# Mathematical Concepts (G6012)

Lecture 13

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# FUNCTIONS

# (Maths) Functions

A function is like a vending machine:

Elements of the **domain** of the function

The elements of the **co-domain (image)** of the function

# Notation

**domain** of the function

**co-domain** or **image** of the function

function name

$$f : \mathbb{R} \to \mathbb{R}_+$$

**argument** of *f*

$$x \mapsto 5x^2 + 1$$

alternative notation:

$$f(x) = 5x^2 + 1$$

Functions are also often called **map**, **mapping** or **transformation.**

# Examples: Polynomials

$$f : \mathbb{R} \to \mathbb{R}_+$$
$$x \mapsto x^2$$

Alternative notation:

$$f(x) = x^2$$

This means …

$$1 \mapsto 1 \qquad f(1) = 1$$
$$\sqrt{2} \mapsto 2 \qquad f(\sqrt{2}) = 2$$
$$2 \mapsto 4 \qquad f(2) = 4$$
$$-3 \mapsto 9 \qquad f(-3) = 9$$
$$4 \mapsto 16 \qquad f(4) = 16$$
$$5 \mapsto \ldots \qquad f(5) = \ldots$$

# More general

A **polynomial** is a function of the form

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = \sum_{n=0}^{k} a_n x^n$$

**degree** of the polynomial

coefficients

# Example: Matrices (more later)

$A \in \mathcal{M}(m,n)$ i.e. A is a m x n matrix

We have seen before that *A* describes a function (transformation) by:

$$A : \mathbb{R}^n \to \mathbb{R}^m$$

$$\vec{x} \mapsto A \cdot \vec{x}$$

Matrix-vector multiplication

# Examples: “Not” function

$$not : \{0,1\} \rightarrow \{0,1\}$$
$$x \mapsto not(x)$$

| x | not(x) |
|---|---|
| 0 | 1 |
| 1 | 0 |

Alternative notation:

$$0 \mapsto 1$$
$$1 \mapsto 0$$

Alternative 2:

$$not(0) = 1$$
$$not(1) = 0$$

# Examples: “+” function

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

$$(x, y) \mapsto x + y$$

This means:

$$(2, 2) \mapsto 2 + 2 = 4$$

$$(2, -3) \mapsto 2 + (-3) = 2 - 3 = -1$$

Alternative notation:

$$+(1, 4) = 5$$

# Please note …

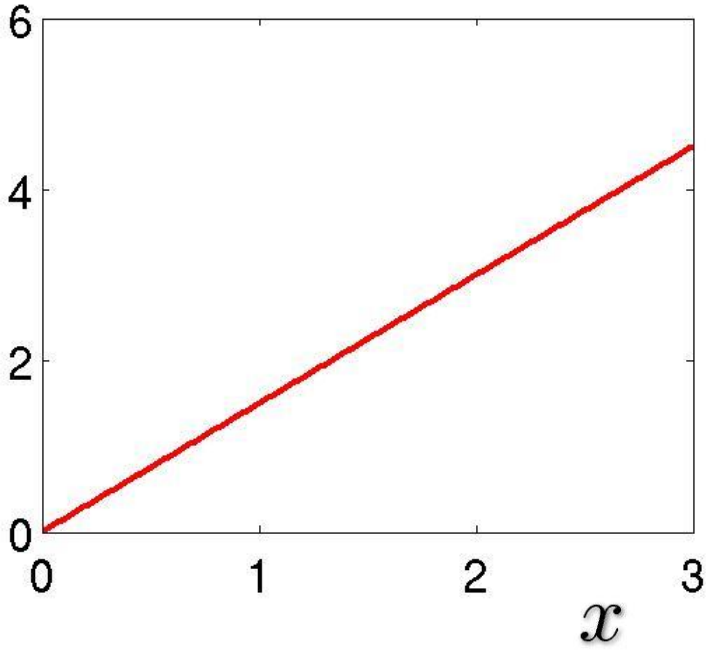
Functions are **single valued,** i.e. for any number *x* from the domain of the function there is **one unique number** $y = f(x)$ from the co-domain.

# BB Graph of a function

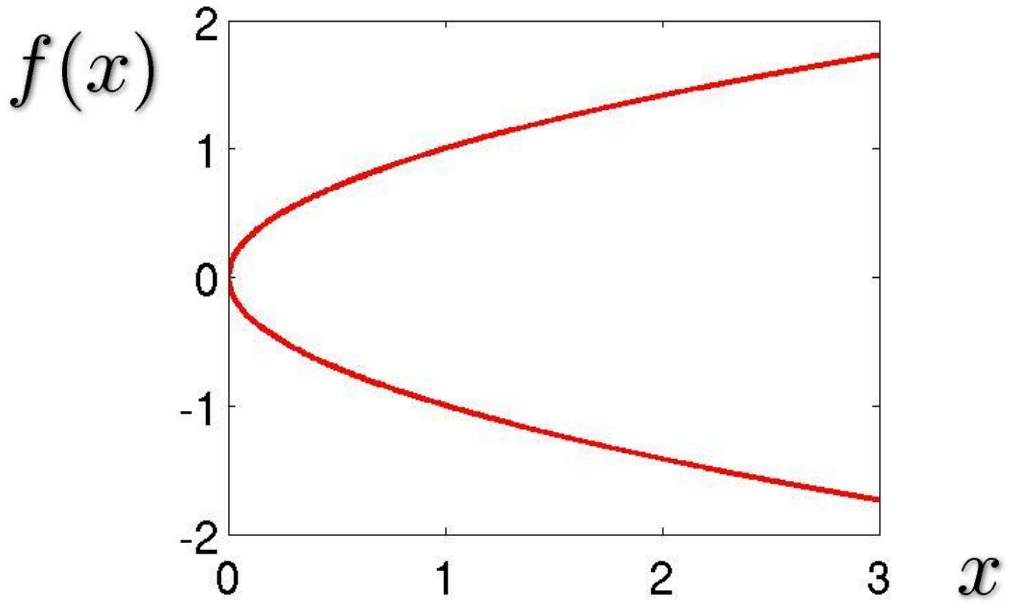
The **graph** of a function are all pairs (points) (x,  f(x) )

$$f(x) = 1.5 \cdot x$$

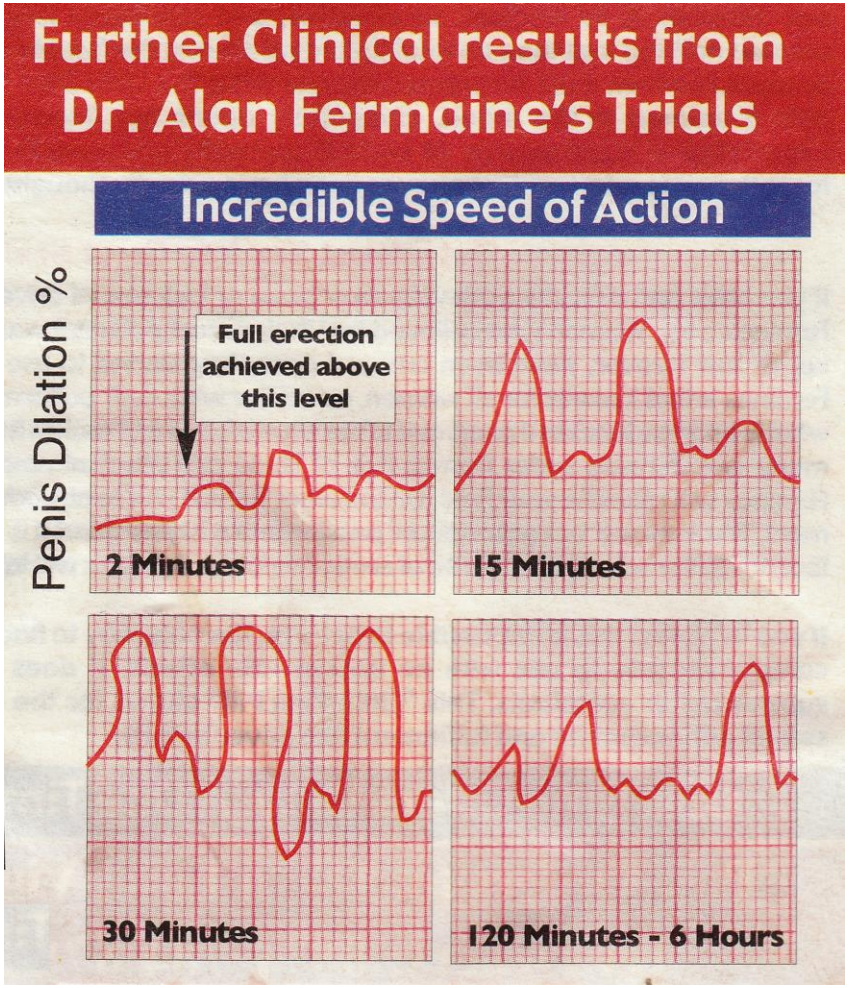
A linear function

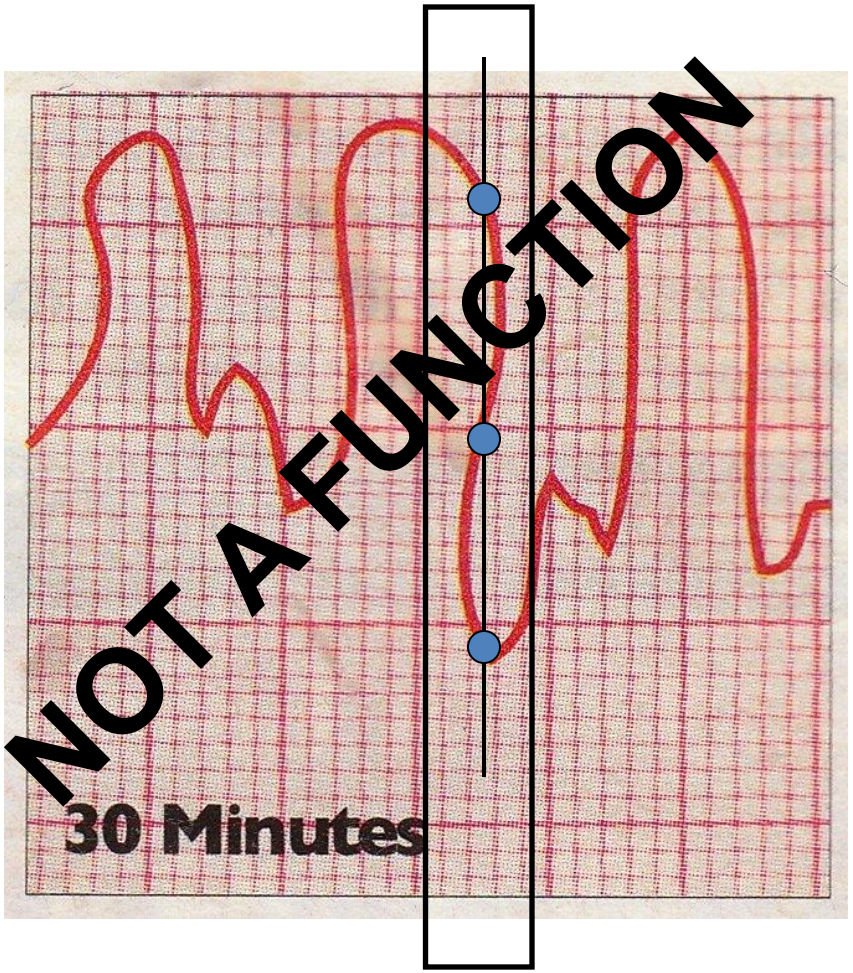


# Warning: Why is this **not** a graph of a function?



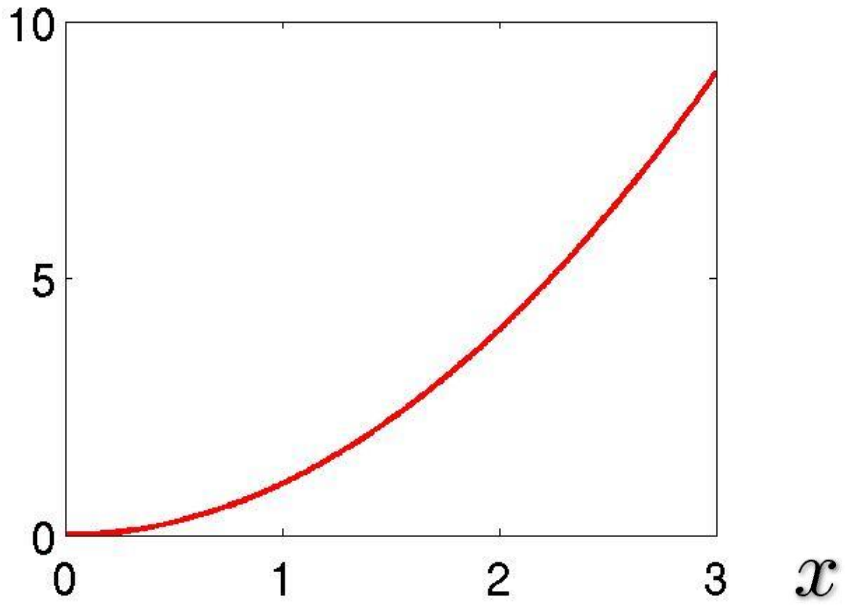
# What about these ones?



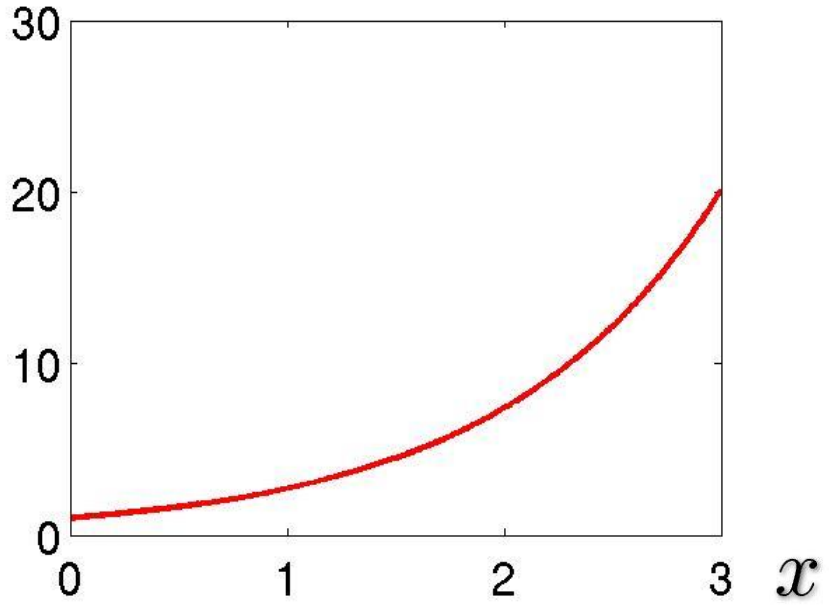
# Some example graphs of functions

$f(x) = x^2$




A parabola

$f(x) = \exp(x) = e^x$

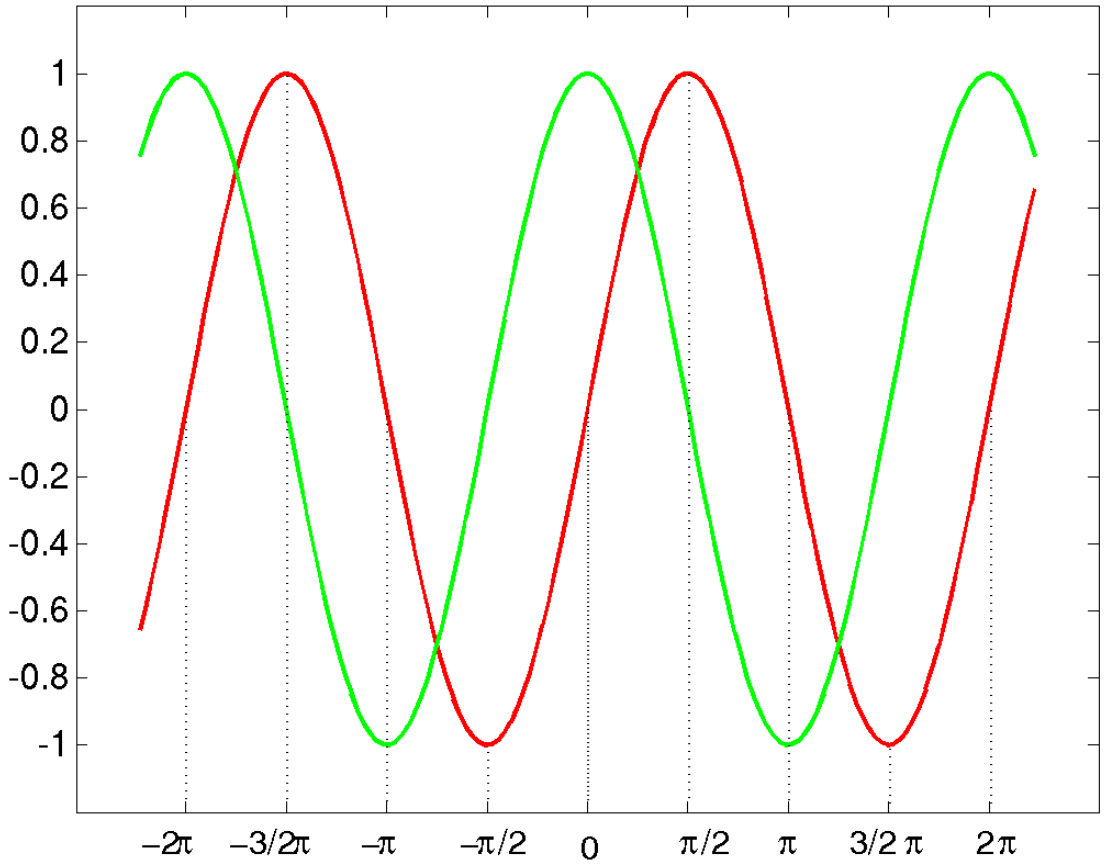



The exponential function

# Other functions you should have seen/know

$$f(x) = \sin(x)$$

$$f(x) = \cos(x)$$



# LIMITS

# The idea

- Problem: Sequence of numbers, e.g.

  $a_n = \frac{1}{n}$ and $n = 1, 2, \ldots$

  What happens to $\frac{1}{n}$ if we make *n* ever larger?

  Mathematicians write $\lim_{n \to \infty} \frac{1}{n} = 0$

  Two *typical* cases:
  The numbers go ever larger (the series **diverges**), or (more interesting) it **converges** to a number (here 0).

# Formal Definition

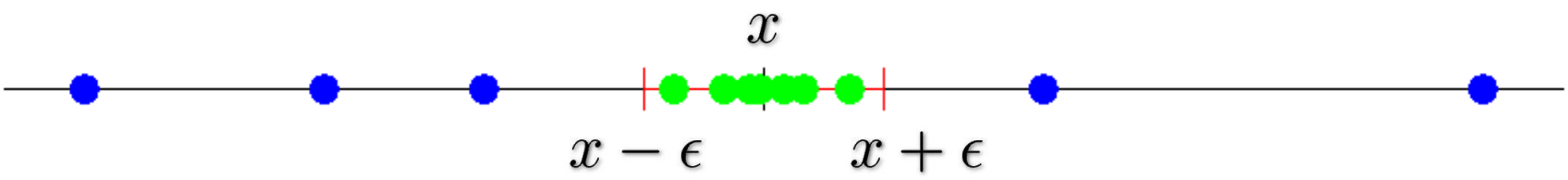
A sequence $a_n$ converges to a number $x \in \mathbb{R}$ if for all (small) $\epsilon > 0$ there is a number $N \in \mathbb{N}$ such that $|a_n - x| < \epsilon$ for all $n \geq N$ .

Intuition: **BB**

# BB Intuition: Convergence

Blue dots: $a_n$ up to some $N \in \mathbb{N}$

Green dots: $a_n$ for all larger $n$



# Examples

$$\lim_{n\to\infty} \frac{1}{n} = 0$$ **BB**

The limit for the sequence $a_n = n$ for $n$ to infinity does not exist, it grows without bound, people write:

$$\lim_{n\to\infty} n = \infty$$

# BB

Claim: $\lim_{n\to\infty} \frac{1}{n} = 0$

Proof: Let $\epsilon > 0$.

We choose $N = \text{floor}\left(\frac{1}{\epsilon}\right) + 1$.

Then, for all $n \geq N$, we know

$$\left|\frac{1}{n} - 0\right| \leq \left|\frac{1}{N}\right| < \left|\frac{1}{\frac{1}{\epsilon}}\right| = \epsilon$$

# Practical rules for limits

- If all limits involved exist,

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(k \cdot a_n) = k \cdot \lim_{n\to\infty} a_n$$

# “Thumb” rules for calculating limits

(Please don’t show this slide to proper mathematicians)

If $\lim_{x \to \infty} x = \infty$

then $\lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0$

# Example

$$f(x) = \exp(-x) = e^{-x}$$

$$\lim_{x\to\infty} \exp(-x) = \lim_{x\to\infty} \frac{1}{\underbrace{\exp(x)}_{\to\infty}} = 0$$

# Some more

(Please don't show this slide to proper mathematicians)

$$k \cdot \infty = \begin{cases} \infty & \text{if } k > 0 \\ -\infty & \text{if } k < 0 \end{cases}$$

$$\frac{k}{0} = \begin{cases} \infty & \text{if } k > 0 \\ -\infty & \text{if } k < 0 \end{cases}$$

# Another example

(Please don't show this slide to proper mathematicians)

$$\lim_{x \to 0} \exp\left(-\frac{1}{x}\right) =$$

$$\exp\left(-\frac{1}{0}\right) = \exp(-\infty) = \frac{1}{\exp(\infty)} = 0$$

# Warning: Not everything converges or diverges (goes to $\infty$ )

What about $\lim_{n\to\infty} \sin(n)$ ? **It does not exist at all!**

# Big O notation: Speed of divergence

- Is used especially in Computer Science
- A function $f$ is $\mathcal{O}(g)$, where $g$ is another function, if $|f(x)| \leq k \cdot |g(x)|$ for $x \to \infty$ and some $k \in \mathbb{R}$ .
- In other words, if $f$ is **on the order of** $g$ for large $x$ (or less than $g$ …)

# Examples

$f(x) = x^2 + 5x + 10$ is $\mathcal{O}(x^2)$

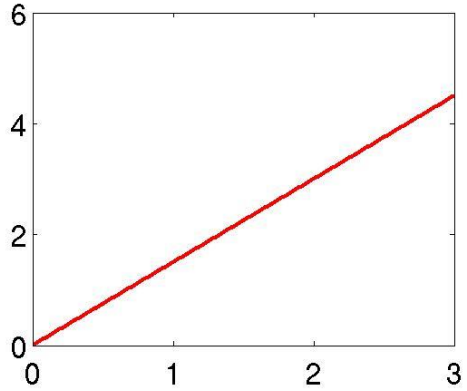
It is also $\mathcal{O}(x^3)$

$x\sqrt{x}$ is $\mathcal{O}(x^2)$

Note: if $f$ is $\mathcal{O}(g)$ and $g$ converges to 0, then $f$ converges to 0 as well!

# DERIVATIVES

# Slope, Derivative




First for **linear functions**:

$$f(x) = a \cdot x$$

The slope or derivative is the ratio of the change of f(x) and the change of x.

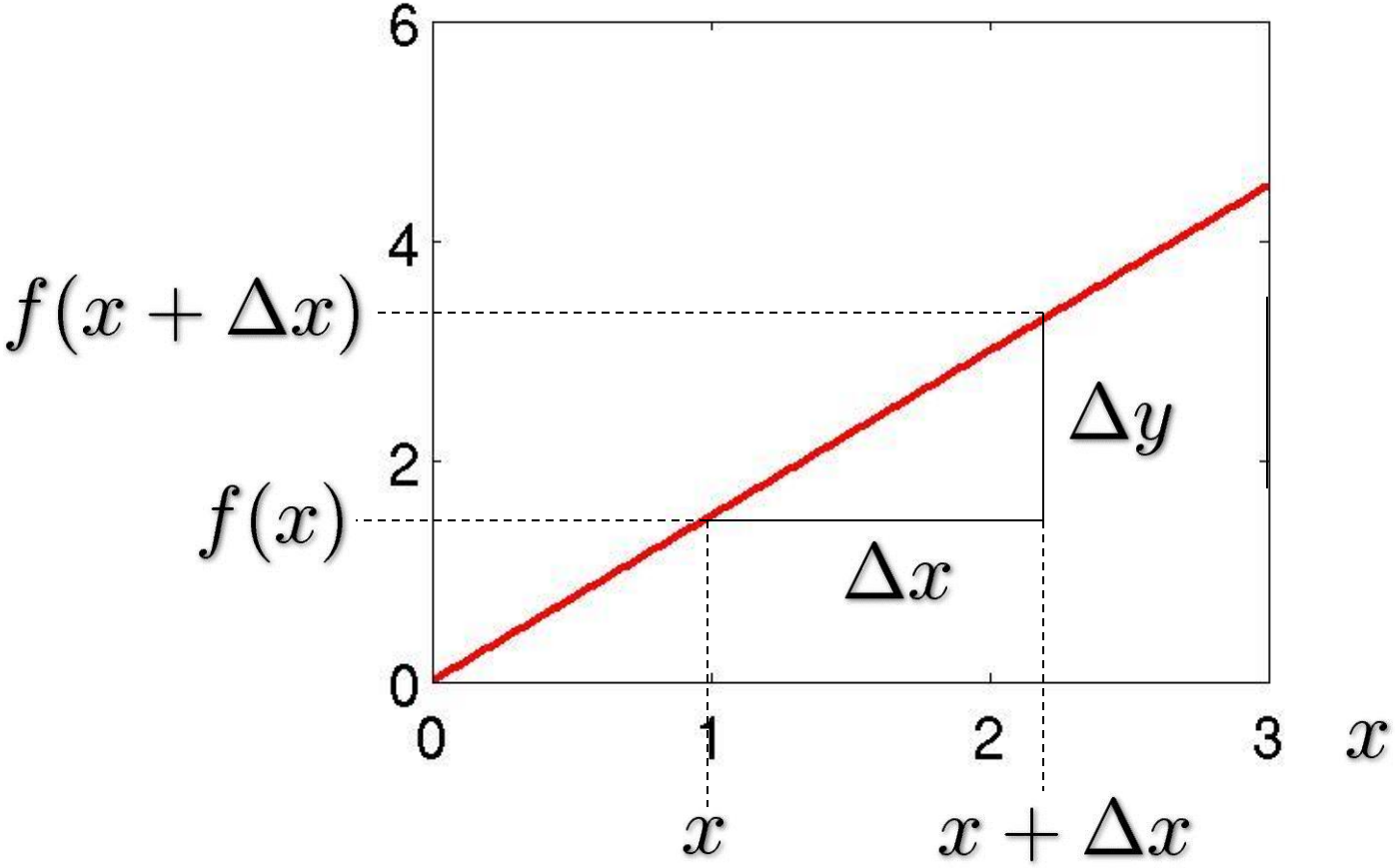
$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

**BB**

**BB**

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f(x) = 1.5 \cdot x$$



# BB Calculating the linear example

$$f(x) = 1.5 \cdot x$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

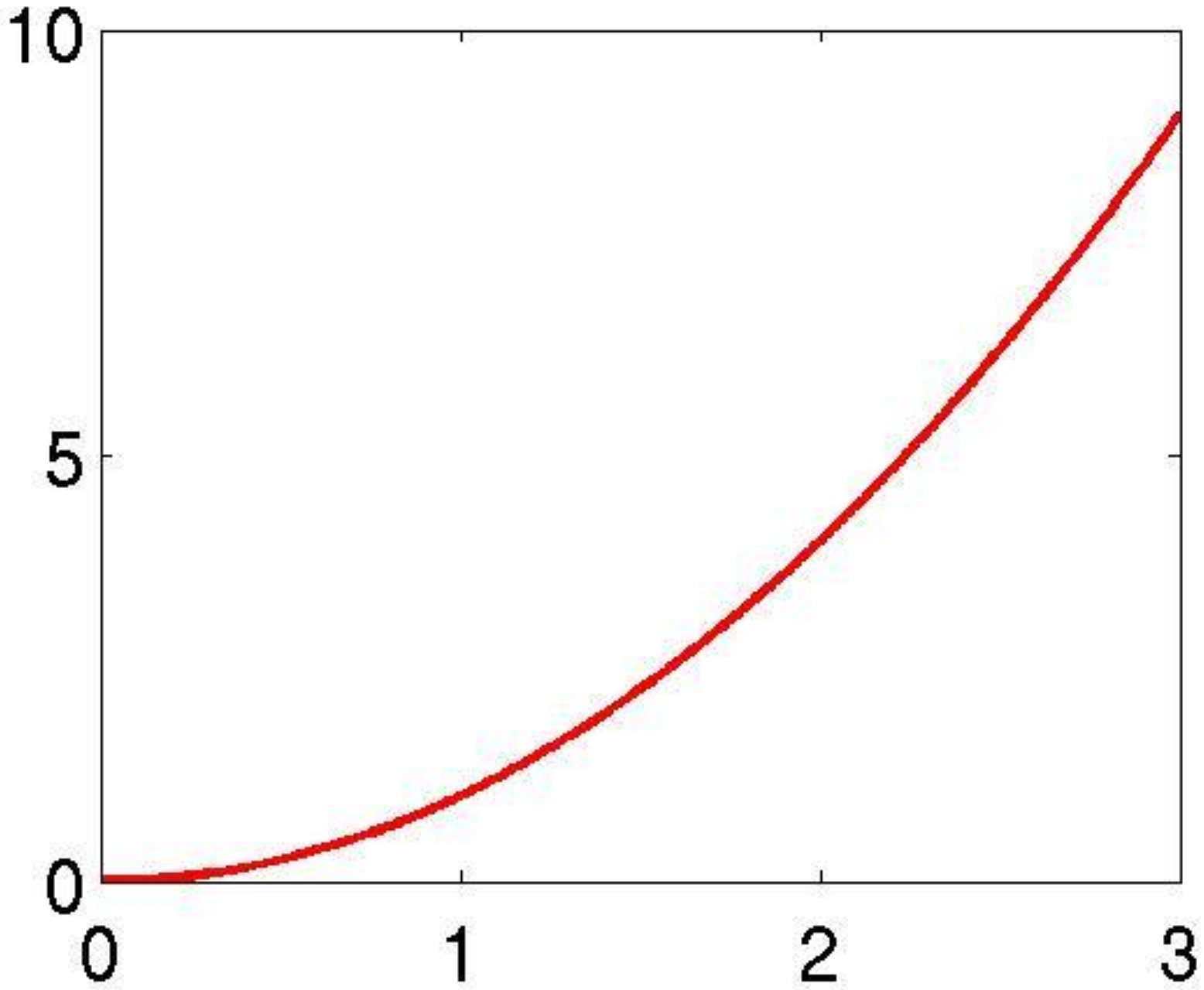
$$= \frac{1.5 \cdot (x + \Delta x) - 1.5 \cdot x}{\Delta x} = 1.5$$

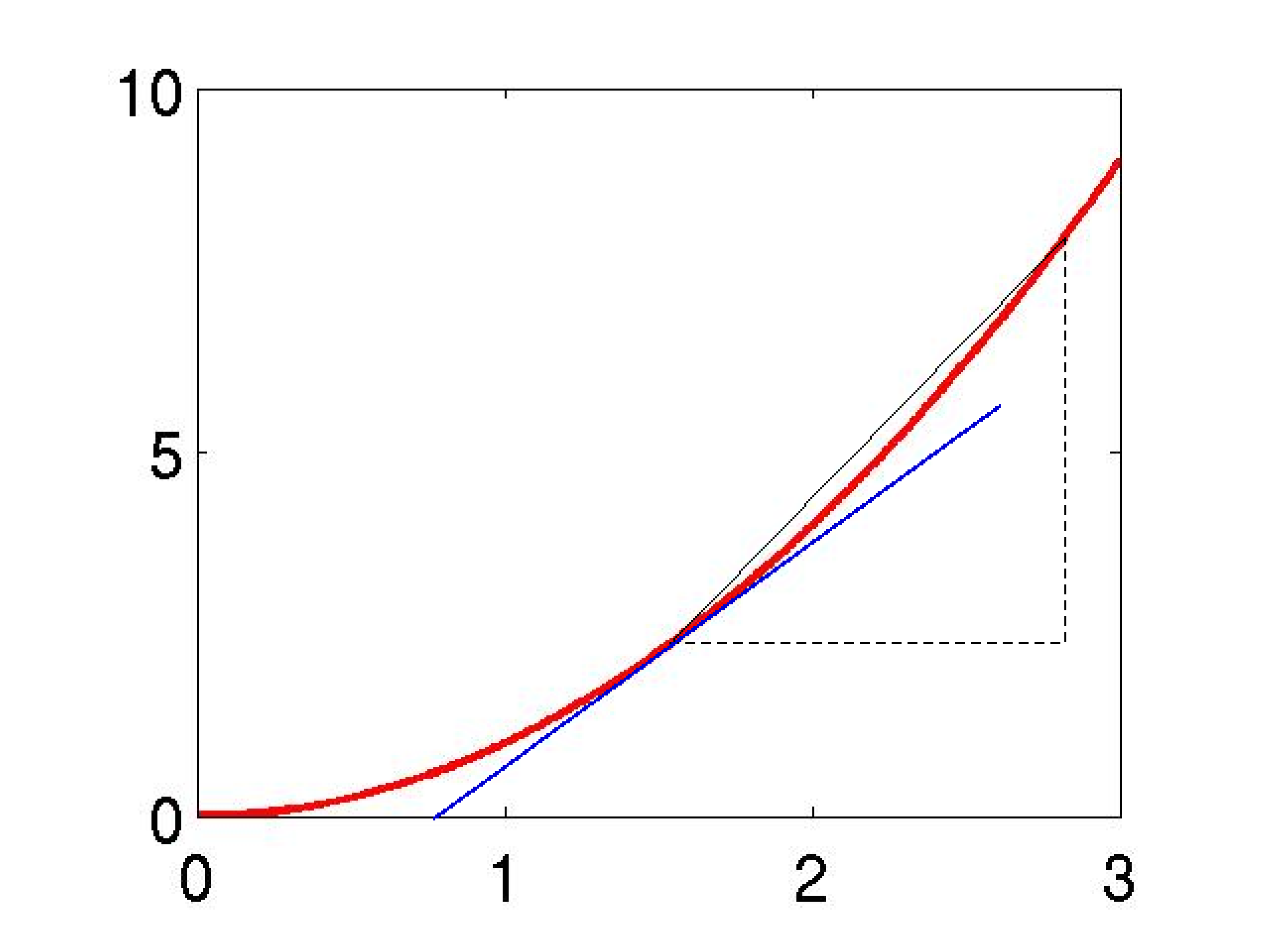
# Linear functions are easy …

- Because the slope is the same everywhere $\Delta x$
- We can make any size we want and get the same value

# Generalisation for any smooth function

- Locally **any** smooth function looks more and more linear the further we zoom in:


10
5
0
0
1
2
3


10
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0
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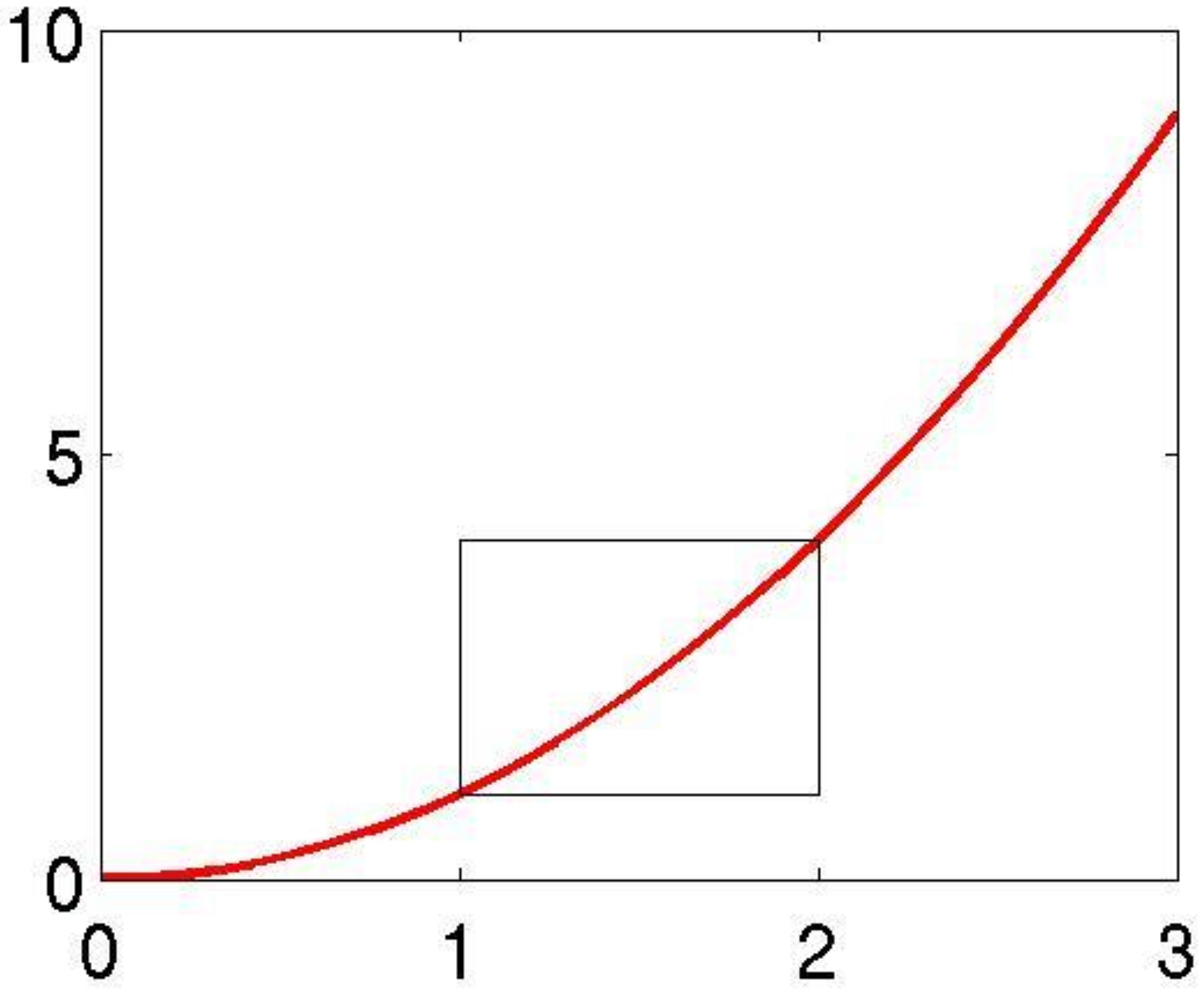
1
2
3


10
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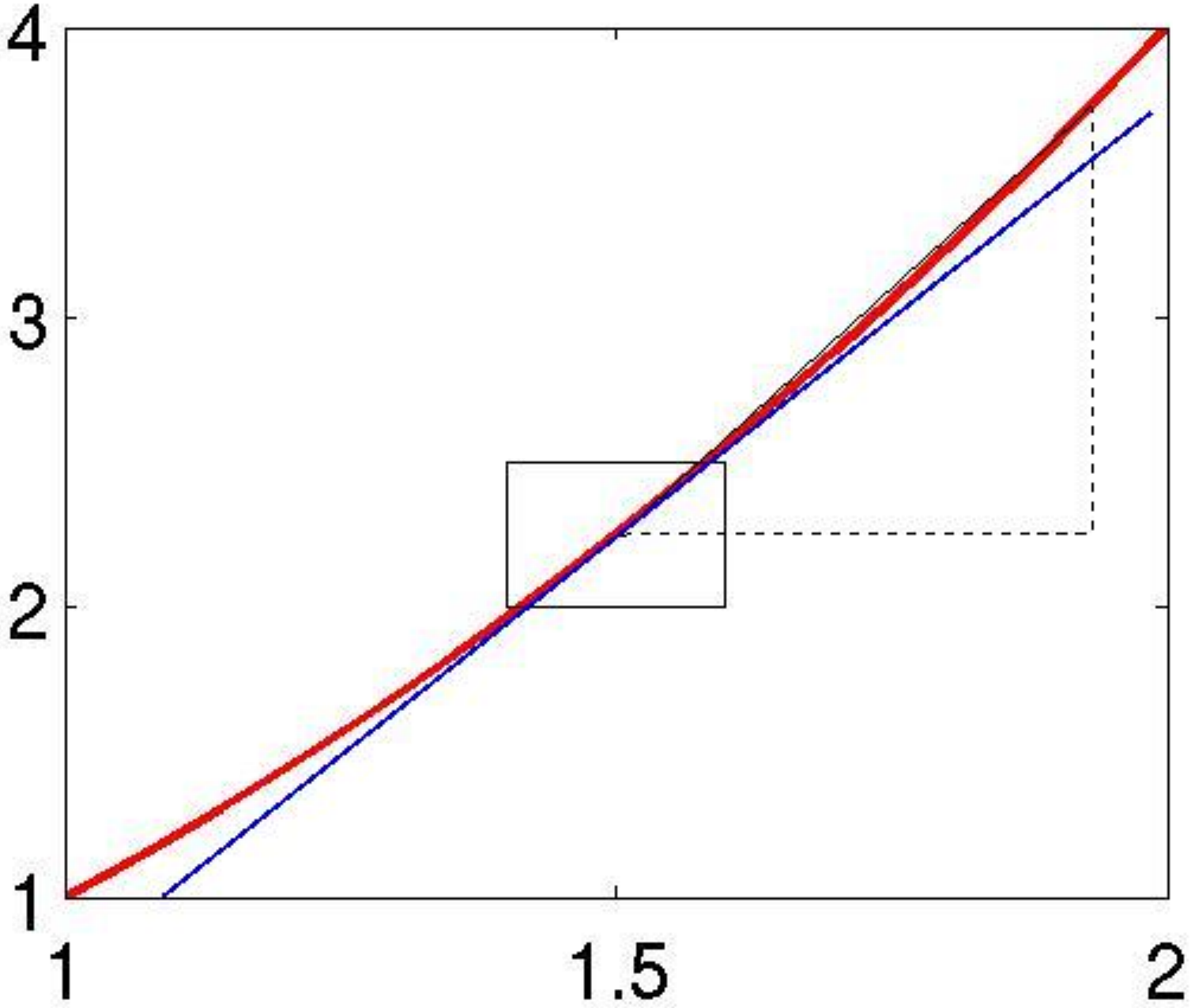
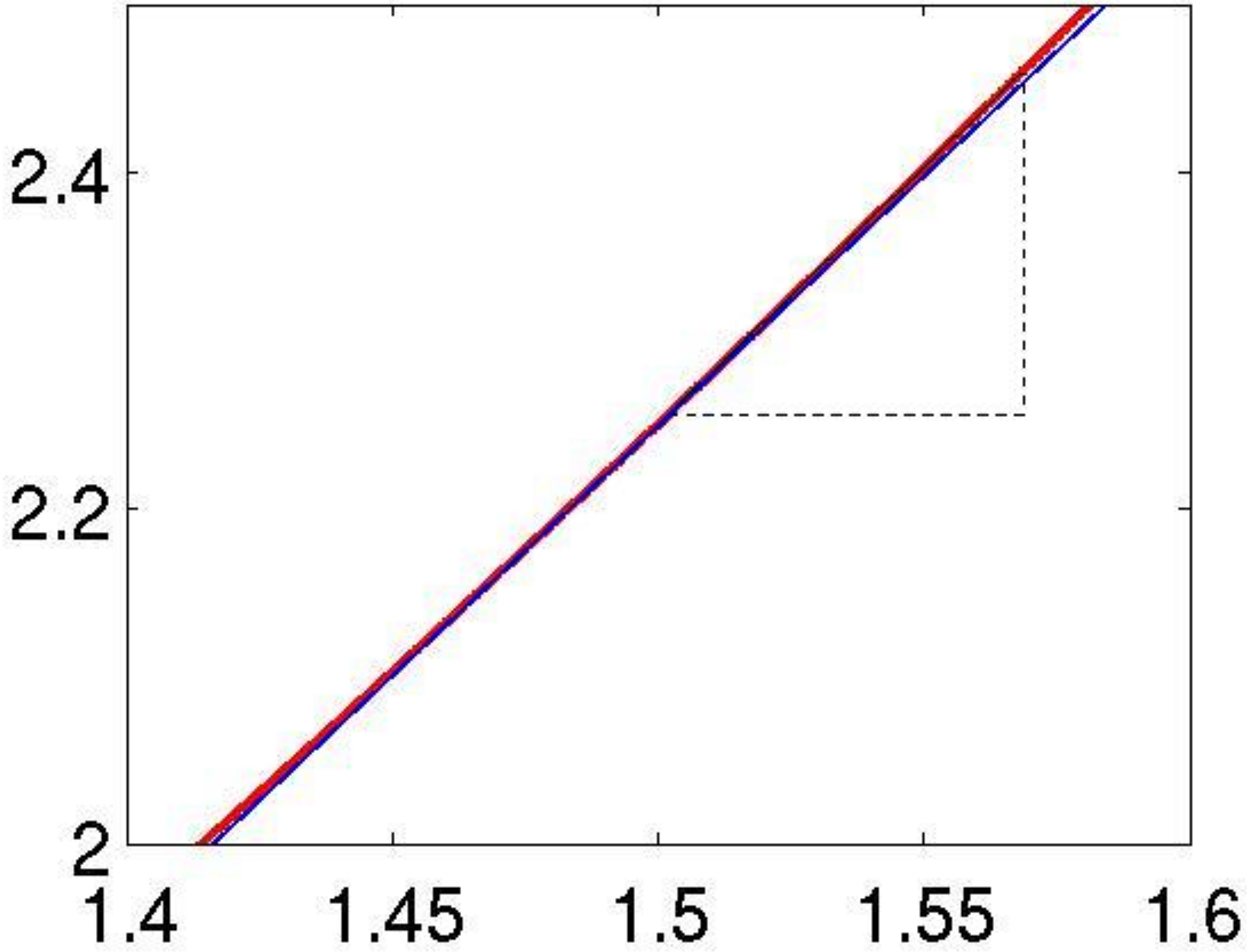

4
3
2
1
1
1.5
2


2.4
2.2
2
1.4
1.45
1.5
1.55
1.6

# Derivative of a smooth function

- The derivative of a smooth function is the value the ratio $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

  converges to for smaller and smaller $\Delta x$ , mathematicians write

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

# In the form as for the limits before

Sequence $a_n = \dfrac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}$

$$f'(x) = \lim_{n\to\infty} \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}$$

# Alternative notations

If $f : \mathbb{R} \to \mathbb{R}$ is a smooth function

$$x \mapsto f(x)$$

Then the derivative of $f$ is denoted as

$$f'(x) = \frac{df(x)}{dx} = \frac{df}{dx} = \frac{d}{dx} f$$

# Note …

The derivative *f'(x)* of a function is again a function because we can calculate it for any point x.

**BB**

# Example - Derivative of $f(x) = x^2$

$$\frac{d}{dx}x^2 = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\,\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

# Applications

- If *f(x)* is your **distance** from home as a function of the time *x*. Then *f'(x)* is the **speed** you are driving towards (or away from) home.
- If you take the derivative of the derivative *f''(x)*, that would be your **acceleration.**

(These are important for animating things!)

# More Applications

- If *f(x)* describes the **height** of a hill, then *f'(x)* is the **steepness.**
- *f(x)* is your **total money** as a function of time, *f'(x)* is your instantaneous **spending rate.**
- (your example here)

# Derivative of a polynomial

- Last time:

  $f(x) = ax$ then $f'(x) = \frac{d}{dx} f(x) = a$

- For $f(x) = x^2$ we saw just now $f'(x) = 2x$

- Generally, for $f(x) = x^n$
  the derivative is $f'(x) = nx^{n-1}$

# For those interested: General case

$$f(x) = x^n$$

$$\frac{d}{dx}x^n = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\,\Delta x + \mathcal{O}((\Delta x)^2) - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} nx^{n-1} + \mathcal{O}(\Delta x) = nx^{n-1}$$

# Derivatives: Basic rules

| Rule name | Function | Derivative |
|---|---|---|
| Polynomials | $f(x) = x^n$ | $f'(x) = n\,x^{n-1}$ |
| Constant factor | $g(x) = a\,f(x)$ | $g'(x) = a\,f'(x)$ |
| Sum and Difference | $h(x) = f(x) + g(x)$ | $h'(x) = f'(x) + g'(x)$ |

# Examples: Polynomial rule

Example 1

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \qquad f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{x}}$$

Example 2

$$g(x) = \frac{1}{x^n} = x^{-n} \qquad g'(x) = -n\,x^{-n-1} = \frac{-n}{x^{n+1}}$$

# Special functions

| Function | Derivative |
| --- | --- |
| $\exp(x) = e^x$ | $\exp(x) = e^x$ |
| $\log(x) = \ln(x)$ | $\frac{1}{x}$ |
| $\sin(x)$ | $\cos(x)$ |
| $\cos(x)$ | $-\sin(x)$ |