Mathematical Concepts (G6012)

Lecture 11

Thomas Nowotny
Chichester I, Room CI-105
Office hours: Tuesdays 15:00-16:45
T.Nowotny@sussex.ac.uk

VECTORS AND MATRICES

Why matrix algebra?

- Multimedia/Design/Art: Computer graphics are 90% vectors and matrices
- AI: Artificial Neural Networks heavily depend on vectors and matrices.
- Music: Discretised sound spectra are vectors; digital filtering & enhancement depend on matrices; modern compression (mp3 etc) is one of the most maths-heavy problems in Informatics

VECTORS AND MATRICES

Vector Notations

$$\begin{pmatrix} -1 \\ 3 \\ 9 \end{pmatrix} \in \mathbb{R}^3$$

column vector

arrow, underline

$$\vec{x} = \mathbf{x} = \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$$

bold

$$x_i \in \mathbb{R}$$

"component"

index

$$\begin{pmatrix} -5 & 3 & 1.1 \end{pmatrix} \in \mathbb{R}^3$$

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \in \mathbb{R}^3$$

Row vector

Similar for all $\mathbb{R}^n$

Matrices

$$\begin{pmatrix} -1 & 5 & -4 \\ 9.1 & 3 & -4.5 \\ 7 & 0.1 & \sqrt{2} \end{pmatrix} \in \mathcal{M}(3,3)$$ is a 3x3 matrix.

capital letter

"entry", "element"

$$A = \underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (a_{ij}) \in \mathcal{M}(3,3)$$

double underscore

index: row first, column second

Adding and subtracting matrices

- Same as for vectors …

  BB

- Interpretation not so direct: Operations on vectors – next time.

BB Example: Subtractring a matrix from an other matrix

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & -2 \\ -2 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 3 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-1 & -2-0 & 3-(-1) \\ 0-0 & 4-2 & -2-2 \\ -2-3 & 2-1 & 1-(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 & 4 \\ 0 & 2 & -4 \\ -5 & 1 & 2 \end{pmatrix}$$

Properties of +, -

$$A, B, C \in \mathcal{M}(m, n)$$

Associativity

$$A + B + C = (A + B) + C = A + (B + C)$$

Commutativity $A + B = B + A$

Properties of scalar multiplication

$$A, B, C \in \mathcal{M}(m,n) \qquad r, s \in \mathbb{R}$$

Compatibility with scalar operations (multiplying with numbers)

$$r \cdot (A + B) = r \cdot A + r \cdot B$$

$$(r + s) \cdot A = r \cdot A + s \cdot A$$

Matrix-Vector Multiplication

$A \in$ M(3,3) and $\vec{x} \in \mathbb{R}^3$

$$A \cdot \vec{x} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$$

BB

BB Matrix-vector multiplication

$$A\vec{x} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

The result is again a vector!

BB Matrix-vector multiplication

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$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

The result is again a vector!

Properties of Matrix-Vector Multiplication

Linear (both ways)

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$$

$$(A + B)\vec{x} = A\vec{x} + B\vec{x}$$

Associative:

$$(A \cdot B)\vec{x} = A(B\vec{x})$$

Interpretation

Matrices are transformations (linear functions)

$$A = \underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \quad \text{M(3,3)}$$

$$A : \mathbb{R}^3 \to \mathbb{R}^3$$

A maps vectors from $\mathbb{R}^3$ to vectors in $\mathbb{R}^3$

$$\vec{x} \mapsto A \cdot \vec{x}$$

$$A\,\vec{x}$$

Matrix- vector multiplication

Matrix as a transformation: Can we see what it does?

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

BB

BB Matrix as a transformation

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

Column of the
matrix are
the images of the
basis vectors!

Matrix as a transformation

The columns of the matrix are the vectors
the basis vectors

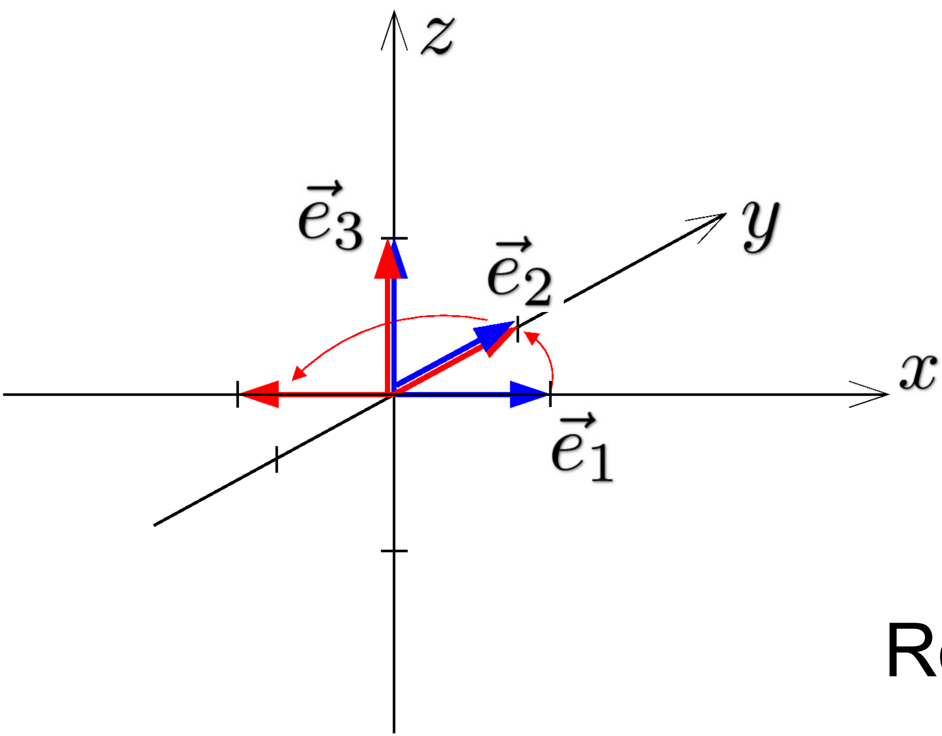
$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

are mapped to!

Example: BB

BB What does this matrix do?

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Rotation by 90° around z axis!

Remember: Basis vectors “span” the space

Every vector can be expressed as the sum of basis vectors:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$$

So we can see how **a matrix defines a mapping of the whole space**.

Reminder: Summation notation

upper limit

Definition: $$\sum_{j=1}^{3} x_j := x_1 + x_2 + x_3$$

summation index

lower limit

Note: Increment always by 1!

It is like a "for" loop:

```
a= 0;
 for ( j=1; j <= 3; j= j + 1) {
    a= a+x_j
 }
```

Matrix-vector Multiplication

$$(A\vec{x})_i = \sum_{j=1}^{3} a_{ij}x_j = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3$$

… simplifies many calculations.

Matrix Multiplication

$$A \cdot B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} =$$

BB

Interpretation: $A \cdot B$ is the transformation of applying B and then A :

$$(\mathbf{A} \cdot \mathbf{B}) \cdot \vec{x} = \mathbf{A} \cdot (\mathbf{B} \cdot \vec{x})$$

BB Matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}$$

The squares illustrate how things combine, analogous for the other fields.

Much easier

$$(A \cdot B)_{ij} = \sum_{k=1}^{3} a_{ik} b_{kj}$$

Summation index
"in the middle"

Sometimes called a
"contraction" over index k

Properties of Matrix Multiplication

Associativity: $A \cdot (B \cdot C) = (A \cdot B) \cdot C$ **BB**

Not commutative (!!!): $A \cdot B \neq B \cdot A$

Under certain circumstances the Inverse of a matrix exists:

$$A \cdot A^{-1} = \mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So-called "right inverse"

BB Proof of associativity in matrix multiplication

$$A \cdot (B \cdot C) = \sum_{k=1}^{n} a_{ik} \sum_{l=1}^{n} b_{kl} c_{lj}$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{n} a_{ik} b_{kl} c_{lj}$$

because $x(y+z) = xy + xz$ for $x,\ y,\ z, \in \mathbb{R}$

BB Proof continued

$$= \sum_{l=1}^{n} \sum_{k=1}^{n} a_{ik} b_{kl} c_{lj}$$ as $x + y = y + x$

$$= \sum_{l=1}^{n} \left(\sum_{k=1}^{n} a_{ik} b_{kl} \right) c_{lj}$$ as $xz + yz = (x + y)z$

$$= (A \cdot B) \cdot C$$

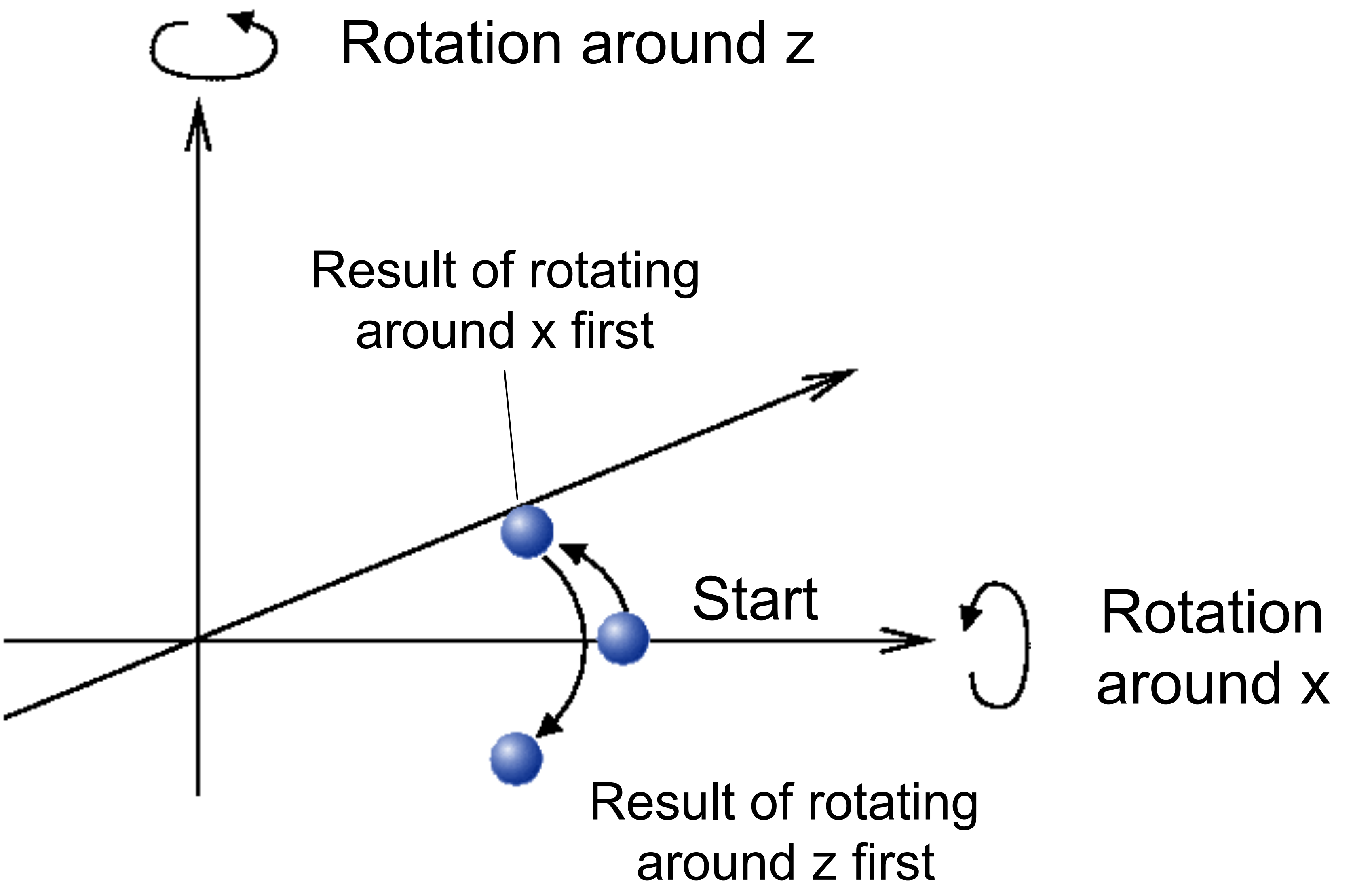
q.e.d.

Example of non-commuting matrices

- Rotation matrices in 3d do not commute:

  BB

BB Non-commutative rotation



Non-square matrices

Matrices do not have to be square:

$$A = \begin{pmatrix} -5 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix} \in \mathcal{M}(2,3)$$

$$B = \begin{pmatrix} 1 & -1 \\ 0 & 3 \\ 2 & -2 \end{pmatrix} \in \mathcal{M}(3,2)$$

$$A \cdot B =$$

BB

BB Multiplication

$$\begin{pmatrix} \boxed{-5 \quad 2 \quad 1} \\ 0 \quad 3 \quad -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 3 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} \boxed{-3} & 9 \\ -2 & 11 \end{pmatrix}$$

Non-square matrices

- An m x n matrix can be multiplied with
  - An n dimensional column vector
  - An m dimensional row vector
  - An n x k matrix, any k > 0
- In other words the summation index must always have the same range.
- An m x n matrix transforms vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$

Matrix transpose

Transposition is the operation where lines and columns are swapped. Or a reflection along the diagonal, if you want:

$$A^T = (a_{ij})^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^T$$

$$= \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = (a_{ji})$$

Properties of transposition

- In “component notation” it looks quite minimal:

$$A = (a_{ij})\ , \quad B = (b_{ij}) = A^T$$

$$\Rightarrow b_{ij} = a_{ji}$$

- Row vectors become column vectors (and vice versa)

- m x n matrix becomes a n x m matrix

Scalar product

The scalar product of two vectors is defined as

$$\vec{x} \cdot \vec{y} := \sum_{i=1}^{3} x_i y_i$$ also denoted as $\langle \vec{x}, \vec{y} \rangle$

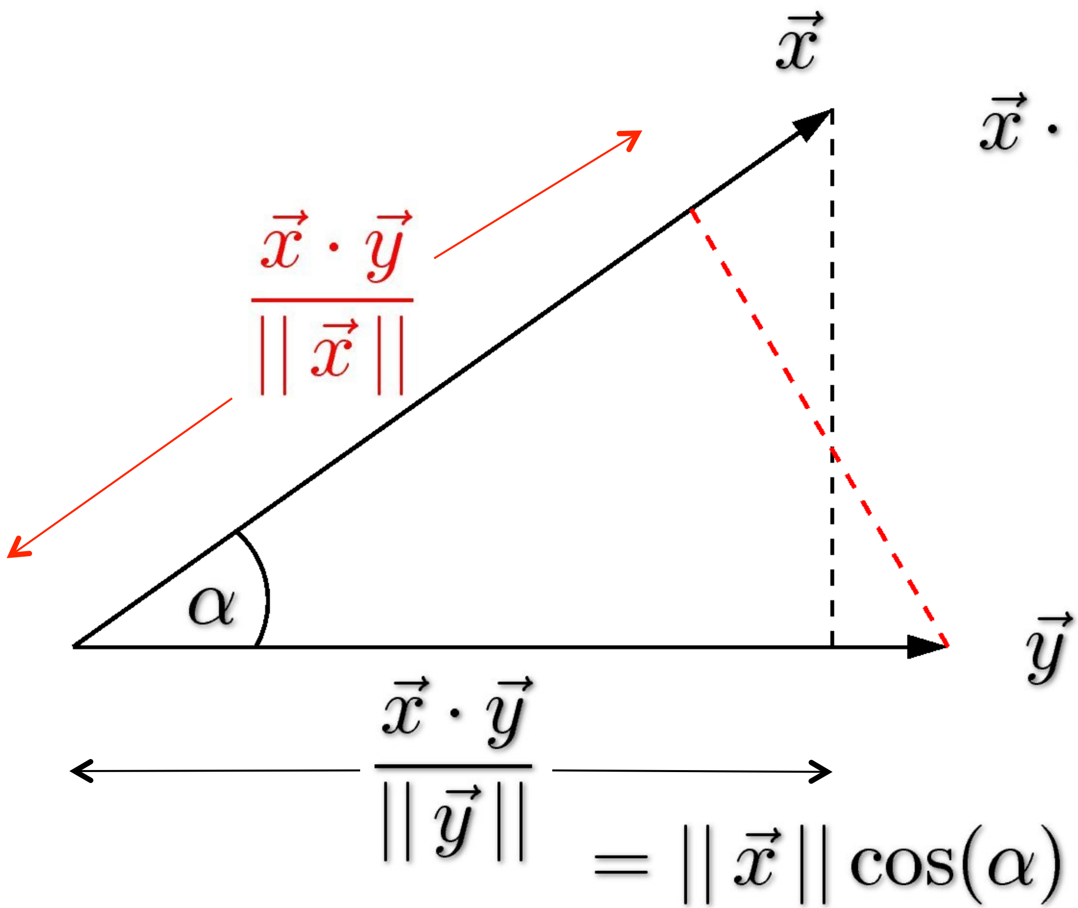
Interpretation: *(in a moment) …*

It is a special case of Matrix multiplication:

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

Interpretation of scalar product

$$\vec{x} \cdot \vec{y} = ||\vec{x}||\ ||\vec{y}||\ \cos(\alpha)$$



Strictly speaking one should write

...

$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \cdot \vec{y}$ for the scalar product.

Length and distances

- Euclidean norm (length)

$\vec{x} \in \mathbb{R}^n$

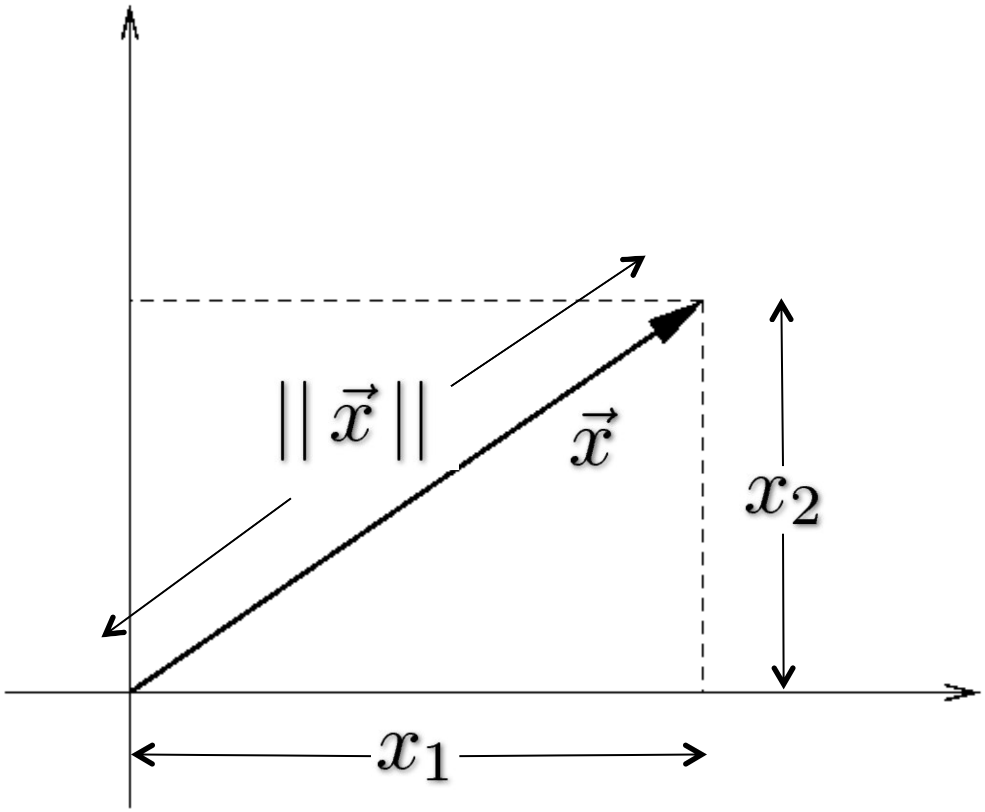
Norm of $\vec{x}$ is $||\vec{x}|| := \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{\vec{x} \cdot \vec{x}}$

It is also called “2-norm”. Why this is our “natural” notion of length: BB

Remark: There are many other notions of length

BB

Why the definition of length matches our intuition for length



$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$||\vec{x}||^2 = x_1^2 + x_2^2$$

This is the **Pythagorean theorem** (check Wikipedia if never heard of it)

How does Length become Distance?

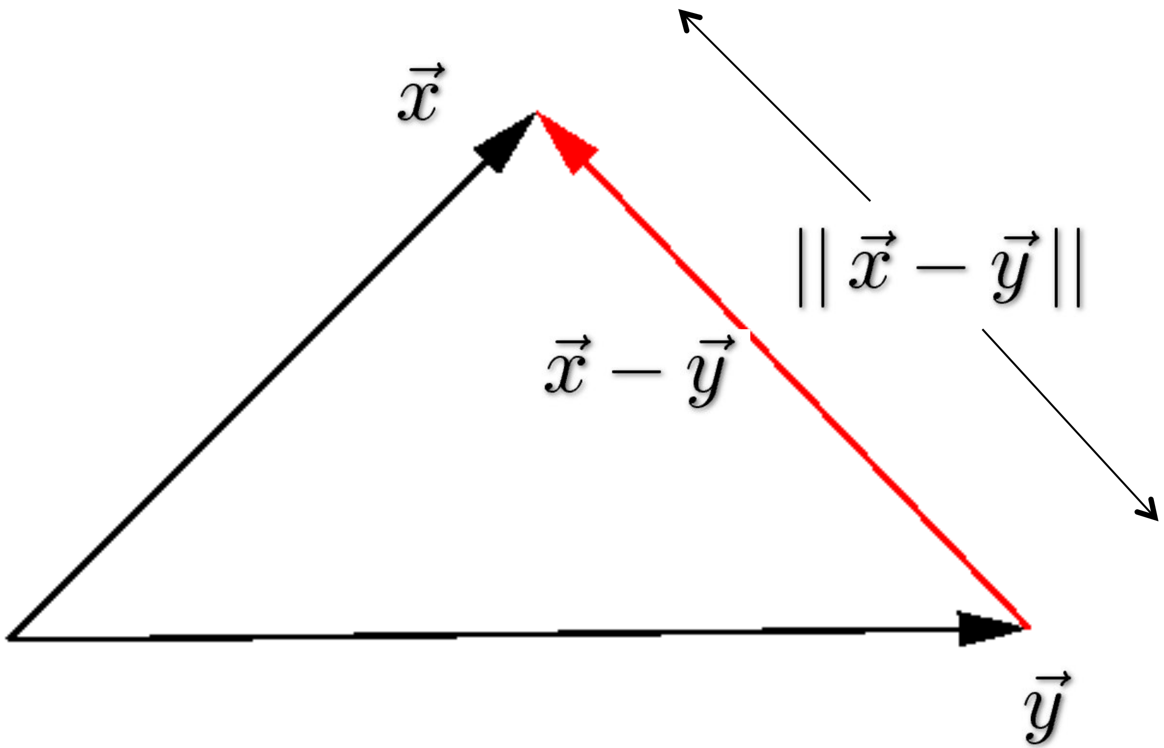
The distance between two vectors (points) is the length of the difference:

$$\vec{x},\ \vec{y} \in \mathbb{R}^n$$

$$d(\vec{x}, \vec{y}) = ||\vec{x} - \vec{y}||$$

It is called **Euclidean distance**. Geometric interpretation: BB

BB Distance



The norm (length) $||\vec{x}-\vec{y}||$ of $\vec{x}-\vec{y}$ is the distance from $\vec{x}$ to $\vec{y}$

Some properties you would like to know

A norm or distance is always positive or 0.

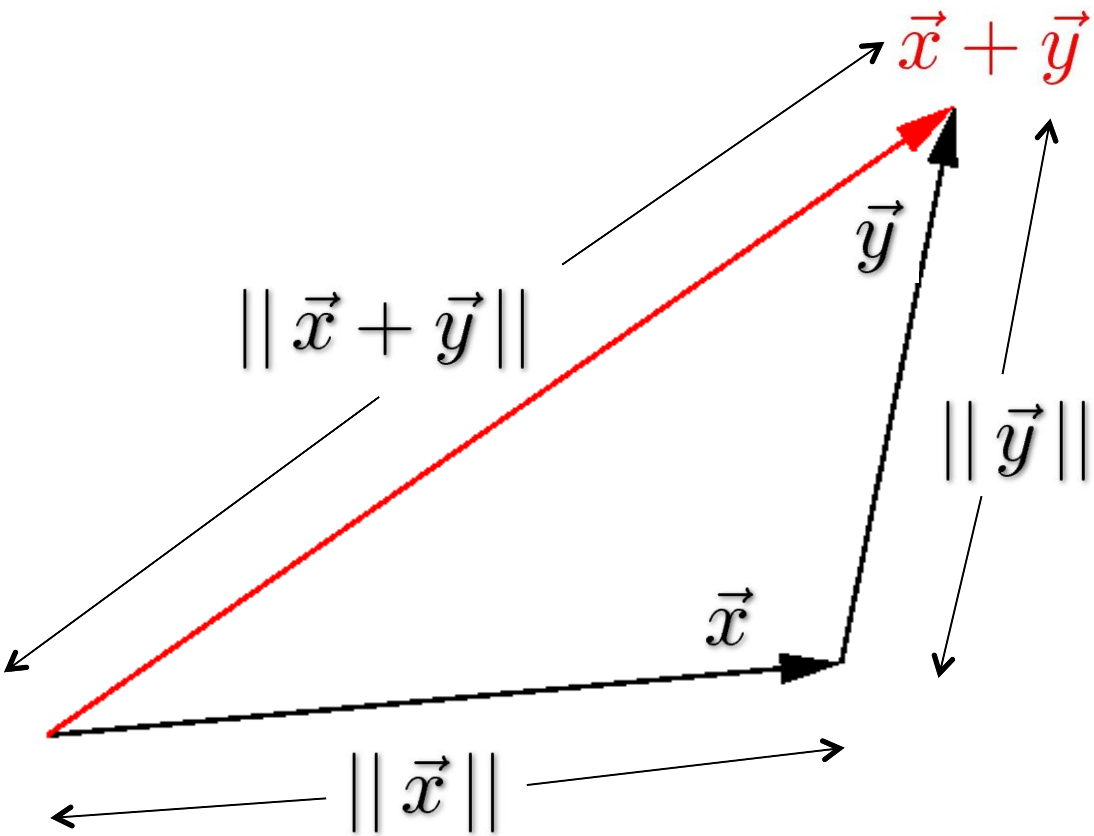
Scaling vectors: $||\,a\,\vec{x}\,|| = |a|\;||\vec{x}\,||$

Triangle inequality:

$$||\,\vec{x} + \vec{y}\,|| \;\leq\; ||\,\vec{x}\,|| + ||\,\vec{y}\,||$$

Geometric meaning … **BB**

BB Geometric meaning of Triangle Inequality



The triangle inequality means that going along the direct way ($||\vec{x}+\vec{y}||$) in a triangle is always shorter than (or equal to) going along the two other sides ($||\vec{x}||+||\vec{y}||$)