Pregeometric Concepts on Graphs and Cellular Networks as Possible Models of Space-Time at the Planck-Scale[1]

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Abstract

Starting from the working hypothesis that both physics and the corresponding mathematics have to be described by means of discrete concepts on the Planck-scale, one of the many problems one has to face is to find the discrete protoforms of the building blocks of continuum physics and mathematics. In the following we embark on developing such concepts for irregular structures like (large) graphs or networks which are intended to emulate (some of) the generic properties of the presumed combinatorial substratum from which continuum physics is assumed to emerge as a coarse grained and secondary model theory. We briefly indicate how various concepts of discrete (functional) analysis and geometry can be naturally constructed within this framework, leaving a larger portion of the paper to the systematic developement of dimensional concepts and their properties, which may have a possible bearing on various branches of modern physics beyond quantum gravity.

[1]Invited paper to appear in the special issue of the Journal of Chaos, Solitons and Fractals on: "Superstrings, M, F, S ... Theory" (M. S. El Naschie, C. Castro, Editors)

1 Introduction

There exists a certain suspicion in parts of the scientific community that nature may be discrete or rather "behaves discretely" on the Planck scale. But even if one is willing to agree with this "working philosophy", it is far from being evident what this vague metaphor actually has to mean or how it should be implemented into a concrete and systematic inquiry concerning physics and mathematics in the Planck regime.

There are basically two overall attitudes as to "discreteness on the Planck scale", the one comprising approaches which start (to a greater or lesser degree) from continuum concepts (or more specifically: concepts being more or less openly inspired by them) and then try to detect or create modes of "discrete behavior" on very fine scales, typically by imposing quantum theory in full or in parts upon the model system or framework under discussion.

There are prominent and very promising candidates in this class like e.g. *'string theory'* or *'loop quantum gravity'*. Somewhat intermediate is a more recent version (or rather: aspect) of the latter approach, its *'polymer'* respectively *'spin network'* variants. As these approaches are widely known and will probably get their fair share in this volume anyhow, we refrain from citing from the vast corresponding literature. We only recommend, as more recent reviews to the latter approach, containing some cursory remarks about the former together with a host of references, [5] and [6] and, as a beautiful introduction to the whole field, [7].

On the other hand one could adopt an even more speculative and radical attitude and approach the Planck regime from more or less the opposite direction by developing a framework almost from scratch which has "discreteness" already built in into its very building blocks and then try to reconstruct, working, so to speak, "bottom up", all the continuum concepts of ordinary space-time physics as sort of *'collective quantities'* like e.g. *'collective excitations'* via the cooperation of many microscopic (discrete) degrees of freedom. If one is very bold one could even entertain the idea that the quantum phenomena as such are perhaps not the eternal and irreducible principles as they are still viewed today by the majority of physicists but, to the contrary, may emerge as derived and secondary concepts, together with gravitation, from a more primordial truly discrete and *'combinatorial'* theory.

It goes without saying that such a radical approach is beyond the reach of direct experimental verification in the strict sense for the foreseeable future (as is the case with the other frameworks mentioned above). As a substitute one rather has to rely on inner theoretical criteria, among other things the capability to generate the hierarchy of complex patterns we are observing in nature or in our present day *'effective theories'* of the various regimes many orders of magnitude away from the Planck scale while introducing as few simple and elementary assumptions as possible. More specifically, one would like such a framework to provide clues how the continuum concepts of ordinary space-time physics/mathematics may emerge in a natural manner from their respective *'discrete protoforms'*.

Another more aesthetic criterion would be a kind of natural convergence of the different approaches towards a common substructure which is discrete in a really primordial way. Indications for such a convergence can, in our view, be detected in the various lines of research going on presently. *'spin networks'* or *'polymer states'* are e.g. cases in point where modes of discreteness do emerge from a at first glance continuous environment. Furthermore it may well be that *'string field theory'* will turn out to live on a more discrete and singular substratum than presently suspected (some speculative remarks pointing in this direction can e.g. be found at the end of [8]), a catchword being *'fractal geometry'*. A brief but illuminating analysis concerning such a possible convergence in the future towards a decidedly discrete and *'combinatorial'* common limit has been given in section 8 of [7]. A bundle of related ideas with which we sympathize is developed by Nottale (see e.g. [16]).

In the following we will try to give a very brief survey over our personal variant of Planck scale physics and mathematics, which has a pronouncedly combinatorial flavor (employing, among other things, tools from *'algebraic combinatorics'* and *'(random) graph theory'*). Interesting enough, there exist also close ties to *'noncommutative geometry'* (a general source being [9]).

2 The Cellular Network Environment

In this section we will sketch the type of model systems on which the following analysis will be based. As already said in the introduction we will start from a rather primordial level, trying to make no allusions whatsoever to continuum concepts. We will then show how protoforms of ideas and notions, playing a key role in ordinary continuum physics/mathematics emerge in a relatively natural and unforced way from this framework. Cases in point are e.g. concepts like *'dimension'*, *'differential structure'*, the idea of *'physical points'* (being endowed with an internal structure), the web of which establishes the substratum of macroscopic space-time, and other geometrical/topological notions. The framework turns out to be even rich enough to support a full fledged kind of *'discrete functional analysis'*, comprising e.g. *'Laplace-, Dirac operators'* and all that. It is perhaps particularly noteworthy that an advanced structure like *'Connes' spectral triple'* shows up in a very natural way within this context.

Besides the reconstruction of basic concepts of continuum physics/ mathematics another goal is to describe the micro dynamics going on in this discrete substratum over (in) which macroscopic space-time is floating as a kind of coarse grained *'superstructure'*, the formation of *'physical points'* and their mutual entanglement, yielding the kind of *'near-/far-order'* or *'causal structure'* we are used to from continuum space-time.

To this end we view this substratum as, what we like to call, a *'cellular network'*, consisting of *'nodes'* and *'bonds'*. The nodes are assumed to represent certain elementary modules (cells or "monads") having a discrete, usually simple, internal state structure, the bonds modeling elementary direct interactions among the nodes. As an important ingredient, these bonds are dynamical insofar as they are capable to be in a (typically limited) number of *'bond states'*, thus implementing the varying strength of the mutual interactions among the cells.

It is a further characteristic of our model class that these interactions are not only allowed to vary in strength but, a fortiori, can be switched off or on, depending on the state of their local environment. In other words, bonds can be, stated in physical terms, created or annihilated in the course of network evolution, which (hopefully) enables the system to undergo *'geometric phase transitions'* being accompanied by an *'unfolding'* and *'pattern formation'*, starting e.g. from a less structured chaotic initial phase. To put it briefly: in contrast to, say, *'cellular automata'*, which are relatively rigid and regular in their wiring and geometric structure (in particular: with the bonds typically being non-dynamical), our cellular networks do not carry such a rigid overall order as an external constraint (e.g. a regular lattice structure); their "wiring" is dynamical and thus behaves randomly to some extent. The clue is that order and modes of regularity are hoped to emerge via a process of *'self-organization'*.

Definition 2.1 (Class of Cellular Networks).

1. *"Geometrically" our networks represent at each fixed* 'clock time' *a* 'labeled graph'*, i.e. they consist of nodes $\{n_i\}$ and bonds $\{b_{ik}\}$, with the bond b_{ik} connecting the nodes (cells) n_i, n_k. We assume that the graph has neither elementary loops nor multi-bonds, that is, only nodes with $i \neq k$ are connected by at most one bond.*

2. *At each site n_i we have a local node state $s_i \in q \cdot \mathbb{Z}$ with q, for the time being, a certain not further specified elementary quantum. The bond variables J_{ik}, attached to b_{ik}, are in the most simplest cases assumed to be two- or three-valued, i.e. $J_{ik} \in \{\pm 1\}$ or $J_{ik} \in \{\pm 1, 0\}$*

Remark.

1. In the proper graph context the notions *'vertex'* and *'edge'* are perhaps more common (see e.g. [10]). As to some further concepts used in graph theory see below.

2. These are, in some sense, the most simple choices one can make. It is an easy matter to employ instead more complicated internal state spaces like, say, groups, manifolds etc. One could in particular replace $\mathbb{Z}$ by one of its subgroups or impose suitable boundary conditions.

3. In the following section we will give the bonds b_{ik} an *'orientation'*, i.e. (understood in an precise algebraic/geometric sense) $b_{ik} = -b_{ki}$. This implies the compatibility conditions $J_{ik} = -J_{ki}$.

In a next step we have to impose a dynamical law on our model network. In doing this we are of course inspired by *'cellular automaton laws'* (see e.g. [11]). The main difference is however that in our context also the bonds are dynamical degrees of freedom and that, a fortiori, they can become dead or alive (active or inactive), so that the whole net is capable of performing drastic topological/geometrical changes in the course of clock time.

A particular type of a dynamical *'local law'* is now introduced as follows: We assume that all the nodes/bonds at *'(clock) time'* $t+\tau$, τ an elementary clock time step, are updated according to a certain local rule which relates for each given node n_i and bond b_{ik} their respective states at time $t+\tau$ with the states of the nodes/bonds of a certain fixed local neighborhood at time t.

It is important that, generically, such a law does not lead to a reversible time evolution, i.e. there will typically exist attractors in total phase space (the overall configuration space of the node and bond states).

A crucial ingredient of our network laws is what we would like to call a *'hysteresis interval'*. We will assume that our network starts from a densely entangled *'initial phase'* QX_0, in which practically every pair of nodes is on average connected by an *'active'* bond, i.e. $J_{ik} = \pm 1$. Our dynamical law will have a built-in mechanism which switches bonds off (more properly: sets $J_{ik} = 0$) if local fluctuations among the node states become too large. Then there is hope that this mechanism may trigger an *'unfolding phase transition'*, starting from a local seed of spontaneous large fluctuations towards a new phase (an attractor) carrying a certain *'super structure'*, which we would like to relate to the hidden discrete substratum of space-time (points).

One example of such a law is given in the following definition.

Definition 2.2 (Local Law). *At each clock time step a certain 'quantum' q is transported between, say, the nodes n_i, n_k such that*

$$s_i(t+\tau) - s_i(t) = q \cdot \sum_k J_{ki}(t) \tag{1}$$

(i.e. if $J_{ki} = +1$ a quantum q flows from n_k to n_i etc.)
The second part of the law describes the back reaction on the bonds (and is, typically, more subtle). This is the place where the so-called 'hysteresis interval' enters the stage. We assume the existence of two 'critical parameters' $0 \leq \lambda_1 \leq \lambda_2$ with:

$$J_{ik}(t+\tau) = 0 \quad \text{if} \quad |s_i(t) - s_k(t)| =: |s_{ik}(t)| > \lambda_2 \tag{2}$$

$$J_{ik}(t+\tau) = \pm 1 \quad \text{if} \quad 0 < \pm s_{ik}(t) < \lambda_1 \tag{3}$$

with the special proviso that

$$J_{ik}(t+\tau) = J_{ik}(t) \quad \text{if} \quad s_{ik}(t) = 0 \tag{4}$$

On the other side

$$J_{ik}(t+\tau) = \begin{cases} \pm 1 & J_{ik}(t) \neq 0 \\ 0 & J_{ik}(t) = 0 \end{cases} \quad \text{if} \quad \lambda_1 \leq \pm s_{ik}(t) \leq \lambda_2 \ . \tag{5}$$

In other words, bonds are switched off if local spatial charge fluctuations are too large, switched on again if they are too small, their orientation following the sign of local charge differences, or remain inactive.

Remark.

1. The reason why we do not choose the "current" $q \cdot J_{ik}$ proportional to the "voltage difference" $(s_i - s_k)$ as e.g. in Ohm's law is that we favor a non-linear (!) network which is capable of self-excitation and self-organization rather than self-regulation around a relatively uninteresting equilibrium state! The balance between dissipation and amplification of spontaneous fluctuations has however to be carefully chosen ("complexity at the edge of chaos")

2. We presently have emulated these local network laws on a computer. It is not yet clear whether this simple network law already does what we expect. In any case, it is fascinating to observe the enormous capability of such intelligent networks to find attractors very rapidly, given the enormous accessible phase space

3. In the above class of laws a direct bond-bond-interaction is not yet implemented. We are prepared to incorporate such a contribution in a next step if it turns out to be necessary. In any case there are not so many ways to do this in a sensible way. Stated differently, the class of possible physically sensible interactions is perhaps not so numerous.

4. Note that – in contrast to e.g. Euclidean lattice field theory – the so-called *'clock time' t* is, for the time being, not standing on the same footing as potential "coordinates" in the network (e.g. curves of nodes/bonds). Anyhow We suppose that so-called *'physical time'* will emerge as sort of a secondary collective variable in the network, i.e. being different from the clock time (while being of course functionally related to it).

In our view 4 is consistent with the spirit of relativity. What Einstein was really teaching us is that there is a (dynamical) interdependence between what we experience as space respectively time, not that they are absolutely identical! In any case the assumption of an overall clock time is at the moment only made just for convenience in order to make the model system not too complicated. If our understanding of the complex behavior of the network dynamics increases, this assumption may be weakened in favor of a possibly local and/or dynamical clock frequency. A similar attitude should be adopted concerning concepts like *'Lorentz-(In)Covariance'* which we also consider as *'emergent'* properties. It is needless to say that it is of tantamount importance to understand the way how these patterns do emerge from the relatively chaotic background which will be attempted in future work.

As can be seen from the definition of the cellular network, a full scale investigation of its behavior separates quite naturally into two parts of both a different mathematical and physical nature. The first one comprises its more geometric/algebraic content in form of large static graphs and their intricate structure (at, say, arbitrary but fixed clock time), thus neglecting the details of the internal states of bonds and nodes, the other one conveys a more dynamical flavor, i.e. analyzing topological/geometrical change and pattern formation in the course of clock time.

Due to lack of space we cannot treat all these different aspects in any detail, as their proper discussion would require, among other things, the development of a fair amount of relatively advanced (discrete) mathematics. Therefore we prefer to make only a couple of provisional remarks as to some of the necessary building blocks of our framework in the following section, referring to our other papers for more details, and using the remaining space for a discussion of one single aspect in slightly more depth. This aspect is the development of a suitable concept of *'dimension'* on graphs and similar erratic structures.

3 A brief Discussion of various Geometric and Topological Concepts on Graphs and Networks

In a first step we would like to have something like a *'discrete differential calculus'* and *'functional analysis on graphs'* as kind of protoforms of the corresponding continuum concepts. We recently have shown, that this is in fact possible and leads to quite interesting mathematical structures. As to differential calculus and related aspects see the corresponding sections in [12] and references given there as well as [17] for a complimentary but slightly different approach. Functional analysis on graphs is developed in [13]. As to some other aspects of discrete noncommutative geometry refer to the work of Sorkin and Balachandran et al [18], [19] .

The starting point of our approach is the introduction of a differential calculus on *'node-'* and *'bond functions'* as elements of the *'node-'* and *'bond space'* of a given fixed graph G. To put it briefly, one can define the derivative of elementary node functions n_i (the function with the value one on the node n_i, zero elsewhere) and extend it by linearity to general node functions $f := \sum f_i \cdot n_i$, the sum taken over the set of nodes. It is then an easy matter to define e.g. L^p-spaces etc. The crucial point is that the derivative dn_i is represented as the sum over the oriented bonds being incident with n_i, i.e.

$$dn_i := \sum_k b_{ki} \tag{6}$$

(where the n_i's and b_{ik} are viewed to generate complex vector spaces). In other words, d maps the node space into the bond space.

It turns out to make sense to introduce in addition to the oriented bonds, b_{ik}, *'directed bonds'*, d_{ik}, having a fixed direction, pointing from node n_i to node n_k with

$$b_{ik} := d_{ik} - d_{ki} \tag{7}$$

This yields:

$$df = \sum_{ik} (f_k - f_i) d_{ik} \tag{8}$$

If one develops this discrete calculus further quite a few interesting aspects will emerge with links to various areas of modern mathematics, catchwords being: *'non-commutative geometry'*, *'modules'*, *'groupoids'* etc. ([12]).

It is also possible to develop something like discrete functional analysis on graphs ([13]). Among other things a *'graph Laplacian'* does exist which turns out to be intimately related with the *'adjacency matrix'* of graph theory.

$$(9) \qquad -\Delta f := -\sum_i (\sum_k (f_k - f_i)) n_i \quad \text{and} \quad -\Delta = V - A \;,$$

with A the adjacency matrix of a graph (entries $a_{ik} = 1$ or 0 depending on whether the nodes n_i and n_k are connected by a bond or not). V is the *'vertex degree matrix'*, its diagonal elements v_{ii} counting the number of bonds being incident with n_i (for more details see the literature cited in [13]). It is particularly noteworthy that $-\Delta$ or A encode many geometric/combinatorial graph properties being of general interest.

4 The Random Graph Aspect of the Dynamical Network and the Notion of Physical Points

We again have to be very brief; for more details we refer to [14], which represents, however, given the rapid development of this new field, only a preliminary draft. A more up to date version is forthcoming. The underlying idea is the following: Instead of studying the extremely complicated network dynamics in full, it is tempting to try to catch only its generic qualitative behavior. Following this idea a statistical approach suggests itself. One could e.g. assume that the network dynamics is sufficiently random so that *'graph properties'* can be modeled as *'random functions'* over a certain probability space. In a completely different spirit similar ideas have been developed quite some time ago by Erdős and Renyi and more recently by e.g. Bollobás (see [15]). It was then an important observation that many graph properties have a so-called *'threshold function'*, which is very reminiscent of a *'phase transition line'* in statistical physics.

We already remarked above that we are particularly interested in the possibility of geometric phase transitions in our network. Our hope is that something like a protoform of space-time may emerge as kind of a superstructure in the network. The elementary building blocks of this fabric, which we like to call '*physical points*' we expect to be made up of densely wired '*sub-clusters*' of nodes/bonds. These, on their side, are then assumed to establish a kind of near-/far-order in the network, thus generating something like a causal structure. One possibility to associate what we like to call '*physical points*' with a certain class of subgraphs is it to define them as the '*maximally connected subgraphs*' (i.e. subgraphs which are maximal simplices) or what graph theorist call a '*clique*'. Such cliques can be constructed or found in an algorithmic way starting from an arbitrary node ([14]).

5 More Graph Theoretical Definitions

In this section we give some more definitions of ordinary graph theory to discuss one of the aspects mentioned above, the dimensional concept on graphs, in more detail in the next section. Most of the notions are well known in graph theory but we nevertheless want to repeat them to avoid any confusion concerning the exact definitions.

We already introduced the undirected simple graph as the geometric aspect of a cellular network. In the following $\mathcal{G} = (N, B)$ will always be an undirected simple graph. We also need the notion of the degree of a node $n_i \in N$.

Definition 5.3 (Degree). *The* degree *of a node $n_i \in N$ is the number of bonds incident with it, i.e. the number of bonds which have n_i at one end. We count b_{ik} and b_{ki} only once as we interpret them as the same bond.*

We assume the node degree of any node $n_i \in N$ of the graphs under consideration to be finite. The next step is to define a metric structure on $\mathcal{G}$. To this end we need to define paths in $\mathcal{G}$ and their length.

Definition 5.4 (Path). *A* path *γ of length l in $\mathcal{G}$ is an ordered $(l+1)$ tuple of nodes $n_i \in N$, $i \in I$, $I = \{0, \ldots, l\}$ with the properties $n_{i+1} \neq n_i$ and $b_{i\,i+1} \in B$.*

A single node $n_i \in N$ is a path of length 0. This definition encodes the obvious idea of a path in $\mathcal{G}$ allowing multiple transversals of nodes or bonds. Jumps across non-existent bonds and stays at a single node are not allowed. Sometimes this notion of a path is also called a *bond sequence*.

We will call a path with the property that all $n_i \in \gamma$ are pairwise different a *simple path*.

The concept of paths on $\mathcal{G}$ now leads to a natural definition for the distance of two nodes n_i and $n_j \in N$, namely the length of the shortest path connecting n_i and n_j.

Definition 5.5 (Metric). *Let $l(\gamma)$ denote the length of γ. A* metric *d on $\mathcal{G}$ is*

$$d(n_i, n_j) := \begin{cases} \min\{l(\gamma) : n_i, n_j \in \gamma\} & \textit{if such } \gamma \textit{ exist} \\ \infty & \textit{otherwise.} \end{cases} \tag{10}$$

That this actually defines a metric is easily established. Finally we need the notion of neighborhoods which follows canonically from the metric.

Definition 5.6 (Neighborhood). *Let $n_i \in N$ be an arbitrary node in $\mathcal{G}$. An n- neighborhood of n_i is the set $\mathcal{U}_n(n_i) := \{n_j \in N : d(n_i, n_j) \leq n\}$.*

Remark. The topology generated by the n-neighborhoods is the discrete topology as should be expected from the construction and the discreteness of graphs.

We will denote the *surface* or *boundary* of the neighborhood $\mathcal{U}_n(n_i)$ as $\partial\mathcal{U}_n(n_i) := \mathcal{U}_n(n_i) \setminus \mathcal{U}_{n-1}(n_i)$, $\partial\mathcal{U}_0(n_i) = \{n_i\}$ and the cardinality of $\mathcal{U}_n(n_i)$ and $\partial\mathcal{U}_n(n_i)$ as $|\mathcal{U}_n(n_i)|$ and $|\partial\mathcal{U}_n(n_i)|$ respectively.

6 Dimensions of Graphs and Networks

Definition 6.7 (Internal Scaling Dimension). *Let $x \in N$ be an arbitrary node of $\mathcal{G}$. Consider the sequence of real numbers $D_n(x) := \frac{\ln|\mathcal{U}_n(x)|}{\ln(n)}$. We say $\underline{D}_S(x) := \liminf_{n\to\infty} D_n(x)$ is the* lower *and $\overline{D}_S(x) := \limsup_{n\to\infty} D_n(x)$ the* upper internal scaling dimension *of $\mathcal{G}$* starting from *x. If $\underline{D}_S(x) = \overline{D}_S(x) =: D_S(x)$ we say $\mathcal{G}$ has internal scaling dimension $D_S(x)$ starting from x. Finally, if $D_S(x) = D_S\ \forall x$, we simply say $\mathcal{G}$ has* internal scaling dimension *D_S.*

A second notion of dimension we want to introduce is the *connectivity dimension* which is based on the surfaces of neighborhoods $\partial\mathcal{U}_n(n_i)$ rather than on the whole neighborhoods $\mathcal{U}_n(n_i)$.

Definition 6.8 (Connectivity Dimension). *Let $x \in N$ again be an arbitrary node of $\mathcal{G}$. We set $\tilde{D}_n(x) := \frac{\ln|\partial\mathcal{U}_n(x)|}{\ln(n)} + 1$ and $\underline{D}_C(x) := \liminf_{n\to\infty} \tilde{D}_n(x)$ as the* lower *and $\overline{D}_C(x) := \limsup_{n\to\infty} \tilde{D}_n(x)$ as the* upper connectivity dimension. *If lower and upper dimension coincide, we say $\mathcal{G}$ has* connectivity dimension *$D_C(x) := \overline{D}_C(x) = \underline{D}_C(x)$* starting from *$x$. If $D_C(x) = D_C$ for all $x \in N$ we call D_C simply the* connectivity dimension *of $\mathcal{G}$.*

One could easily think that both definitions are equivalent. This is however not the case as one definition is stronger than the other. We will discuss this in 6.2.

The internal scaling dimension is rather a mathematical concept and is related to well known dimensional concepts in fractal geometry as we will see in 7.2. The connectivity dimension on the other hand seems to be a more physical concept as it measures more precisely how the graph is connected and thus how nodes can influence each other.

6.1 Basic Properties of the Internal Scaling Dimension

The first lemma gives us a criterion for the uniform convergence of $\underline{D}_S(x)$ or $\overline{D}_S(x)$ to some common $\underline{D}_S$ or $\overline{D}_S$ for all nodes x in $\mathcal{G}$.

Lemma 6.9. *Let $x,y \in N$ be two arbitrary nodes in $\mathcal{G}$ with $d(x,y) < \infty$. Then $\underline{D}_S(y) = \underline{D}_S(x)$* and *$\overline{D}_S(y) = \overline{D}_S(x)$.*

Proof. Let $a := d(x,y)$ be the distance of the nodes x and y. We have

$$\mathcal{U}_{n-a}(y) \subseteq \mathcal{U}_n(x) \subseteq \mathcal{U}_{n+a}(y) \tag{11}$$

$$\Longrightarrow \quad \frac{\ln|\mathcal{U}_{n-a}(y)|}{\ln(n)} \le \frac{\ln|\mathcal{U}_n(x)|}{\ln(n)} \le \frac{\ln|\mathcal{U}_{n+a}(y)|}{\ln(n)} \tag{12}$$

$$\Longrightarrow \quad \frac{\ln|\mathcal{U}_{n-a}(y)|}{\ln(n-a) + \ln\left(\frac{n}{n-a}\right)} \le \frac{\ln|\mathcal{U}_n(x)|}{\ln(n)} \le \frac{\ln|\mathcal{U}_{n+a}(y)|}{\ln(n+a) - \ln\left(\frac{n+a}{n}\right)} \tag{13}$$

$$\Longrightarrow \quad \underline{D}_S(x) = \liminf_{n\to\infty} \frac{\ln|\mathcal{U}_n(x)|}{\ln(n)} = \liminf_{n\to\infty} \frac{\ln|\mathcal{U}_n(y)|}{\ln(n)} = \underline{D}_S(y)\ . \tag{14}$$

Similarly we get $\overline{D}_S(x) = \overline{D}_S(y)$. □

Another rather technical lemma provides us with a convenient method to calculate the dimension of certain graphs, e.g. the self-similar or hierarchical graphs we construct in 7.2. It shows that under one technical assumption the convergence of a subsequence of $D_n(x)$ is sufficient for the convergence of $D_n(x)$ itself.

Lemma 6.10. *Let $x \in N$ be an arbitrary node of $\mathcal{G}$ and let $(|\mathcal{U}_{n_k}(x)|)_{k\in\mathbb{N}}$ be a subsequence of $(|\mathcal{U}_n(x)|)_{n\in\mathbb{N}}$. There may exist a number $1 > c > 0$ such that $\frac{n_k}{n_{k+1}} \geq c$ holds for all $k \geq K \in \mathbb{N}$. Then $\liminf_{k\to\infty} \frac{\ln|\mathcal{U}_{n_k}(x)|}{ln(n_k)} = \liminf_{n\to\infty} D_n(x) = \underline{D}_S(x)$ and similar for $\overline{D}_S(x)$.*

Proof. The sequence of the neighborhood sizes $|\mathcal{U}_n(x)|$ is monotone such that $|\mathcal{U}_{n_k}(x)| \leq |\mathcal{U}_n(x)| \leq |\mathcal{U}_{n_{k+1}}(x)|$ for $n_k \leq n \leq n_{k+1}$. A short calculation yields

$$\frac{\ln|\mathcal{U}_{n_k}(x)|}{\ln(n_k)+\ln(\frac{1}{c})} \leq \frac{\ln|\mathcal{U}_n(x)|}{\ln(n)} \leq \frac{\ln|\mathcal{U}_{n_{k+1}}(x)|}{\ln(n_{k+1})+\ln(c)} \;, \tag{15}$$

which implies the conjecture. □

This result is well known in the context of calculation schemes for dimensions in fractal geometry, see e.g. [2].

Naturally one also may ask how the internal scaling dimension behaves under insertion of bonds into $\mathcal{G}$. We were able to show that it is pretty much stable under any local changes. We state this in the following lemma.

Lemma 6.11. *Let $k \in \mathbb{N}$ be a positive natural number and $x \in N$ a node in $\mathcal{G}$. Insertion of bonds between arbitrary many pairs of nodes (y, z) obeying the relation $d(y,z) \leq k$ does not change $\underline{D}_S(x)$ or $\overline{D}_S(x)$.*

Proof. We denote the new graph built by insertion of new bonds into $\mathcal{G}$ as $\mathcal{G}'$ and accordingly the neighborhoods in $\mathcal{G}'$ as $\mathcal{U}'_n(\cdot)$. Being a node in $\mathcal{G}$, x is also a node in $\mathcal{G}'$. The restriction on the choice of additional bonds in $\mathcal{G}'$ implies that even if we connect every node $y \in N$ with every node in $\mathcal{U}_k(y)$, which is the maximum we are allowed to do, we still can't get beyond $\mathcal{U}_n(x)$ with less or equal $\lfloor\frac{n}{k}\rfloor$[2] steps,

$$\mathcal{U}_{\lfloor\frac{n}{k}\rfloor}(x) \subseteq \mathcal{U}'_{\lfloor\frac{n}{k}\rfloor}(x) \subseteq \mathcal{U}_n(x) \;. \tag{16}$$

A short calculation yields the equality $\liminf_{n\to\infty} \frac{\ln|\mathcal{U}'_n(x)|}{\ln(n)} = \liminf_{n\to\infty} \frac{\ln|U_n(x)|}{\ln(n)}$. The same holds for lim sup. □

Remark. Obviously the insertion of a *finite* number of additional bonds between nodes y and z with $d(y,z) < \infty$ doesn't change the internal scaling dimension either. Therefore we can slightly generalize lemma 6.11 by changing our requirements to the following. Only bonds between nodes of finite distance and only finitely many bonds between nodes of distance $d(y,z) > k$ are inserted into $\mathcal{G}$ to form $\mathcal{G}'$. Then $\mathcal{G}'$ still has the same internal scaling dimensions $\underline{D}_S$ and $\overline{D}_S$ as $\mathcal{G}$.

[2]The floor-symbol, $\lfloor x \rfloor$, denotes the largest integer below x, see e.g. [4]

Conclusions. We have seen that the internal scaling dimension does not depend on the node from which we start our calculation and that under not too strong conditions even the convergence of a subsequence of the relevant sequence $D_n(x)$ is sufficient to calculate $\underline{D}_S$ and $\overline{D}_S$. Furthermore the dimension is stable under local changes in the wiring of the graph. This is a very desirable feature for physical reasons. Furthermore it shows that a mechanism inducing dimensional phase transitions has to relate nodes of increasing distance, i.e. has to change the graph non-locally.

6.2 Relations Between Internal Scaling Dimension and Connectivity Dimension

As already stated above the two concepts of dimension we introduced are not equivalent. In the following lemma we show that the existence of the connectivity dimension implies the existence of the internal scaling dimension and that they then have the same value.

Lemma 6.12. *Let $x \in N$ again be an arbitrary node in $\mathcal{G}$. In the case that the limit $\lim_{n\to\infty} \frac{\ln|\partial\mathcal{U}_n(x)|}{\ln(n)} =: D_C(x) - 1$ exists with $D_C(x) > 1$, $\mathcal{G}$ has internal scaling dimension $D_S(x) = D_C(x)$ starting from x.*

Proof. The rather lengthy proof is given in detail in [1]. The main idea is that $|\mathcal{U}_n(x)| = \sum_{j=0}^{n} |\partial\mathcal{U}_j(x)|$ such that $|\mathcal{U}_n(x)|$ can be approximated with the knowledge of the behavior of $|\partial\mathcal{U}_n(x)|$. After some calculations one gets the desired result from this. □

Inversely, the existence of the internal scaling dimension does not imply the existence of the connectivity dimension. There are examples in which the scaling dimension has a well defined value but the connectivity dimension does not exist. Even neither the upper nor the lower connectivity dimension need to coincide with the scaling dimension. In [1] we gave an example in which $D_S = D, \underline{D}_C = 0$ and $\overline{D}_C = D + 1$.

The only always valid assertion is $\overline{D}_C + 1 \leq D_S(x)$.

7 Construction of Graphs

In the following we want to show how to construct graphs of arbitrary real internal scaling dimension. We also want to investigate the connections between the internal scaling dimension of graphs and the box counting dimension of fractal sets. As will been seen below there is a strong relationship between self similar sets and what we also want to call self similar graphs with non-integer internal scaling dimension.

7.1 Conical Graphs with Arbitrary Dimension

For the sake of simplicity we concentrate our discussion on graphs with dimension $1 \leq D \leq 2$. Graphs with higher dimension are easily constructed using a nearly identical scheme.

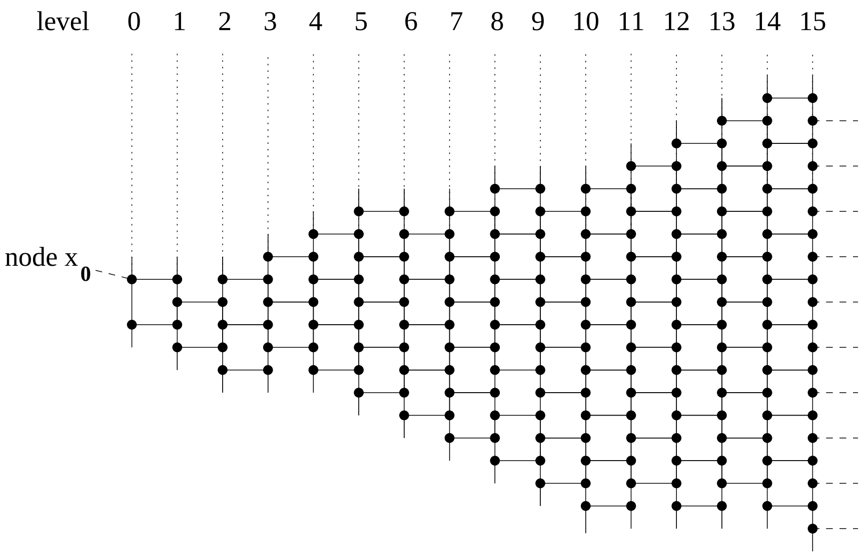

Figure 1: Example of a $\frac{5}{3}$ dimensional conical graph

Let $1 \leq D \leq 2$ be an arbitrary real number. Now we construct the graph like in figure 1. On level m we use a height of $\lfloor (2m-1)^{D-1} \rfloor$ boxes. The construction is continued "to the right" infinitely. To calculate the dimension we observe that starting from x_0 we reach level m after $n = 2m - 1$ steps. Thus we get with $n_k := 2k - 1$

$$(17) \qquad |\partial \mathcal{U}_{n_k}(x_0)| = \lfloor n_k^{D-1} \rfloor \quad \Longrightarrow \quad \lim_{k \to \infty} \frac{\ln |\partial \mathcal{U}_{n_k}(x_0)|}{\ln(n_k)} = D - 1 \ .$$

Using lemmas 6.12, 6.9 and 6.10 we see that this graph has internal scaling dimension $D_S = D$. If we close the construction vertically, i.e. introduce bonds between the uppermost and the lowest nodes on each level we even can achieve a completely homogeneous node degree $d = 3$.

Remark. Locally the constructed conical graph is completely isomorphic to a two-dimensional lattice. The non-integer dimension is only implemented as a global property of the graph.

7.2 Self-Similar Graphs

It is well known in graph theory that it is notoriously difficult to construct large graphs with prescribed properties. It also proved quite difficult to construct graphs with a prescribed (internal scaling) dimension $D_S = D$ which don't exhibit the disadvantages of the conical graphs described above. The main idea which solves the problem is to use the well known theory of self similar sets or fractals and their dimension theory. In the following we want to show how this works and that we indeed can construct adjoint graphs to self similar sets which have internal scaling dimension equal to the box counting dimension of the self similar sets.

Given a strictly self similar set in $\mathbb{R}^p$ we canonically construct an adjoint graph which also will be called self-similar. The construction principle is based on an algorithm to compute the box counting dimension of a self-similar set.

For details concerning self-similar sets and dimensions of fractals see [2].

7.2.1 Construction Based on Self-Similar Sets

Let M be a strictly self-similar set with similarity transforms S_i, $i \in I$, $I \subset \mathbb{N}$ and $|I| < \infty$. The contraction factors c_i of S_i may all be equal, $c_i = c \in (0, 1)$. Now we cover M with cubic lattices $L_n \subset \mathbb{R}^p$ with closed cubes of edge length c^n, $n \in \mathbb{N}$, and replace every cube which has non-void intersection with M by a node. Nodes will be connected iff the corresponding cubes in the covering cubic lattices have a non-void intersection, i.e. have a common corner or edge.

By this construction we get a finite graph $\mathcal{G}_n$ for each $n \in \mathbb{N}$. The degree of these $\mathcal{G}_n$ is uniformly bounded because an n-dimensional cube can only touch a finite number of neighbor cubes in the cubic lattice. The graph we are interested in is $\mathcal{G}_\infty$, the graph we get through infinite continuation of our construction.

Remark.

1. No problems arise from the infinite continuation of the construction steps.

2. The self-similarity of M transfers to $\mathcal{G}$ in the sense that we can also define an equivalent of the similarity transforms of the self-similar set M.

3. Connected self-similar sets produce connected self-similar graphs. The inverse is not true in general though.

7.2.2 Self-Contained Construction Algorithm

There also are self-contained construction algorithms for self-similar or hierarchical graphs. One possibility is the following algorithm:

1. We start with a single node, $\mathcal{G}_0 = (\{n_0\}, \emptyset)$.

2. $\mathcal{G}_1$ is the so-called generator, some finite graph.

3. We construct $\mathcal{G}_{n+1}$ from $\mathcal{G}_n$ by replacing every node in $\mathcal{G}_n$ by the generator $\mathcal{G}_1$ and interpret the original bonds in $\mathcal{G}_n$ as bonds between some "marginal" nodes of the different copies of $\mathcal{G}_1$.

The construction is not unique. The result strongly depends on the choice of the nodes in $\mathcal{G}_{n+1}$ which carry the bonds of $\mathcal{G}_n$. In our example all "marginal" nodes of the generator are equivalent because of the symmetry of the generator and therefore the construction is unique.

A slightly different construction algorithm with identical results is also possible. We will not describe it here for lack of space. Refer to [1] for more details.

7.2.3 Dimension of Self-Similar Graphs

Now we calculate the dimension of the graphs we get by the above construction using some self-similar set M. For the sake of simplicity we assume that $\mathcal{G}_1$ has a central node x_0 in the sense that all "marginal" nodes which carry the "outer" bonds have all the same distance r to this node. We further assume that $\frac{1}{c}$ (c the contraction parameter) is a natural number which is true in most of the well known examples of self-similar sets and finally that the self-similar set produces a connected adjoint graph. Then it is easy to see that starting from node x_0 we can exactly reach all nodes of construction step $k+1$ after $n_{k+1} = r + 2r\, n_k + n_k = (2r+1)\, n_k + r$ steps in the graph, with $n_0 = 0$. Thus $|\mathcal{U}_{n_k}(x_0)|$ is equal to the number of nodes in construction step k, i.e. $|\mathcal{U}_{n_k}(x_0)| = N_{\delta_k} = N_{c^k}$.[3] Explicitly we get for n_k

$$n_k = \sum_{j=0}^{k-1} (2r+1)^j\, r = r\,\frac{(2r+1)^k - 1}{2r} \qquad \forall k \geq 1 \;. \tag{18}$$

Now let us relate r to the contraction parameter c of the self-similar set. We assumed that the graph constructed from the self-similar set is connected. This implies that there are $\frac{1}{c}$ nodes on the "diagonal" of the generator, i.e. $2r+1 = \frac{1}{c}$. Now we have for the internal scaling dimension of $\mathcal{G}$

$$\lim_{k\to\infty} D_{n_k}(x_0) = \lim_{k\to\infty} \frac{\ln(N_{c^k})}{\ln\left(r\frac{(2r+1)^k-1}{2r}\right)} \tag{19}$$

$$= \lim_{k\to\infty} \frac{\ln(N_{c^k})}{\ln((2r+1)^k) + \ln\left(\frac{1-(2r+1)^{-k}}{2r}\right)} \tag{20}$$

$$= \lim_{k\to\infty} \frac{\ln(N_{c^k})}{-\ln(c^k) + \ln\left(\frac{1-(2r+1)^{-k}}{2r}\right)} = \dim_B(M) \tag{21}$$

in which $\dim_B(M)$ is the box counting dimension of M. Of course lemmas 6.9 and 6.10 provide us with the knowledge that this is the dimension of $\mathcal{G}$ starting from any node.

Thus we established equality of the box counting dimension of self-similar sets and the internal scaling dimension of the adjoint self-similar graphs under the assumptions stated above.

Remark. The assumed existence of a central node x_0 is not essential for the equality of the dimensions of the fractal and the graph. The equality still holds in a more general context, e.g. for fractals like the Sirpinski Triangle. It is difficult though to give a general proof for arbitrary self-similar sets.

7.2.4 Approximation of a Two Dimensional Lattice

In this paragraph we want to show how it now becomes possible to do a dimensional approximation of a n-dimensional cubic lattice. Again, for the sake

[3] N_{δ_k} is the number of cubes of edge length δ_k intersecting M, see the calculation of the box counting dimension in e.g. [2].

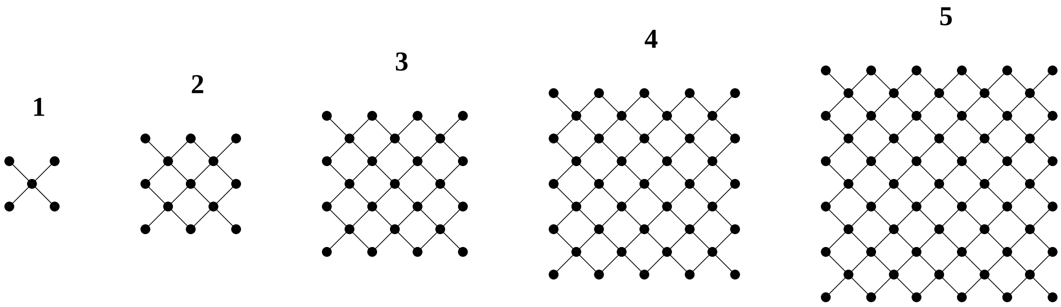

Figure 2: Some generators to approximate $\mathbb{Z}^2$

of simplicity, we discuss the idea only with a two-dimensional lattice but the generalization to n dimensions is obvious.

We introduce generators as shown in figure 2. With these we get graphs of dimensions

$$D_S^{(l)} = \frac{\ln(2l^2 + 2l + 1)}{\ln(2l + 1)} \tag{22}$$

in which l is the number which labels the generators in figure 2. Obviously we have

$$\lim_{l \to \infty} D_S^{(l)} = \lim_{l \to \infty} \frac{\ln(2l^2 + 2l + 1)}{\ln(2l + 1)} = \lim_{l \to \infty} \frac{2\ln(l) + \ln(2 + \frac{2}{l} + \frac{1}{l^2})}{\ln(l) + \ln(2 + \frac{1}{l})} = 2\ . \tag{23}$$

In this sense we have a dimensional approximation of a two-dimensional lattice as alleged above. This might have some relevance in connection with the dimensional regularization used in many renormalization approaches to quantum field theory.

Remark. The generators above correspond to fractal sets known as "sponges", see e.g. [3]. We can construct such "sponges" for any dimension n, we just need to modify the generators appropriately.

References

[1] T. Nowotny, M. Requardt, Göttingen preprint (to appear in J. Phys. A), hep-th/9707082

[2] K. J. Falconer, "Fractal Geometry. Mathematical Foundations and Applications", J. Wiley & Sons, Chichester 1990

[3] G. A. Edgar, "Measure, Topology and Fractal Geometry", Springer, New York 1990

[4] R.L. Graham, D.E. Knuth, O. Patashnik, "Concrete Mathematics", Edison-Wesley, NY 2nd Edition 1988

[5] C.Rovelli, "Loop Quantum Gravity", gr-qc/9710008

[6] L.Smolin, "The Future of Spin Networks", gr-qc/9702030

[7] C.J.Isham, "Structural Issues in Quantum Gravity", gr-qc/9510063 (Lecture given at the GR 14 conference, Florence 1995)

[8] C.Castro, "Beyond Strings ... ", hep-th/9707171

[9] A.Connes, "Non-commutative Geometry", Acad.Pr. N.Y. 1994

[10] Bollobás, "Graph Theory", Graduate Texts in Mathematics, Springer N.Y. 1979

[11] T.Toffoli, N.Margolus, "Cellular Automaton Machines", MIT Pr., Cambridge Massachusetts 1987

[12] M.Requardt, "Discrete Mathematics and Physics on the Planck-Scale ... ", hep-th/9605103

[13] M.Requardt, "A New Approach to Functional Analysis on Graphs ... ", hep-th/9708010

[14] M.Requardt, "Emergence of Space-Time on the Planck Scale ... ", hep-th/9610055

[15] B.Bollobás, "Random Graphs", Academic Pr. London 1985

[16] L.Nottale, "Fractal Space Time and Microphysics ... ", World Scientific 1992

[17] A. Dimakis, F. Müller-Hoissen, J.Math.Phys **35** (1994) p. 6703

[18] R. D. Sorkin, Int. J. Theor. Phys. **30** (1991) p. 923

[19] A.P. Balachandran et al, J. Geom. Phys. **18** (1996) p. 163