Mathematical Concepts (G6012)

Lecture 10

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Church/Turing Thesis

- Every computable function can be computed by a Turing Machine
- I.e.: Turing Machines are universal computing machines
- Every problem that can be solved by an algorithm can be solved by a Turing machine
- Where is the power coming from?

  The read/write input/output tape !

More about TM

- The tape can be used to record any data for later access
- There is always space available after last non-blank location
- There is no limit how often the tape is accessed
- Your PC is less powerful than a TM – why? Because it has finite memory

Efficiency

- TM are universal but not efficient
- Progress can be really slow
- Looking up memory involves sequential access – the opposite of efficiency

Managing complexity

- One can encapsulate useful functionality in “separate” sub-routines
- Collection of states set aside for each subroutine
- (similar to structured programming approach)
- However, TM are mainly useful as a theoretical concept, not for solving real world problems!

Variations

- There are common variants of TM:
    - Multiple tapes
    - Single-side infinite tape
    - Non-deterministic TM
- It can be shown that these have all equivalent power to the TM discussed here.

Example: Non-deterministic TM

- To simulate a non-deterministic TM:
  - 3 tapes:
  - One tape for original input
  - One tape for the choice sequence: (2,3,1,2)
  - One tape to run on current choice sequence
- For this to work we need to enumerate all possible sequences of choices (ok, as states are finite)

Another equivalence

- The “2 pushdown” automaton is equivalent to the Turing Machine:
  - One pushdown holds tape contents to left of tape head
  - One pushdown holds tape contents to the right of tape head
  - As tape head moves, symbols shift across from one pushdown to another

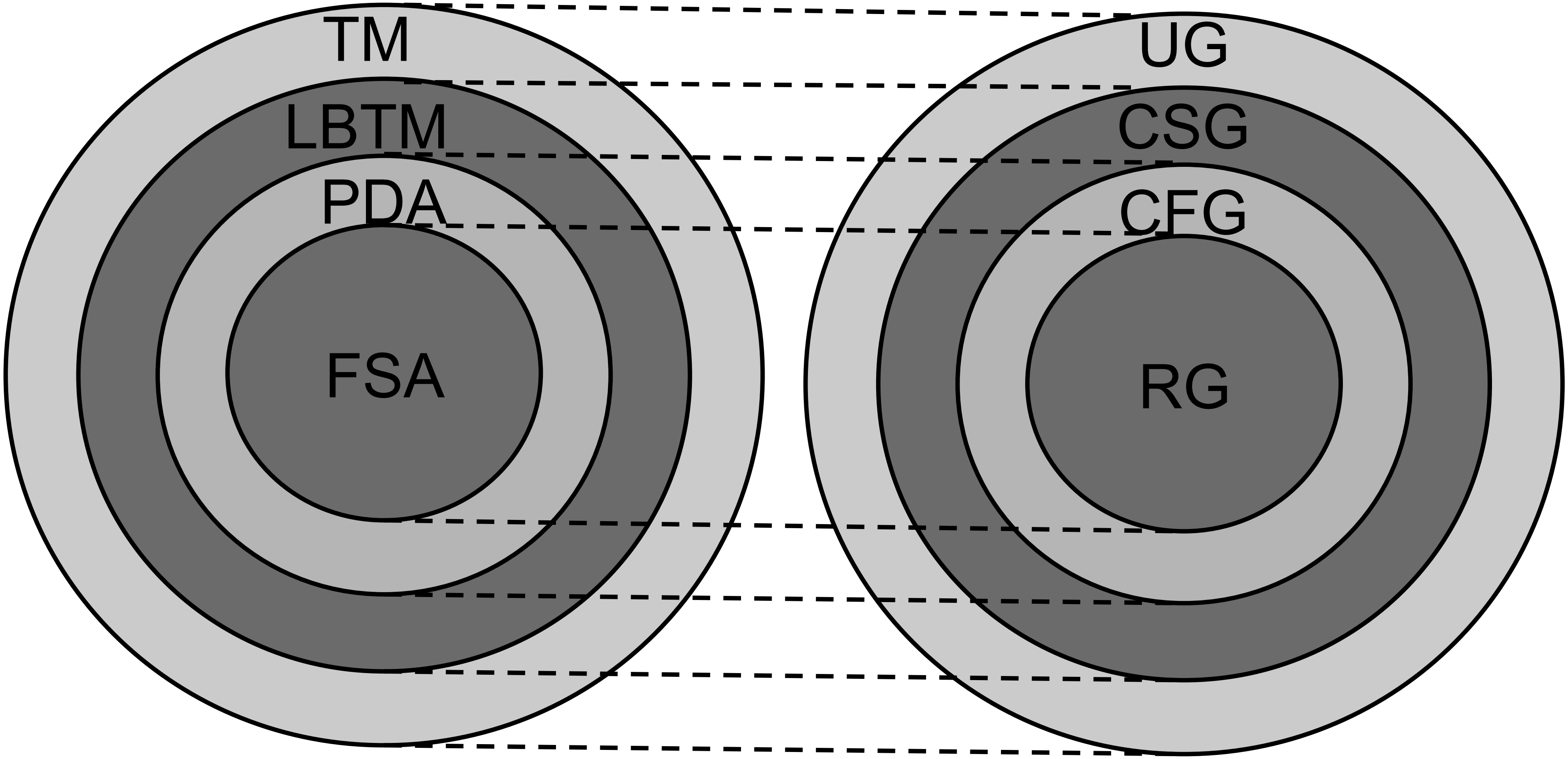
More generally …

- Chomsky Hierarchy (for language classes):
  - Type 0: Languages accepted by Turing Machines
  - Type 1: Languages accepted by Turing Machines with linear bounded storage
  - Type 2: Languages accepted by Pushdown Automata
  - Type 3: Languages accepted by Finite State Automata

Alternative Characterization

- Equivalent grammar formalisms:
  - Type 0: Languages generated by unrestricted grammars
  - Type 1: Languages generated by context-sensitive grammars
  - Type 2: Languages generated by context-free grammars
  - Type 3: Languages generated by regular grammars

Equivalence and Inclusions



BB: Full names and acronyms

TM = Turing Machine

LBTM = Linearly bounded Turing Machine

PDA = Pushdown Automaton

FSA = Finite State Automaton

UG = Unrestricted Grammar

CSG = Context Sensitive Grammar

CFG = Context Free Grammar

RG = Regular Languages

VECTORS AND MATRICES

Why matrix algebra?

- Multimedia/Design/Art: Computer graphics are 90% vectors and matrices
- AI: Artificial Neural Networks heavily depend on vectors and matrices.
- Music: Discretised sound spectra are vectors; digital filtering & enhancement depend on matrices; modern compression (mp3 etc) is one of the most maths-heavy problems in Informatics

VECTORS AND MATRICES

Vectors & Matrices

- A **matrix** or a **vector** is simply a way of representing a structured collection of numbers.
- Vectors are order 1 (rows, columns) and can be used to represent
  - sound samples
  - general arrays
  - lists of things
  - datasets …
  - points in space
  - directions in space
  - velocity

Vectors & Matrices

- **Matrices** are order 2 (rectangles) and can be used to represent
  - Images
  - Datasets
  - Transformations of vectors
  - Parameters of Artificial Neural Networks
  - …

Vector Notations

$$\begin{pmatrix} -1 \\ 3 \\ 9 \end{pmatrix} \in \mathbb{R}^3$$

column vector

arrow

underline

$$\vec{x} = \mathbf{x} = \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$$

bold

"component"

$$x_i \in \mathbb{R}$$

index

$$\begin{pmatrix} -5 & 3 & 1.1 \end{pmatrix} \in \mathbb{R}^3$$

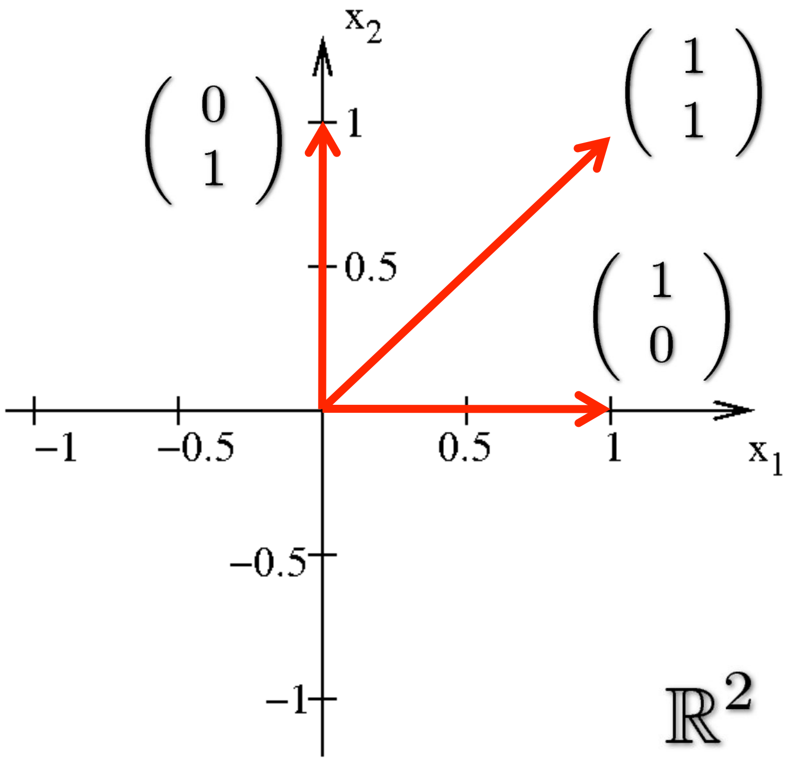
$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \in \mathbb{R}^3$$

Row vector

Similar for all $\mathbb{R}^n$

Geometric interpretation

Vectors are (arrows to) points in space:



Multiplying numbers with vectors

$\vec{x} \in \mathbb{R}^3$ and $a \in \mathbb{R}$

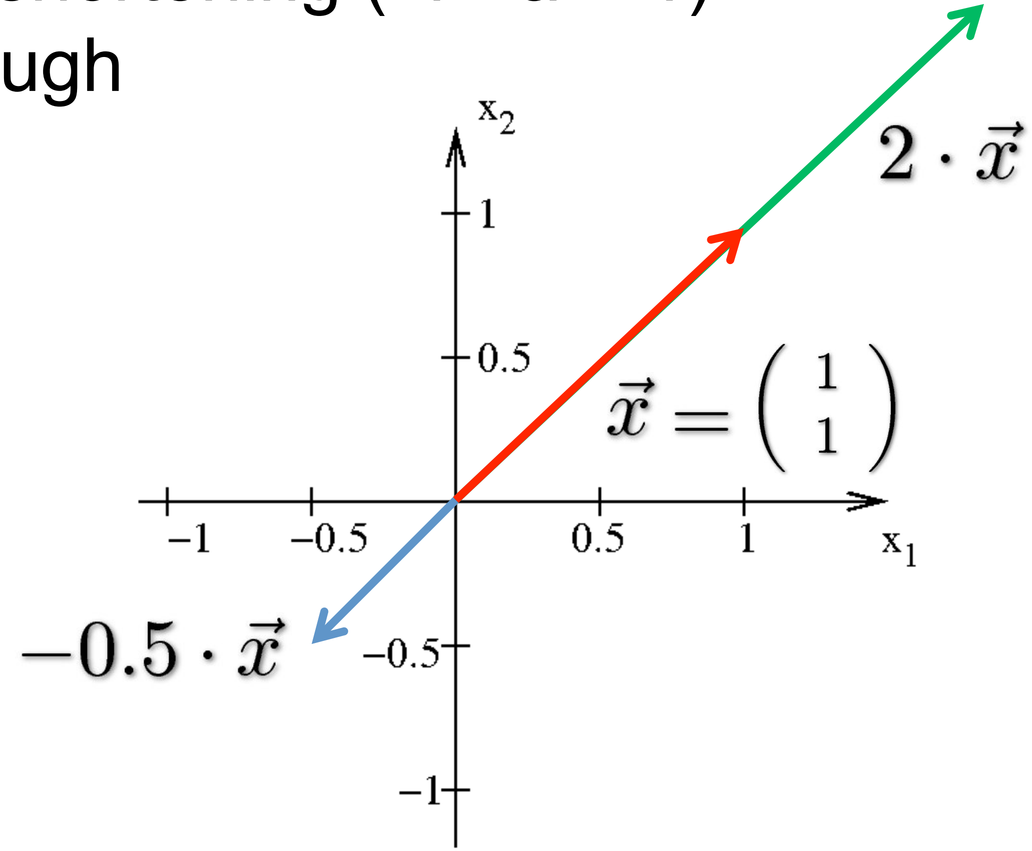
then $a \cdot \vec{x} \in \mathbb{R}^3$

$$a \cdot \vec{x} = a \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \cdot x_1 \\ a \cdot x_2 \\ a \cdot x_3 \end{pmatrix}$$

Interpretation: **BB**

BB Interpretation

Multiplying with $a \in \mathbb{R}$ means stretching (a > 1) or shortening (-1< a < 1) and/or mirroring through the center (a < 0)



Adding vectors

If $\vec{x} \in \mathbb{R}^3$ and $\vec{y} \in \mathbb{R}^3$

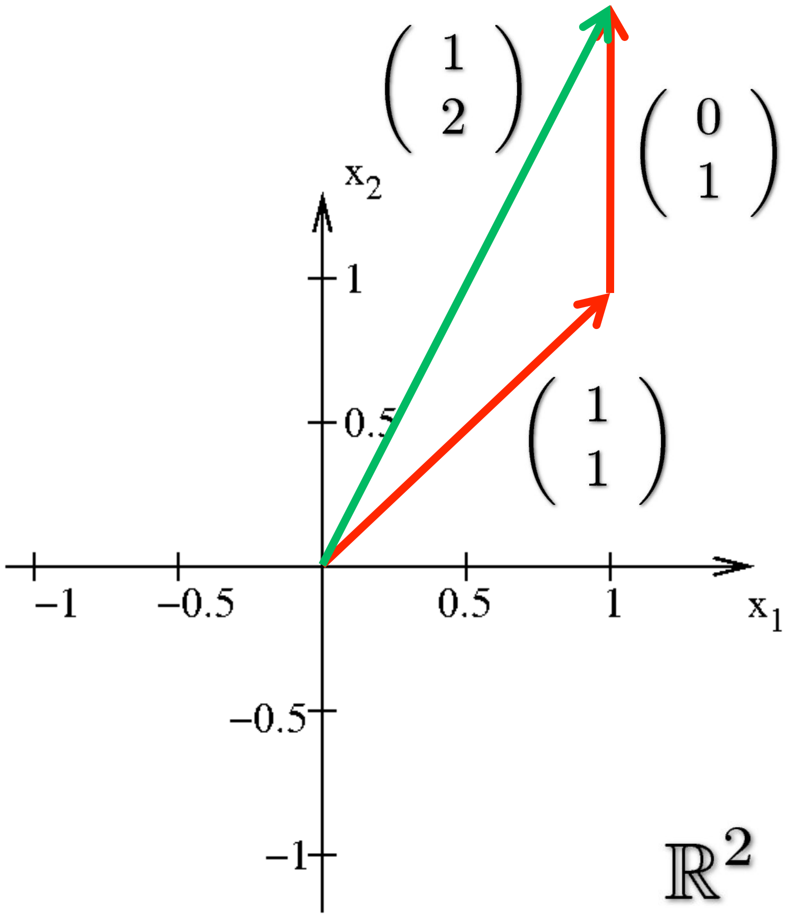
then $\vec{x} + \vec{y} \in \mathbb{R}^3$

$$\vec{x} + \vec{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$$

Example & Interpretation

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1+0 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



Basis vectors

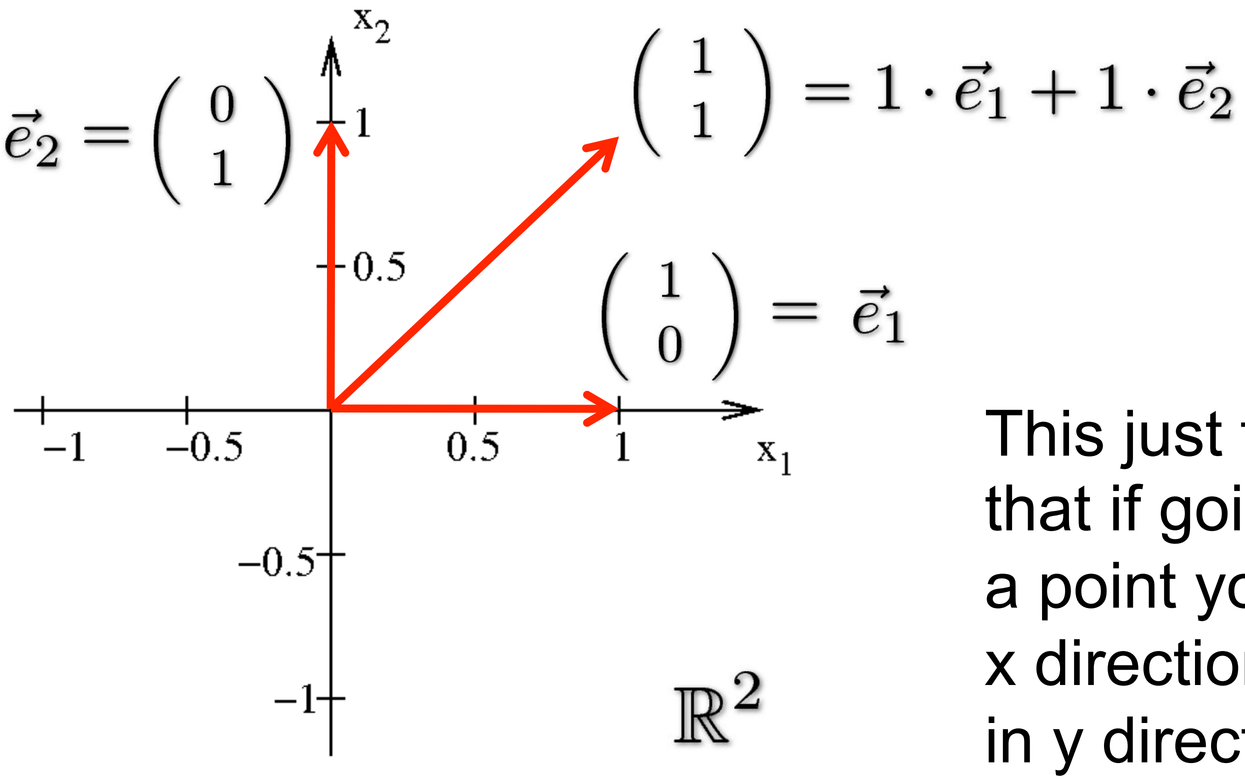
Every vector can be expressed as a combination of basis vectors

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \cdot \vec{e}_1 + x_2 \cdot \vec{e}_2 + x_3 \vec{e}_3$$

Geometric interpretation

Vectors are (arrows to) points in space:



This just formalises that if going from 0 to a point you can go in x direction first, then in y direction instead of going diagonal …

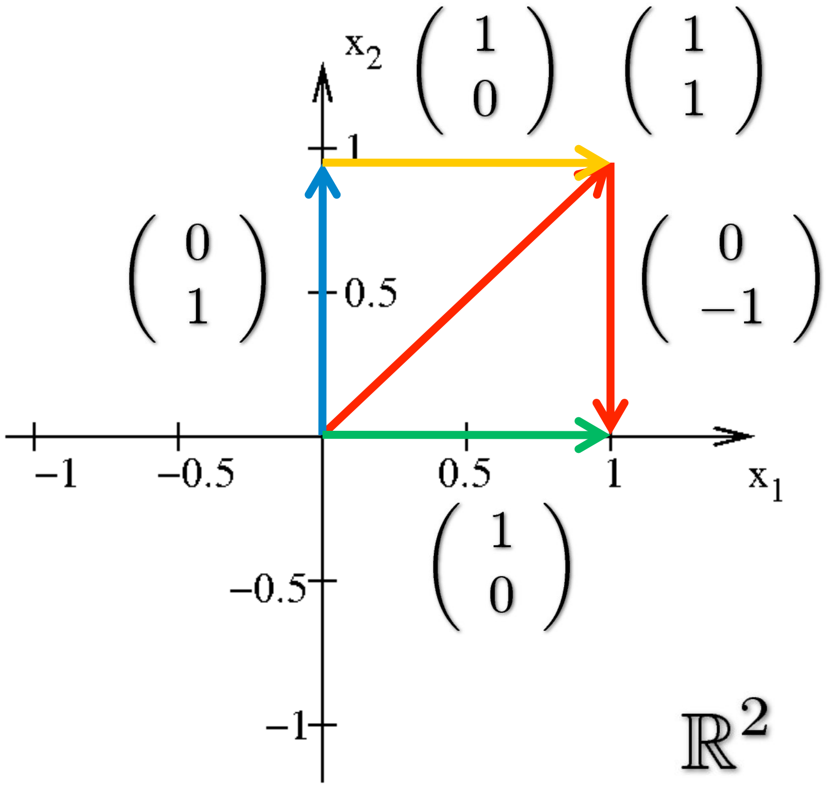
Subtraction

$$\vec{x} - \vec{y} = \vec{x} + (-\vec{y})$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -0 \\ -1 \end{pmatrix} =$$

$$\begin{pmatrix} 1-0 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



Matrices

$$\begin{pmatrix} -1 & 5 & -4 \\ 9.1 & 3 & -4.5 \\ 7 & 0.1 & \sqrt{2} \end{pmatrix} \in \mathcal{M}(3,3)$$ is a 3x3 matrix.

capital letter

“entry”, ”element”

$$A = \underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (a_{ij}) \in \mathcal{M}(3,3)$$

double underscore

index: row first, column second

Tensors

Generalization:
Vector: 1-tensor
Matrix: 2-tensor

Example: 3-tensor

$$A = (a_{ijk})$$

$$i = 1, \ldots, 3$$
$$j = 1, \ldots, 3$$
$$k = 1, \ldots, 3$$

is a 3x3x3 tensor
with 27 entries

(If you write it out it would be a cube of numbers)

Adding and subtracting matrices

- Same as for vectors …

  BB

- Interpretation not so direct: Operations on vectors – next time.

BB Example: Subtractring a matrix from an other matrix

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & -2 \\ -2 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 3 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-1 & -2-0 & 3-(-1) \\ 0-0 & 4-2 & -2-2 \\ -2-3 & 2-1 & 1-(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 & 4 \\ 0 & 2 & -4 \\ -5 & 1 & 2 \end{pmatrix}$$

Properties of +, -

$A, B, C \in \mathcal{M}(m, n)$

Associativity

$$A + B + C = (A + B) + C = A + (B + C)$$

Commutativity $A + B = B + A$

Properties of scalar multiplication

$A, B, C \in \mathcal{M}(m, n) \qquad r, s \in \mathbb{R}$

Compatibility with scalar operations (multiplying with numbers)

$$r \cdot (A + B) = r \cdot A + r \cdot B$$

$$(r + s) \cdot A = r \cdot A + s \cdot A$$

Matrix-Vector Multiplication

$A \in$ M(3,3) and $\vec{x} \in \mathbb{R}^3$

$$A \cdot \vec{x} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$$

BB

BB Matrix-vector multiplication

$$A\vec{x} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$

The result is again a vector!

BB Matrix-vector multiplication

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The result is again a vector!

Properties of Matrix-Vector Multiplication

Linear (both ways)

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$$

$$(A + B)\vec{x} = A\vec{x} + B\vec{x}$$

Associative:

$$(A \cdot B)\vec{x} = A(B\vec{x})$$

Interpretation

Matrices are transformations (linear functions)

$$A = \underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in$$ M(3,3)

$$A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$\vec{x} \mapsto A \cdot \vec{x}$$

$$A\vec{x}$$

A maps vectors from $\mathbb{R}^3$ to vectors in $\mathbb{R}^3$

Matrix- vector multiplication

Matrix as a transformation: Can we see what it does?

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

BB

BB Matrix as a transformation

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

Column of the
matrix are
the images of the
basis vectors!

Matrix as a transformation

The columns of the matrix are the vectors
the basis vectors

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

are mapped to!

Example: BB

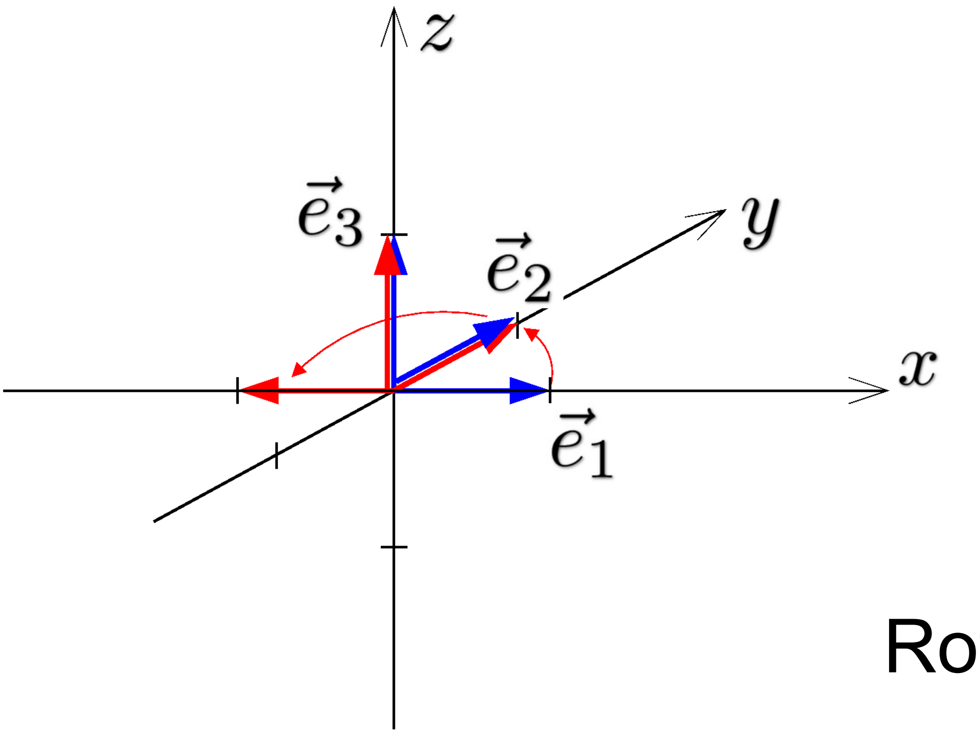
BB What does this matrix do?

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



Rotation by 90° around z axis!

Remember: Basis vectors “span” the space

Every vector can be expressed as the sum of basis vectors:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$$

So we can see how **a matrix defines a mapping of the whole space**.

Reminder: Summation notation

upper limit

Definition: $$\sum_{j=1}^{3} x_j := x_1 + x_2 + x_3$$

summation index

lower limit

Note: Increment always by 1!

It is like a “for” loop:

```
a= 0;
 for ( j=1; j <= 3; j= j + 1) {
    a= a+x_j
 }
```

Some alternative notations **BB**

Matrix-vector Multiplication

$$(A\vec{x})_i = \sum_{j=1}^{3} a_{ij}x_j = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3$$

… simplifies many calculations.