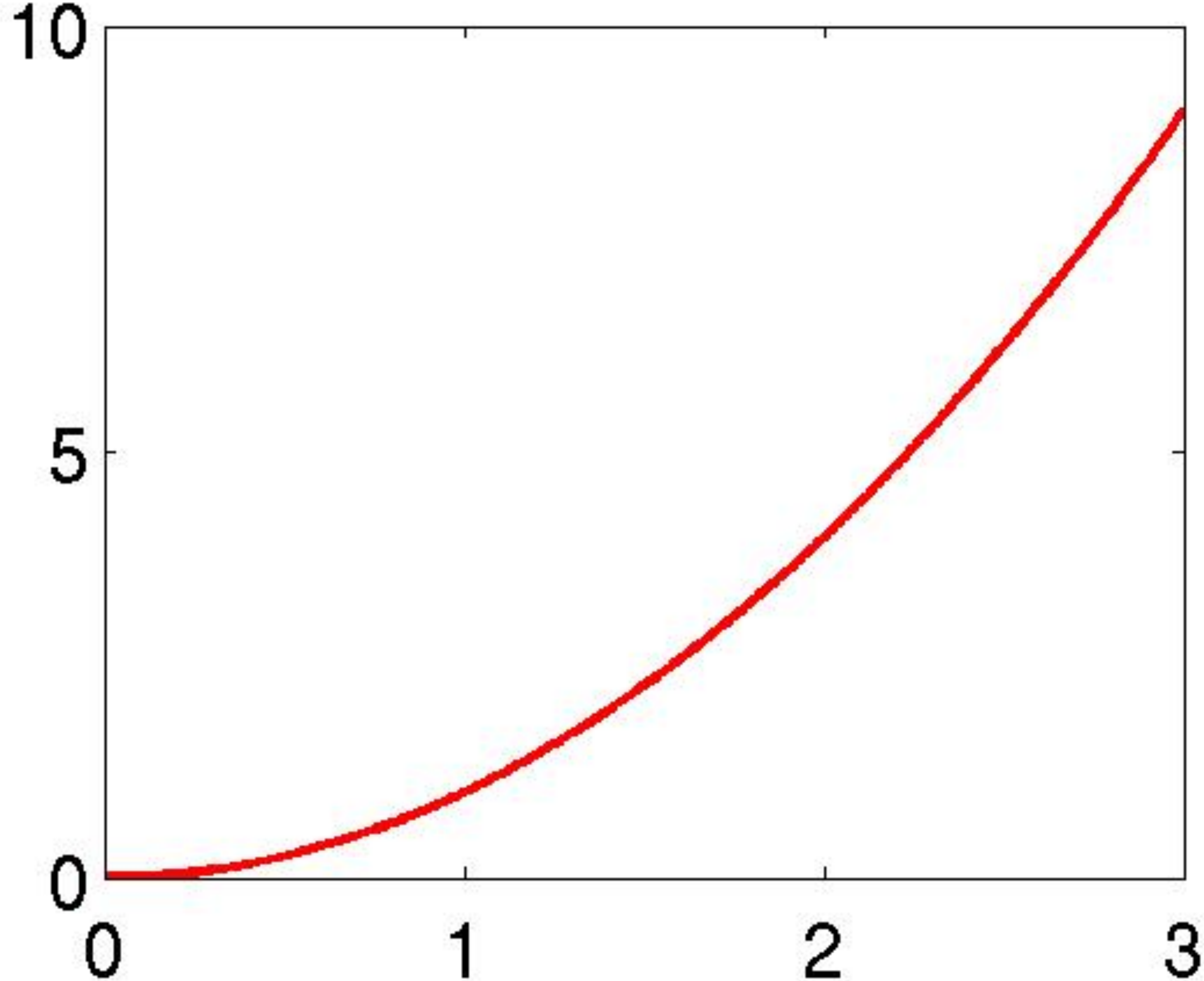
$$= \frac{1.5 \cdot (x + \Delta x) - 1.5 \cdot x}{\Delta x} = 1.5$$

Linear functions are easy …

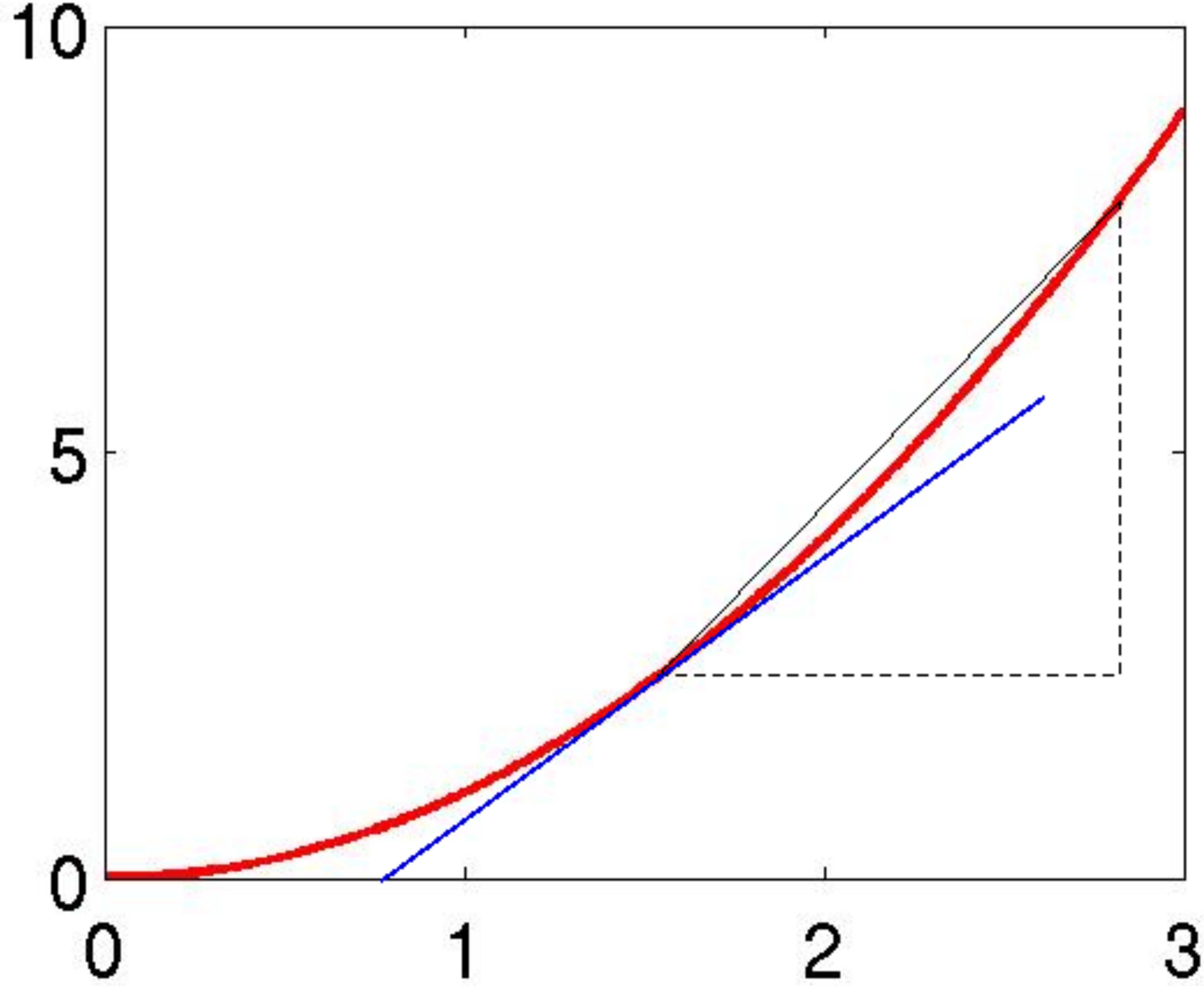
- Because the slope is the same everywhere
- We can make Δx any size we want and get the same value

Generalisation for any smooth function

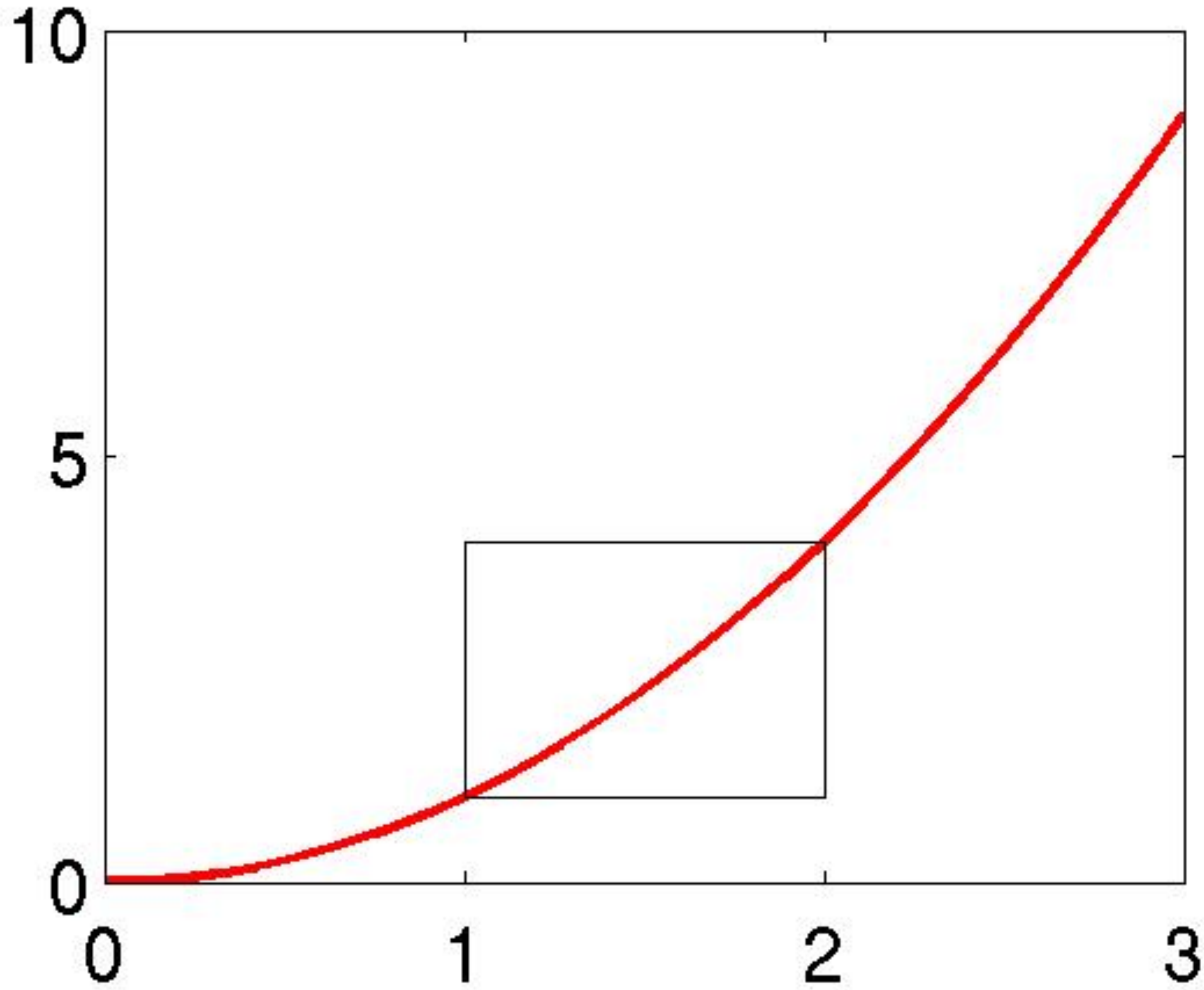
- Locally **any** smooth function looks more and more linear the further we zoom in:



10
5
0
0
1
2
3



10
5
0
0
1
2
3


10
5
0
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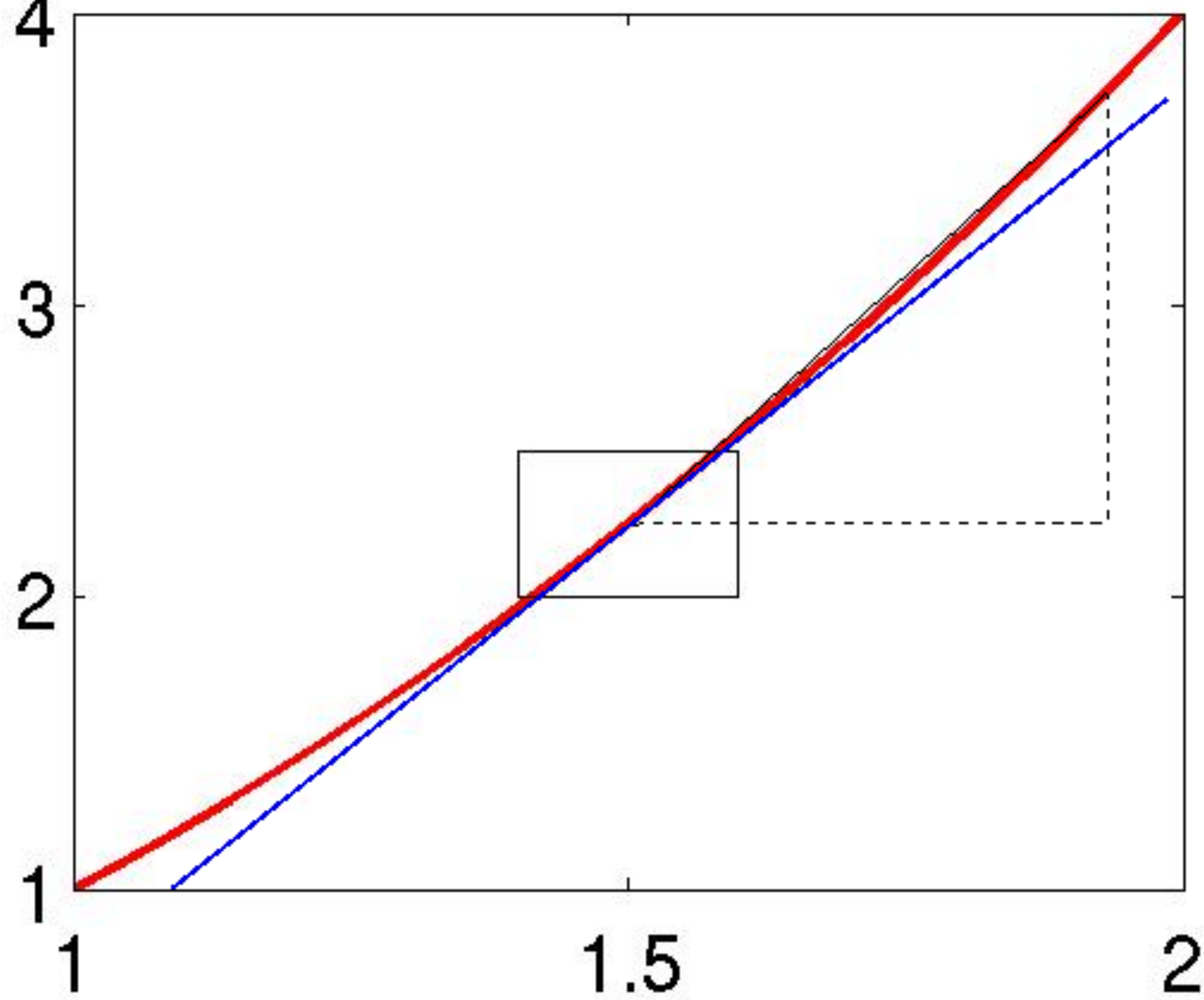
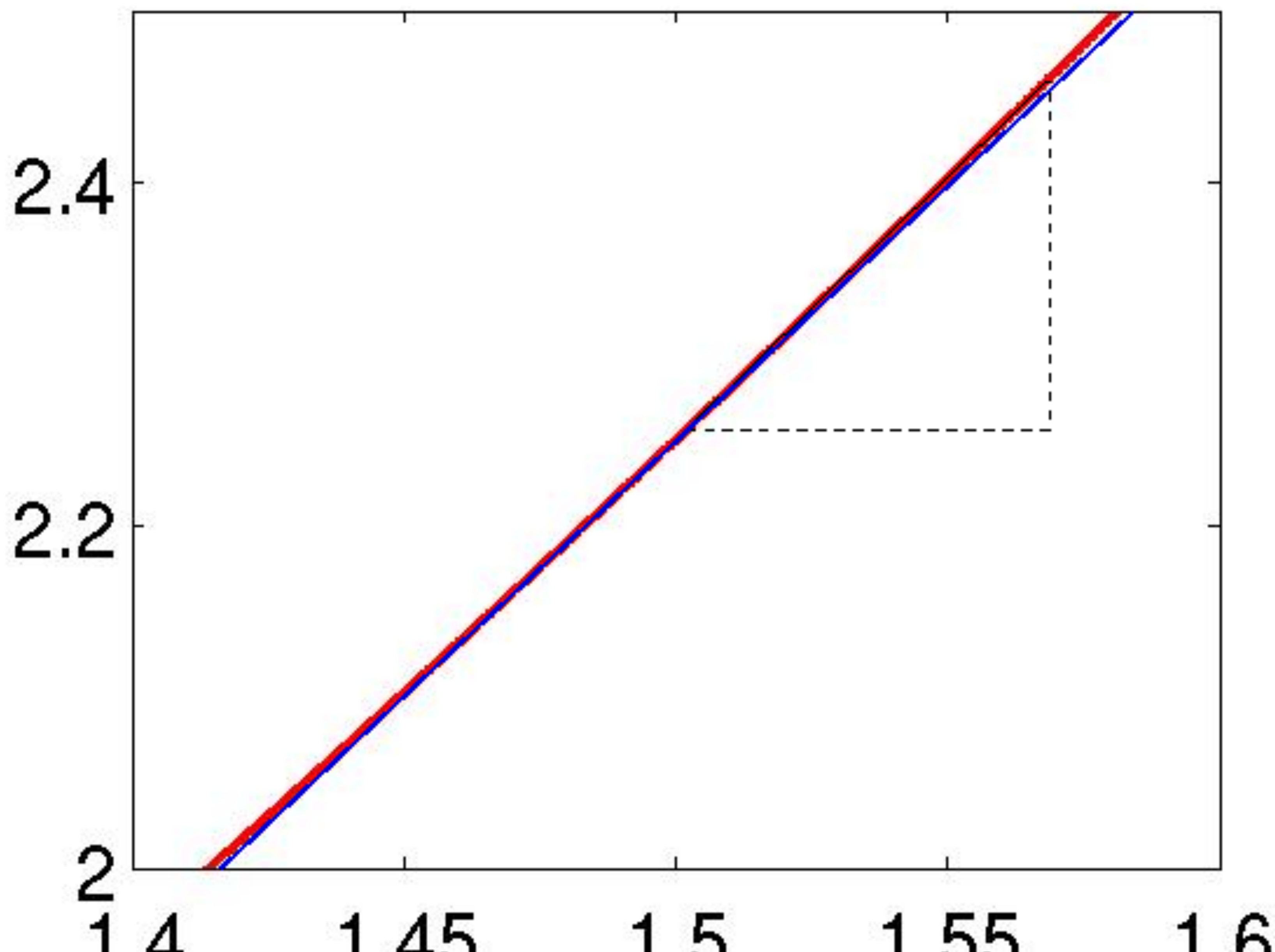
1
2
3


4
3
2
1
1
1.5
2


2.4
2.2
2
1.4
1.45
1.5
1.55
1.6

Derivative of a smooth function

- The derivative of a smooth function is the value the ratio $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ converges to for smaller and smaller Δx , mathematicians write

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

In the form as for the limits before

Sequence $a_n = \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}$

$$f'(x) = \lim_{n\to\infty} \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}$$

Alternative notations

If $f : \mathbb{R} \to \mathbb{R}$ is a smooth function

$$x \mapsto f(x)$$

Then the derivative of f is denoted as

$$f'(x) = \frac{df(x)}{dx} = \frac{df}{dx} = \frac{d}{dx}f$$

Note …

The derivative $f'(x)$ of a function is again a function because we can calculate it for any point x.

BB

Example - Derivative of $f(x) = x^2$

$$\frac{d}{dx}x^2 = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\,\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

Applications

- If *f(x)* is your **distance** from home as a function of the time *x*. Then *f'(x)* is the **speed** you are driving towards (or away from) home.
- If you take the derivative of the derivative *f''(x)*, that would be your **acceleration.**

(These are important for animating things!)

More Applications

- If $f(x)$ describes the **height** of a hill, then $f'(x)$ is the **steepness.**
- $f(x)$ is your **total money** as a function of time, $f'(x)$ is your instantaneous **spending rate.**
- (your example here)

Derivative of a polynomial

- Last time:

$f(x) = ax$ then $f'(x) = \frac{d}{dx}f(x) = a$

- For $f(x) = x^2$ we saw just now $f'(x) = 2x$

- Generally, for $f(x) = x^n$
  the derivative is $f'(x) = nx^{n-1}$

For those interested: General case

$$f(x) = x^n$$

$$\frac{d}{dx}x^n = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\,\Delta x + \mathcal{O}((\Delta x)^2) - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} nx^{n-1} + \mathcal{O}(\Delta x) = nx^{n-1}$$

Derivatives: Basic rules

Rule name	Function	Derivative
Polynomials	$f(x) = x^n$	$f'(x) = n\, x^{n-1}$
Constant factor	$g(x) = a\, f(x)$	$g'(x) = a\, f'(x)$
Sum and Difference	$h(x) = f(x) + g(x)$	$h'(x) = f'(x) + g'(x)$

Examples: Polynomial rule

Example 1

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} \frac{1}{\sqrt{x}}$$

Example 2

$$g(x) = \frac{1}{x^n} = x^{-n}$$

$$g'(x) = -n\, x^{-n-1} = \frac{-n}{x^{n+1}}$$

Special functions

Function	Derivative
$\exp(x) = e^x$	$\exp(x) = e^x$
$\log(x) = \ln(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$