Dimension Theory of Graphs and Networks

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Abstract

Starting from the working hypothesis that both physics and the corresponding mathematics have to be described by means of discrete concepts on the Planck-scale, one of the many problems one has to face in this enterprise is to find the discrete protoforms of the building blocks of continuum physics and mathematics. A core concept is the notion of *dimension*. In the following we develop such a notion for irregular structures like (large) graphs and networks and derive a number of its properties. Among other things we show its stability under a wide class of perturbations which is important if one has '*dimensional phase transitions*' in mind. Furthermore we systematically construct graphs with almost arbitrary '*fractal dimension*' which may be of some use in the context of '*dimensional renormalization*' or statistical mechanics on irregular sets.

1 Introduction

In two recent papers ([1],[2]) we developed a certain framework in form of a class of '*cellular network dynamics*' which are designed to mimic the dynamics of the physical vacuum or space-time on the Planck-scale. In doing this our working philosophy was that both physics and the corresponding mathematics are genuinely discrete on this primordial level. The continuum concepts of ordinary space-time physics are then supposed to emerge from certain discrete patterns via a kind of '*renormalization group process*' on the much coarser scale of resolution given by the comparatively small energies of present day high energy physics. It is one of our aims to find these discrete protoforms.

A crucial concepts in this context is a version of '*intrinsic dimension*' of such discrete irregular networks which geometrically are graphs. This concept should be defined in an intrinsic way, without making open or implicit recourse to continuum concepts whatsoever or kind of an embedding dimension, as we want to understand, among other things, what properties actually are encoded in a notion like dimension on the most fundamental physical level. On the other side, we want to know how the continuum concept of dimension, which is to a large extent of an a priori mathematical viz. geometrical origin, comes into being, starting from an intrinsic property of discrete irregular systems like e.g. general, typically very large and almost randomly organized graphs which are supposed to encode the '*geometrodynamics*' of space-time on Planck scale.

In section 5 of [1] we introduced such a concept which seems suitable to us and which characterizes to some extent the '*wiring*' of the network. At the time of writing [1] we scanned the literature accessible to us in vain for similar ideas and got the impression that such lines of thought had not been pursued in this context. Some time later we were kindly informed by Thomas Filk that a similar concept had been studied by himself and a couple of other physicists (see [3],[4],[5] and further references given there) in an however slightly different context. (They typically investigated the simplicial resolution of continuous manifolds and their numerical treatment via Monte Carlo simulations).

On the other side, at least as far as we can see, this concept had not been systematically developed and many questions of principal interest remained open. In the following we attempt to formulate and solve a couple of problems which naturally emerge in this context, more specifically we embark on developing a full fledged mathematical machinery around this concept which then may be applied to quite diverse fields of physics and mathematics.

Among other things we clarify the somewhat hidden relations to certain parts of '*fractal geometry*' and construct graphs with almost arbitrary '*fractal dimensions*' along these lines. Furthermore we show that the two at first glance almost identical definitions of dimension we introduced in [1] are actually different on certain '*exceptional*' sets while, on the other side, being identical on '*generic*' sets. This is a phenomenon also well known from the various notions of dimension in fractal geometry.

While the first one, which we will call '*internal scaling dimension*' in the following (it is the version which occurs under this label in e.g. [3]), appears to be more natural from a mathematical point of view, the second one, on the other side, is in our opinion more fundamental as far as the encoding of physical data as e.g. the wiring of the graphs under discussion is concerned. For this reason we call it the '*connectivity dimension*' as it reflects to some extent the way the node states are interacting with each other over larger distances via the various bond sequences connecting them.

Another interesting point is the structural stability of such a concept under local and extended perturbations. We showed e.g. that if we start from a given graph with a dimension D this value remains stable under a rather large class of bond insertions. As a consequence one has to add bonds between increasingly distant nodes in order to change the dimension of a graph. This is of some relevance if one wants to invent dynamical mechanisms which are designed to trigger dimensional phase transitions.

Presently we pursue several lines of research concerning applications in quite diverse fields of physics and mathematics as e.g. non-commutative geometry, dimensional phase transitions (see also [2]), statistical mechanics and functional analysis.

2 Graph Theoretical Definitions

In this section we give the necessary definitions to define the internal scaling dimension of graphs. Most of the notions are well known in graph theory but we nevertheless want to repeat them to avoid any confusion concerning the exact definitions.

First of all we need to define an undirected simple graph. This will be our primary object of interest.

Definition 2.1 Undirected Simple Graph. *An* undirected simple graph *consists of two countable sets N and B. We denote the elements of N as n_i with $i \in I$, $I \subseteq \mathbb{N}$. The elements of B are denoted as b_{ik}, $i, k \in I$. The set B is isomorphic to a subset of $N \times N$ and the existence of b_{ik} implies the existence of b_{ki}.*

Remark. Many mathematicians use a slightly different notation. They denote N (nodes) as V (vertices) and B (bonds) as E (edges).

In the following $\mathcal{G} = (N, B)$ will always be an undirected simple graph. We also need the notion of the degree of a node $n_i \in N$.

Definition 2.2 Degree. *The* degree *of a node $n_i \in N$ is the number of bonds incident with it, i.e. the number of bonds which have n_i at one end. We count b_{ik} and b_{ki} only once as we interpret them as the same bond.*

We assume the node degree of any node $n_i \in N$ of the graphs under consideration to be finite. The next step is to define a metric structure on $\mathcal{G}$. To this end we need to define paths in $\mathcal{G}$ and their length.

Definition 2.3 Path. *A* path γ *of length l in $\mathcal{G}$ is an ordered $(l+1)$ tuple of nodes $n_i \in N$, $i \in I$, $I = \{0, \ldots, l\}$ with the properties $n_{i+1} \neq n_i$ and $b_{i\,i+1} \in B$.*

Remark. A single node $n_i \in N$ is a path of length 0.

This definition encodes the obvious idea of a path in $\mathcal{G}$ allowing multiple transversals of nodes or bonds. Jumps across non-existent bonds and stays at a single node are not allowed. Sometimes this notion of a path is also called a *bond sequence*.

Slightly different definitions are also quite common. The path often is restricted to contain any bond in B at most once. Sometimes even the repetition of nodes in a path is excluded. We will call a path with this property – that all $n_i \in \gamma$ are pairwise different – a simple path.

The concept of paths on $\mathcal{G}$ now leads to a natural definition for the distance of two nodes n_i and $n_j \in N$, namely the length of the shortest path connecting n_i and n_j.

Definition 2.4 Metric. *A* metric d *on $\mathcal{G}$ is defined by*

$$d(n_i, n_j) := \begin{cases} \min\{l(\gamma) : n_i, n_j \in \gamma\} & \text{if such } \gamma \text{ exist} \\ \infty & \text{otherwise,} \end{cases} \tag{1}$$

in which $l(\gamma)$ denotes the length of γ.

That this actually defines a metric is easily established. Finally we need the notion of neighborhoods which follows canonically from the metric.

Definition 2.5 Neighborhood. *Let $n_i \in N$ be an arbitrary node in $\mathcal{G}$. An n-neighborhood of n_i is the set $\mathcal{U}_n(n_i) := \{n_j \in N : d(n_i, n_j) \leq n\}$.*

Remark. The topology generated by the n-neighborhoods is the discrete topology as should be expected from the construction and the discreteness of graphs.

We will denote the *surface* or *boundary* of the neighborhood $\mathcal{U}_n(n_i)$ as $\partial\mathcal{U}_n(n_i) := \mathcal{U}_n(n_i) \setminus \mathcal{U}_{n-1}(n_i)$, $\partial\mathcal{U}_0(n_i) = \{n_i\}$ and the cardinality of $\mathcal{U}_n(n_i)$ and $\partial\mathcal{U}_n(n_i)$ as $|\mathcal{U}_n(n_i)|$ and $|\partial\mathcal{U}_n(n_i)|$ respectively.

3 Dimensions of Graphs and Networks

Now we have all the tools to define the central notion of this paper, the notion of the *internal scaling dimension* of $\mathcal{G}$.

Definition 3.6 Internal Scaling Dimension. *Let $x \in N$ be an arbitrary node of $\mathcal{G}$. Consider the sequence of real numbers $D_n(x) := \frac{\ln|\mathcal{U}_n(x)|}{\ln(n)}$. We say $\underline{D}_S(x) := \liminf_{n\to\infty} D_n(x)$ is the* lower *and $\overline{D}_S(x) := \limsup_{n\to\infty} D_n(x)$ the* upper internal scaling dimension *of $\mathcal{G}$* starting from x. *If $\underline{D}_S(x) = \overline{D}_S(x) =: D_S(x)$ we say $\mathcal{G}$ has internal scaling dimension $D_S(x)$ starting from x. Finally, if $D_S(x) = D_S \ \forall x$, we simply say $\mathcal{G}$ has* internal scaling dimension D_S.

A second notion of dimension we want to introduce is the *connectivity dimension* which is based on the surfaces of neighborhoods $\partial\mathcal{U}_n(n_i)$ rather than on the whole neighborhoods $\mathcal{U}_n(n_i)$.

Definition 3.7 Connectivity Dimension. *Let $x \in N$ again be an arbitrary node of $\mathcal{G}$. We set $\tilde{D}_n(x) := \frac{\ln|\partial\mathcal{U}_n(x)|}{\ln(n)} + 1$ and define $\underline{D}_C(x) := \liminf_{n\to\infty} \tilde{D}_n(x)$ as the* lower *and $\overline{D}_C(x) := \limsup_{n\to\infty} \tilde{D}_n(x)$ as the* upper connectivity dimension. *If lower and upper dimension coincide, we say $\mathcal{G}$ has* connectivity dimension $D_C(x) := \overline{D}_C(x) = \underline{D}_C(x)$ starting from x. *If $D_C(x) = D_C$ for all $x \in N$ we call D_C simply the* connectivity dimension *of $\mathcal{G}$.*

One could easily think that both notions of dimension are equivalent. This is however not the case as one definition is stronger than the other which will be shown in detail in 3.2.

The internal scaling dimension is rather a mathematical concept and is related to well known dimensional concepts in fractal geometry as we will see in 4.2. The connectivity dimension on the other hand seems to be a more physical concept as it measures more precisely how the graph is connected and thus how nodes can influence each other.

In the following section we want to establish the basic properties of the internal scaling dimension of graphs.

3.1 Basic Properties of the Internal Scaling Dimension

The first lemma gives us a criterion for the uniform convergence of $\underline{D}_S(x)$ or $\overline{D}_S(x)$ to some common $\underline{D}_S$ or $\overline{D}_S$ for all nodes x in $\mathcal{G}$.

Lemma 3.8. *Let $x,y \in N$ be two arbitrary nodes in $\mathcal{G}$ with $d(x,y) < \infty$. Then $\underline{D}_S(y) = \underline{D}_S(x)$ and $\overline{D}_S(y) = \overline{D}_S(x)$.*

Proof. Let $a := d(x,y)$ be the distance of the nodes x and y. We have

$$\mathcal{U}_{n-a}(y) \subseteq \mathcal{U}_n(x) \subseteq \mathcal{U}_{n+a}(y) \tag{2}$$

$$(3) \qquad \Longrightarrow \quad \frac{\ln|\mathcal{U}_{n-a}(y)|}{\ln(n)} \le \frac{\ln|\mathcal{U}_n(x)|}{\ln(n)} \le \frac{\ln|\mathcal{U}_{n+a}(y)|}{\ln(n)}$$

$$(4) \qquad \Longrightarrow \quad \frac{\ln|\mathcal{U}_{n-a}(y)|}{\ln(n-a)+\ln\left(\frac{n}{n-a}\right)} \le \frac{\ln|\mathcal{U}_n(x)|}{\ln(n)} \le \frac{\ln|\mathcal{U}_{n+a}(y)|}{\ln(n+a)-\ln\left(\frac{n+a}{n}\right)}$$

$$(5) \qquad \Longrightarrow \quad \underline{D}_S(x) = \liminf_{n\to\infty} \frac{\ln|\mathcal{U}_n(x)|}{\ln(n)} = \liminf_{n\to\infty} \frac{\ln|\mathcal{U}_n(y)|}{\ln(n)} = \underline{D}_S(y)\ .$$

Similarly we get $\overline{D}_S(x) = \overline{D}_S(y)$. □

Another rather technical lemma provides us with a convenient method to calculate the dimension of certain graphs, e.g. the self-similar or hierarchical graphs we construct in 4.2. It shows that under one technical assumption the convergence of a subsequence of $D_n(x)$ is sufficient for the convergence of $D_n(x)$ itself.

Lemma 3.9. *Let $x \in N$ be an arbitrary node of $\mathcal{G}$ and let $(|\mathcal{U}_{n_k}(x)|)_{k\in\mathbb{N}}$ be a subsequence of $(|\mathcal{U}_n(x)|)_{n\in\mathbb{N}}$. There may exist a number $1 > c > 0$ such that $\frac{n_k}{n_{k+1}} \ge c$ holds for all $k \ge K \in \mathbb{N}$. Then $\liminf_{k\to\infty} \frac{\ln|\mathcal{U}_{n_k}(x)|}{ln(n_k)} = \liminf_{n\to\infty} D_n(x) = \underline{D}_S(x)$ and similar for $\overline{D}_S(x)$.*

Proof. Let $n \in \mathbb{N}$ be an arbitrary natural number. We find a $k \in \mathbb{N}$ such that $n_k \le n \le n_{k+1}$. As the sequence $(|\mathcal{U}_n(x)|)$ is monotone this implies $|\mathcal{U}_{n_k}(x)| \le |\mathcal{U}_n(x)| \le |\mathcal{U}_{n_{k+1}}(x)|$. Therefore we get

$$(6) \qquad \frac{\ln|\mathcal{U}_{n_k}(x)|}{\ln(n)} \le \frac{\ln|\mathcal{U}_n(x)|}{\ln(n)} \le \frac{\ln|\mathcal{U}_{n_{k+1}}(x)|}{\ln(n)}$$

$$(7) \qquad \Longrightarrow \quad \frac{\ln|\mathcal{U}_{n_k}(x)|}{\ln(n_k)+\ln\left(\frac{n}{n_k}\right)} \le \frac{\ln|\mathcal{U}_n(x)|}{\ln(n)} \le \frac{\ln|\mathcal{U}_{n_{k+1}}(x)|}{\ln(n_{k+1})+\ln\left(\frac{n}{n_{k+1}}\right)}$$

$$(8) \qquad \Longrightarrow \quad \frac{\ln|\mathcal{U}_{n_k}(x)|}{\ln(n_k)+\ln(\frac{1}{c})} \le \frac{\ln|\mathcal{U}_n(x)|}{\ln(n)} \le \frac{\ln|\mathcal{U}_{n_{k+1}}(x)|}{\ln(n_{k+1})+\ln(c)}$$

$$(9) \qquad \Longrightarrow \quad \liminf_{n\to\infty} D_n(x) = \liminf_{k\to\infty} \frac{\ln|\mathcal{U}_{n_k}(x)|}{\ln(n_k)}\ .$$

The same proof holds for lim sup. □

This result is well known in the context of calculation schemes for dimensions in fractal geometry, see e.g. [6].

Naturally one also may ask how the internal scaling dimension behaves under insertion of bonds into $\mathcal{G}$. We were able to show that it is pretty much stable under any local changes. We state this in the following lemma.

Lemma 3.10. *Let $k \in \mathbb{N}$ be a positive natural number and $x \in N$ a node in $\mathcal{G}$. Insertion of bonds between arbitrary many pairs of nodes (y, z) obeying the relation $d(y, z) \leq k$ does not change $\underline{D}_S(x)$ or $\overline{D}_S(x)$.*

Proof. We denote the new graph built by insertion of new bonds into $\mathcal{G}$ as $\mathcal{G}'$ and accordingly the neighborhoods in $\mathcal{G}'$ as $\mathcal{U}'_n(\cdot)$. Being a node in $\mathcal{G}$, x is also a node in $\mathcal{G}'$. The restriction on the choice of additional bonds in $\mathcal{G}'$ implies that even if we connect every node $y \in N$ with every node in $\mathcal{U}_k(y)$, which is the maximum we are allowed to do, we still can't get beyond $\mathcal{U}_n(x)$ with less or equal $\lfloor \frac{n}{k} \rfloor$ steps,

$$\mathcal{U}_{\lfloor \frac{n}{k} \rfloor}(x) \subseteq \mathcal{U}'_{\lfloor \frac{n}{k} \rfloor}(x) \subseteq \mathcal{U}_n(x) \tag{10}$$

$$\Longrightarrow \quad \frac{\ln |\mathcal{U}_{\lfloor \frac{n}{k} \rfloor}(x)|}{\ln(\lfloor \frac{n}{k} \rfloor)} \leq \frac{\ln |\mathcal{U}'_{\lfloor \frac{n}{k} \rfloor}(x)|}{\ln(\lfloor \frac{n}{k} \rfloor)} \leq \frac{\ln |\mathcal{U}_n(x)|}{\ln(\lfloor \frac{n}{k} \rfloor)} \,. \tag{11}$$

Because $\lfloor \frac{n}{k} \rfloor \geq \frac{n}{2k}$ for sufficiently large n, we immediately get

$$\frac{\ln |\mathcal{U}_{\lfloor \frac{n}{k} \rfloor}(x)|}{\ln(\lfloor \frac{n}{k} \rfloor)} \leq \frac{\ln |\mathcal{U}'_{\lfloor \frac{n}{k} \rfloor}(x)|}{\ln(\lfloor \frac{n}{k} \rfloor)} \leq \frac{\ln |\mathcal{U}_n(x)|}{\ln(n) - \ln(2k)} \tag{12}$$

$$\Longrightarrow \quad \liminf_{n\to\infty} \frac{\ln |\mathcal{U}'_n(x)|}{\ln(n)} = \liminf_{n\to\infty} \frac{\ln |U_n(x)|}{\ln(n)} \tag{13}$$

where in the last step lemma 3.9 has been used. The identical result holds for lim sup.

□

Remark. Obviously the insertion of a *finite* number of additional bonds between nodes y and z with $d(y, z) < \infty$ doesn't change the internal scaling dimension either. Therefore we can slightly generalize lemma 3.10 by changing our requirements to the following. Only bonds between nodes of finite distance and only finitely many bonds between nodes of distance $d(y, z) > k$ are inserted into $\mathcal{G}$ to form $\mathcal{G}'$. Then $\mathcal{G}'$ still has the same internal scaling dimensions $\underline{D}_S$ and $\overline{D}_S$ as $\mathcal{G}$.

Conclusions. We have seen that the internal scaling dimension does not depend on the node from which we start our calculation and that under not too strong conditions even the convergence of a subsequence of the relevant sequence $D_n(x)$ is sufficient to calculate $\underline{D}_S$ and $\overline{D}_S$. Furthermore the dimension is stable under local changes in the wiring of the graph. This is a very desirable feature for physical reasons. Furthermore it shows that a mechanism inducing dimensional phase transitions has to relate nodes of increasing distance, i.e. has to change the graph non-locally. We will illustrate this fact with an example in 4.2.5.

3.2 Relations Between Internal Scaling Dimension and Connectivity Dimension

As already stated above the two concepts of dimension we introduced are not equivalent. In the following lemma we show that the existence of the connectivity dimension implies the existence of the internal scaling dimension and that they then have the same value.

Lemma 3.11. *Let $x \in N$ again be an arbitrary node in $\mathcal{G}$. In the case that the limit $\lim_{n\to\infty} \frac{\ln|\partial\mathcal{U}_n(x)|}{\ln(n)} =: D_C(x) - 1$ exists with $D_C(x) > 1$, $\mathcal{G}$ has internal scaling dimension $D_S(x) = D_C(x)$ starting from x.*

Proof. We know that $D_C(x) > 1$ exists and have to show that this implies the existence of $\lim_{n\to\infty} \frac{\ln|\mathcal{U}_n(x)|}{\ln(n)}$ and that the limit is $D_C(x)$. Let $D := D_C(x)$ and $\epsilon > 0$ be an arbitrary positive number small enough such that $D - 1 - \epsilon > 0$. From the convergence of $\frac{\ln|\partial\mathcal{U}_n(x)|}{\ln(n)}$ we know that we can find $N \in \mathbb{N}$ such that

$$\left| \frac{\ln|\partial\mathcal{U}_n(x)|}{\ln(n)} - D + 1 \right| < \epsilon \qquad \forall n \geq N \tag{14}$$

$$\Longrightarrow \quad -\epsilon < \frac{\ln|\partial\mathcal{U}_n(x)|}{\ln(n)} - D + 1 < \epsilon \tag{15}$$

$$\Longrightarrow \quad (D-1-\epsilon)\ln(n) < \ln|\partial\mathcal{U}_n(x)| < (D-1+\epsilon)\ln(n) \tag{16}$$

$$\Longrightarrow \quad n^{D-1-\epsilon} < |\partial\mathcal{U}_n(x)| < n^{D-1+\epsilon}\ . \tag{17}$$

On the other hand we naturally have

$$|\mathcal{U}_n(x)| = \sum_{j=0}^{n} |\partial\mathcal{U}_j(x)| \tag{18}$$

$$\Longrightarrow \quad K(N) + \sum_{j=N+1}^{n} j^{D-1-\epsilon} \leq |\mathcal{U}_n(x)| \leq K(N) + \sum_{j=N+1}^{n} j^{D-1+\epsilon} \tag{19}$$

in which $K(N) = \sum_{j=0}^{N} |\partial\mathcal{U}_j(x)|$. Now we can give a lower bound for the sum on the left hand side and an upper bound for the one on the right hand side by replacing them with integrals.

$$\sum_{j=N+1}^{n} j^{D-1-\epsilon} \geq \int_N^n j^{D-1-\epsilon} \mathrm{d}j = \left. \frac{j^{D-\epsilon}}{D-\epsilon} \right|_N^n \tag{20}$$

$$\sum_{j=N+1}^{n} j^{D-1+\epsilon} \leq \int_{N+1}^{n+1} j^{D-1+\epsilon} \mathrm{d}j = \left. \frac{j^{D+\epsilon}}{D+\epsilon} \right|_{N+1}^{n+1} \tag{21}$$

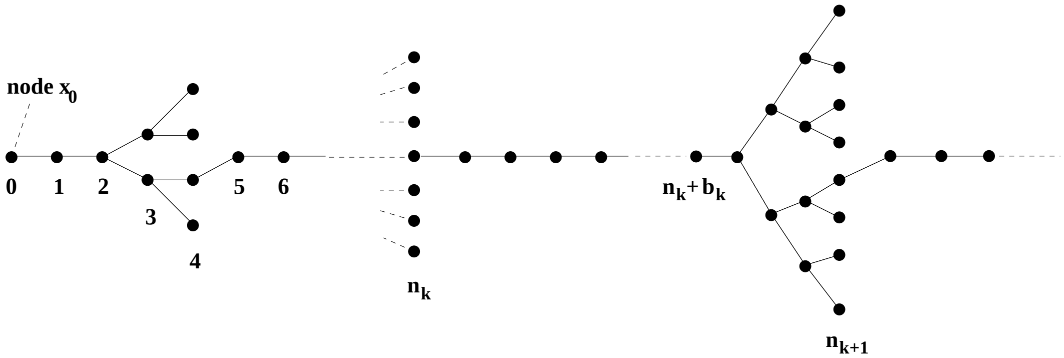

Figure 1: Example of a graph with strange behavior of $\tilde{D}_n(x_0) = \frac{\ln|\partial\mathcal{U}_n(x_0)|}{\ln(n)}$

With these bounds we get

$$(22)\qquad \ln\left(K(N) + \frac{n^{D-\epsilon}-N^{D-\epsilon}}{D-\epsilon}\right) \le \ln|\mathcal{U}_n| \le \ln\left(K(N) + \frac{(n+1)^{D+\epsilon}-(N+1)^{D+\epsilon}}{D+\epsilon}\right)$$

$$(23)\qquad \Longrightarrow \quad \ln(n^{D-\epsilon}) + \ln\left(\frac{K(N)}{n^{D-\epsilon}} + \frac{1}{D-\epsilon}\left(1 - \frac{N^{D-\epsilon}}{n^{D-\epsilon}}\right)\right) \le \ln|\mathcal{U}_n|$$
$$\le \ln\left((n+1)^{D+\epsilon}\right) + \ln\left(\frac{K(N)}{(n+1)^{D+\epsilon}} + \frac{1}{D+\epsilon}\left(1 - \frac{(N+1)^{D+\epsilon}}{(n+1)^{D+\epsilon}}\right)\right)$$

Because the arguments of the second logarithm on each side are uniformly bounded for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n)} = 1$, we can find an $N' \in \mathbb{N}$, $N' \ge N$ such that $\forall n \ge N'$

$$(24)\qquad D - \epsilon + \frac{\ln\left(\frac{K(N)}{n^{D-\epsilon}} - \frac{1}{D-\epsilon}\left(1 - \frac{N^{D-\epsilon}}{n^{D-\epsilon}}\right)\right)}{\ln(n)} \ge \; D - 2\epsilon \quad \text{and}$$

$$(25)\qquad (D+\epsilon)\frac{\ln(n+1)}{\ln(n)} + \frac{\ln\left(\frac{K(N)}{(n+1)^{D+\epsilon}} + \frac{1}{D+\epsilon}\left(1 - \frac{(N+1)^{D+\epsilon}}{(n+1)^{D+\epsilon}}\right)\right)}{\ln(n)} \le \; D + 2\epsilon \; .$$

From this we immediately find

$$(26)\qquad \left|\frac{\ln|\mathcal{U}_n|}{\ln(n)} - D\right| \le 2\epsilon \quad \forall n \ge N' \; .$$

□

Inversely, the existence of the internal scaling dimension does not imply the existence of the connectivity dimension. We illustrate this fact with the following example.

Example 3.1. We will construct a graph $\mathcal{G}$ with uniformly bounded node degree, degree of $x \in N$ less or equal $d \geq 3$, which has internal scaling dimension $D_S = D > 1$ but the connectivity dimension $\lim_{n\to\infty} \frac{\ln|\partial\mathcal{U}_n(x_0)|}{\ln(n)}$ does not exist and even $\limsup_{n\to\infty} \frac{\ln|\partial\mathcal{U}_n(x_0)|}{\ln(n)} = D \neq D-1$, i.e. $\overline{D}_C(x_0) = \overline{D}_S(x_0) + 1$. To this end we construct a "linear graph" in the fashion depicted in figure 1. In the figure d is equal to 3. The main idea of the construction is to let $|\partial\mathcal{U}_n(x_0)|$ oscillate so much that $\lim_{n\to\infty} \tilde{D}_n(x_0)$ does not exist any more but we still can have convergence of $D_n(x_0)$ and thus the internal scaling dimension exists.

We choose the numbers n_k such that $n_{k+1} = c\,n_k$ with some $c > 0$. For technical reasons we choose $c > d^{1/D}$. With this choice we already fulfill the prerequisite to use lemma 3.9.

Let us denote the "leftmost" node as x_0. All distances will refer to x_0 as the origin. The construction is determined by the following requirements. From distance n_k to $n_k + b_k$ the graph is a simple string of nodes and from distance $n_k + b_k + 1$ to n_{k+1} a complete[1] $(d-1)$-nary[2] tree graph. b_k is chosen to be $b_k = \max\{b \in \{0, \ldots, n_{k+1} - n_k\} : |\mathcal{U}_{n_{k+1}}| \geq (n_{k+1})^D\}$. This means that we start the $(d-1)$-nary tree as late as possible to still be sure to surpass our aim of $|\mathcal{U}_{n_{k+1}}| = (n_{k+1})^D$. It is easily established that $n_{k+1} - n_k$ gets large enough for $n_k \geq N$ with some $N \in \mathbb{N}$ to contain the necessary $(d-1)$-nary tree. A necessary and sufficient condition for this is

$$(d-1)^{n_{k+1}-n_k} \geq n_{k+1}^D - n_k^D \tag{27}$$

$$\Longleftrightarrow (d-1)^{cn_k - n_k} \geq c^D n_k^D - n_k^D \tag{28}$$

$$\Longleftrightarrow (d-1)^{n_k(c-1)} \geq (c^D - 1) n_k^D \tag{29}$$

which certainly holds for any $n_k \geq N$ with sufficiently large $N \in \mathbb{N}$ because the exponential function grows faster than any polynomial. The part of the graph where $n_{k+1} - n_k$ might be to small for the above construction, we choose to be of arbitrary form with $|\mathcal{U}_{n_k}| = \lfloor n_k^D \rfloor$.

Now we calculate the internal scaling dimension of the constructed graph. We know $\forall n_k \geq N$

$$\frac{\ln|\mathcal{U}_{n_k}(x_0)|}{\ln(n_k)} = \frac{\ln(n_k^D + \Delta_k)}{\ln(n_k)}, \tag{30}$$

where Δ_k is the additional number of nodes we get because of the usage of *complete* tree graphs. From the construction principle we know

$$\Delta_k \leq |\partial\mathcal{U}_{n_k}(x_0)| \leq (d-1)|\partial\mathcal{U}_{n_k-1}(x_0)| \leq (d-1)|\mathcal{U}_{n_k-1}(x_0)| \leq (d-1)n_k^D, \tag{31}$$

[1] In a complete tree graph every node has maximal degree.

[2] In a $(d-1)$-nary tree graph every node has $(d-1)$ or less children such that the degree of each node is bounded by d.

which is a rather crude estimate. Nonetheless we get

$$\frac{\ln(n_k^D)}{\ln(n_k)} \leq \frac{\ln|\mathcal{U}_{n_k}(x_0)|}{\ln(n_k)} \leq \frac{\ln(dn_k^D)}{\ln(n_k)} \tag{32}$$

$$\Longrightarrow \quad \lim_{k\to\infty} \frac{\ln|\mathcal{U}_{n_k}(x_0)|}{\ln(n_k)} = D \ . \tag{33}$$

Using lemma 3.9 we get

$$D_S(x_0) = \lim_{n\to\infty} \frac{\ln|\mathcal{U}_n(x_0)|}{\ln(n)} = D \ . \tag{34}$$

Finally we apply lemma 3.8 and get the dimension D starting from any node.

On the other hand we have to consider lim inf and lim sup of the sequence $\frac{\ln|\partial\mathcal{U}_n(x_0)|}{\ln(n)}$. The lim inf is trivial because $|\partial\mathcal{U}_{n_k+1}(x_0)| = 1$ which implies that $\liminf_{n\to\infty} \frac{\ln|\partial\mathcal{U}_n(x_0)|}{\ln(n)} = 0$. As far as the lim sup is concerned we know

$$|\mathcal{U}_{n_{k+1}}(x_0)| - |\mathcal{U}_{n_k}(x_0)| = b_k + \sum_{j=0}^{a_k} (d-1)^j = b_k + \frac{(d-1)^{a_k+1} - 1}{d-2} \tag{35}$$

with $a_k = n_{k+1} - (n_k + b_k)$. On the other hand

$$|\mathcal{U}_{n_{k+1}}(x_0)| - |\mathcal{U}_{n_k}(x_0)| = n_{k+1}^D + \Delta_{k+1} - (n_k^D + \Delta_k) \ . \tag{36}$$

Using (35), (36), $\Delta_k \leq (d-1)n_k^D$, $b_k \leq n_{k+1} - n_k$, $c > d^{1/D}$ and $|\partial\mathcal{U}_{n_{k+1}}|(x_0) = (d-1)^{a_k}$, we get after a short calculation that

$$D + \frac{\ln\left(\frac{1}{d-1} - \frac{d-2}{d-1}(1-\frac{1}{c})n_{k+1}^{1-D}\right)}{\ln(n_{k+1})} \leq \frac{\ln|\partial\mathcal{U}_{n_{k+1}}(x_0)|}{\ln(n_{k+1})} \tag{37}$$

$$\Longrightarrow \quad \limsup_{k\to\infty} \frac{\ln|\partial\mathcal{U}_n(x_0)|}{\ln(n)} \geq D \ . \tag{38}$$

But we always have

$$\frac{\ln|\partial\mathcal{U}_n(x_0)|}{\ln(n)} \leq \frac{\ln|\mathcal{U}_n(x_0)|}{\ln(n)} \tag{39}$$

$$\Longrightarrow \quad \limsup_{n\to\infty} \frac{\ln|\partial\mathcal{U}_n(x_0)|}{\ln(n)} \leq D \ . \tag{40}$$

Taking this together with (38) we finally get

$$\limsup_{n\to\infty} \frac{\ln|\partial\mathcal{U}_n(x_0)|}{\ln(n)} = D \ . \tag{41}$$

This example shows that we can't get much information about the behavior of $|\partial\mathcal{U}_n(x_0)|$ from the existence and value of the internal scaling dimension D_S of $\mathcal{G}$. The only always valid assertion is $\limsup_{n\to\infty} \frac{\ln|\partial\mathcal{U}_n(x)|}{\ln(n)} \leq D_S(x) \ \forall x \in N$.

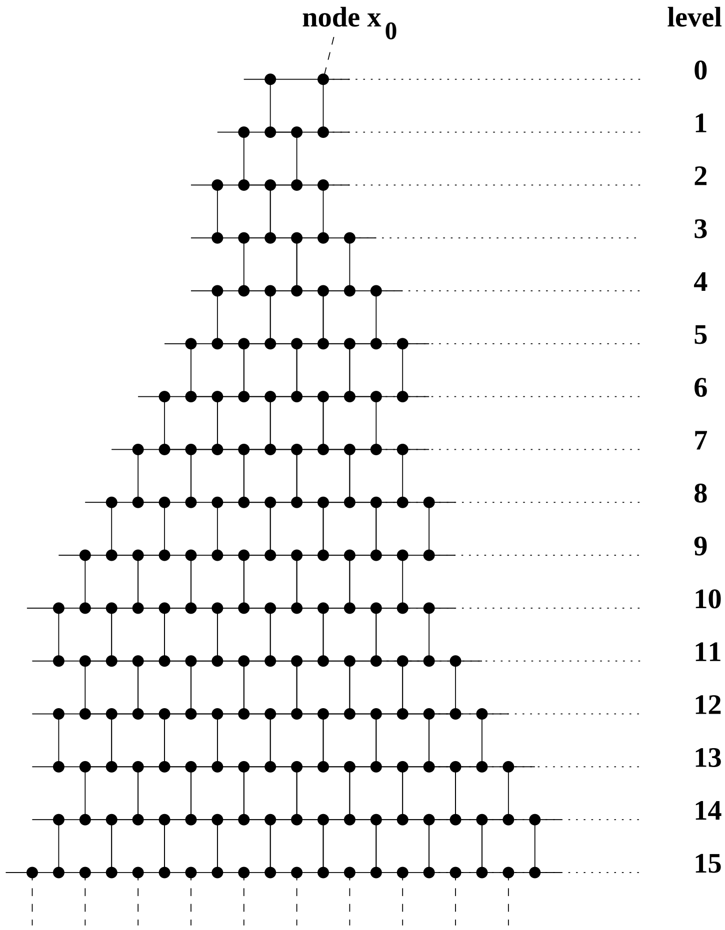

Figure 2: Example of a $\frac{5}{3}$ dimensional conical graph

4 Construction of Graphs

In the following we want to show how to construct graphs of arbitrary real internal scaling dimension. We also want to investigate the connections between the internal scaling dimension of graphs and the box counting dimension of fractal sets. As will been seen below there is a strong relationship between self similar sets and what we also want to call self similar graphs with non-integer internal scaling dimension.

4.1 Conical Graphs with Arbitrary Dimension

For the sake of simplicity we concentrate our discussion on graphs with dimension $1 \leq D \leq 2$. Graphs with higher dimension are easily constructed using a nearly identical scheme.

Let $1 \leq D \leq 2$ be an arbitrary real number. Now we construct the graph like in figure 2. On level m we use a width of $\lfloor (2m-1)^{D-1} \rfloor$ boxes. The construction is continued "downwards" to infinity. To calculate the dimension we observe that starting from x_0 we reach level m after $n = 2m-1$ steps. Thus we get with $n_k := 2k-1$

$$|\partial\mathcal{U}_{n_k}(x_0)| = \lfloor n_k^{D-1} \rfloor \quad \Longrightarrow \quad \lim_{k\to\infty} \frac{\ln|\partial\mathcal{U}_{n_k}(x_0)|}{\ln(n_k)} = D-1 \ . \tag{42}$$

Using lemmas 3.11, 3.8 and 3.9 we see that this graph has internal scaling dimension $D_S = D$. If we close the construction horizontally, i.e. introduce bonds between the leftmost and the rightmost nodes on each level we even can achieve a completely homogeneous node degree $d = 3$.

Remark. 1. The constructed graph has privileged nodes, the one we denoted as node x_0 and its counterpart on the same level.

2. Locally the constructed conical graph is completely isomorphic to a two-dimensional lattice. The non-integer dimension is only implemented as a global property of the graph.

4.2 Self-Similar Graphs

It is well known in graph theory that it is notoriously difficult to construct large graphs with prescribed properties. It also proved quite difficult to construct graphs with a prescribed (internal scaling) dimension $D_S = D$ which don't exhibit the disadvantages of the conical graphs described above. The main idea which solves the problem is to use the well known theory of self similar sets or fractals and their dimension theory. In the following we want to show how this works and that we indeed can construct adjoint graphs to self similar sets which have internal scaling dimension equal to the box counting dimension of the self similar sets.

Given a strictly self similar set in $\mathbb{R}^p$ we canonically construct an adjoint graph which also will be called self-similar. The construction principle is based on an algorithm to compute the box counting dimension of a self-similar set. We will illustrate our proceedings with one main example. We construct a self-similar set generated with the open unit square in $\mathbb{R}^2$ with lower left corner at the origin and the similarity transforms

(43)

$$S_1 : \underline{x} \longmapsto \frac{1}{3}\underline{x} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \quad S_2 : \underline{x} \longmapsto \frac{1}{3}\underline{x} + \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix} , \quad S_3 : \underline{x} \longmapsto \frac{1}{3}\underline{x} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

(44)

$$S_4 : \underline{x} \longmapsto \frac{1}{3}\underline{x} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix} , \quad S_5 : \underline{x} \longmapsto \frac{1}{3}\underline{x} + \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} .$$

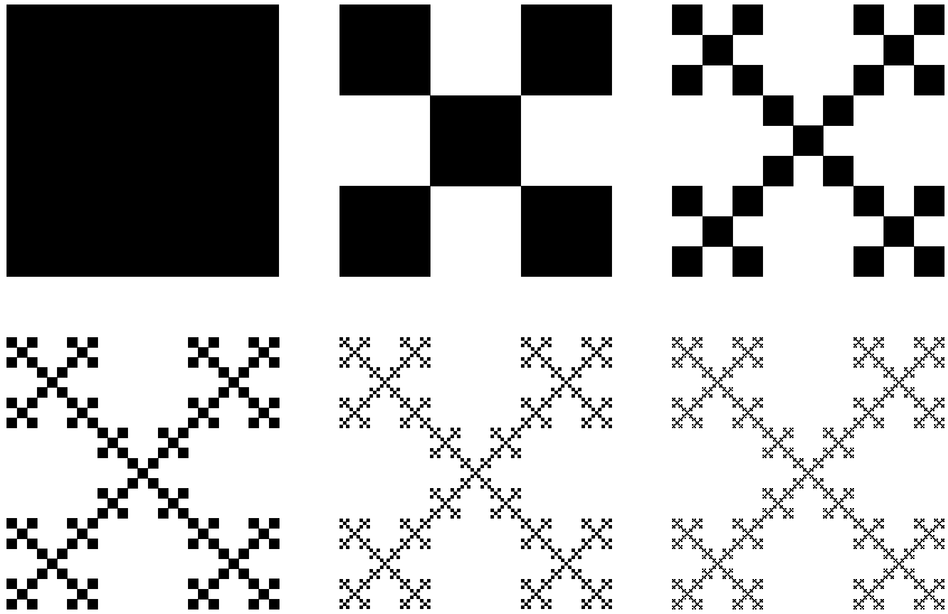

Figure 3: Construction steps of the example self-similar set

This set is sometimes called *Maltese Cross*, cf. [7]. The first construction steps are shown in figure 3. For details concerning self-similar sets and dimensions of fractals see [6].

4.2.1 Construction Based on Self-Similar Sets

Let M be a strictly self-similar set with similarity transforms S_i, $i \in I$, $I \subset \mathbb{N}$ and $|I| < \infty$. The contraction factors c_i of S_i may all be equal, $c_i = c \in (0,1)$. Now we cover M with cubic lattices $L_n \subset \mathbb{R}^p$ with closed cubes of edge length c^n, $n \in \mathbb{N}$, and replace every cube which has non-void intersection with M by a node. Nodes will be connected iff the corresponding cubes in the covering cubic lattices have a non-void intersection, i.e. have a common corner or edge.

By this construction we get a finite graph $\mathcal{G}_n$ for each $n \in \mathbb{N}$. The degree of these $\mathcal{G}_n$ is uniformly bounded because an n-dimensional cube can only touch a finite number of neighbor cubes in the cubic lattice. The graph we are interested in is $\mathcal{G}_\infty$, the graph we get through infinite continuation of our construction. The first steps of this construction scheme for our example are shown in figure 4.

Remark. 1. We will see later on, that no problems arise from the infinite continuation of the construction steps.

2. The self-similarity of M transfers to $\mathcal{G}$ in the sense that we can also define an equivalent of the similarity transforms of the self-similar set M. Details will become clear when we give a self-contained algorithm for the construction of self-similar graphs.

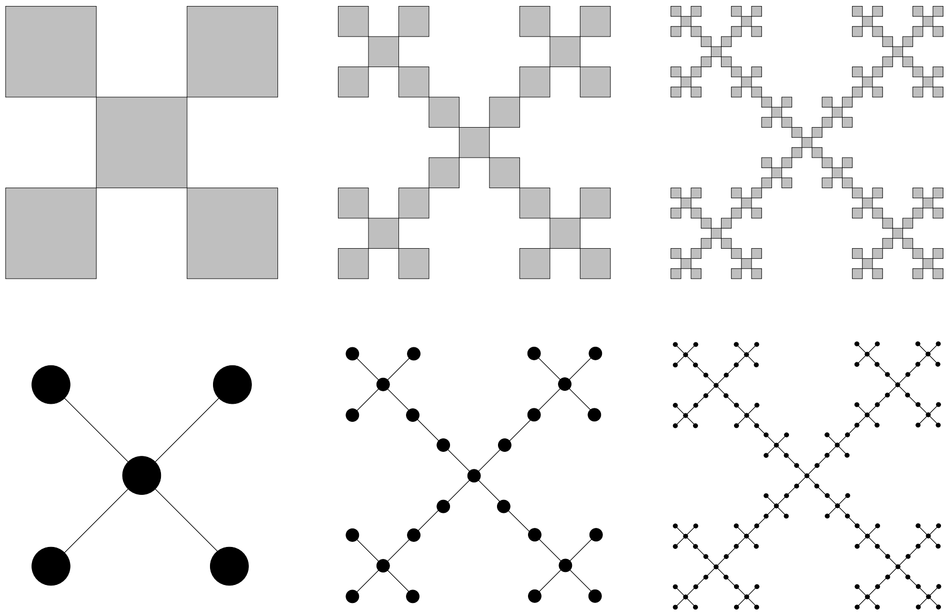

Figure 4: Construction of graphs from self-similar sets

3. Connected self-similar sets produce connected self-similar graphs. The inverse is not true in general as our example shows. Here $\mathcal{G}$ is connected but the self similar set we started with is not.

4.2.2 Self-Contained Construction Algorithm

We want to illustrate two different views of a self-contained construction algorithm for self-similar or hierarchical graphs.

1. Construction by insertion:

   (a) We start with a single node, $\mathcal{G}_0 = (\{n_0\}, \emptyset)$.

   (b) $\mathcal{G}_1$ is the so-called generator, some finite graph. We denote the number of nodes in $\mathcal{G}_1$ as N_g.

   (c) We construct $\mathcal{G}_{n+1}$ from $\mathcal{G}_n$ by replacing every node in $\mathcal{G}_n$ by the generator $\mathcal{G}_1$ and interpret the original bonds in $\mathcal{G}_n$ as bonds between some "marginal" nodes of the different copies of $\mathcal{G}_1$. In figure 5 we have drawn the first construction steps of our example.

2. Construction by "copy and paste":

   (a) and (b) are identical to 1.

   (c) We construct $\mathcal{G}_{n+1}$ from $\mathcal{G}_n$ by copying $\mathcal{G}_n$ N_g times and pasting these copies together in the same fashion as the nodes of the generator are

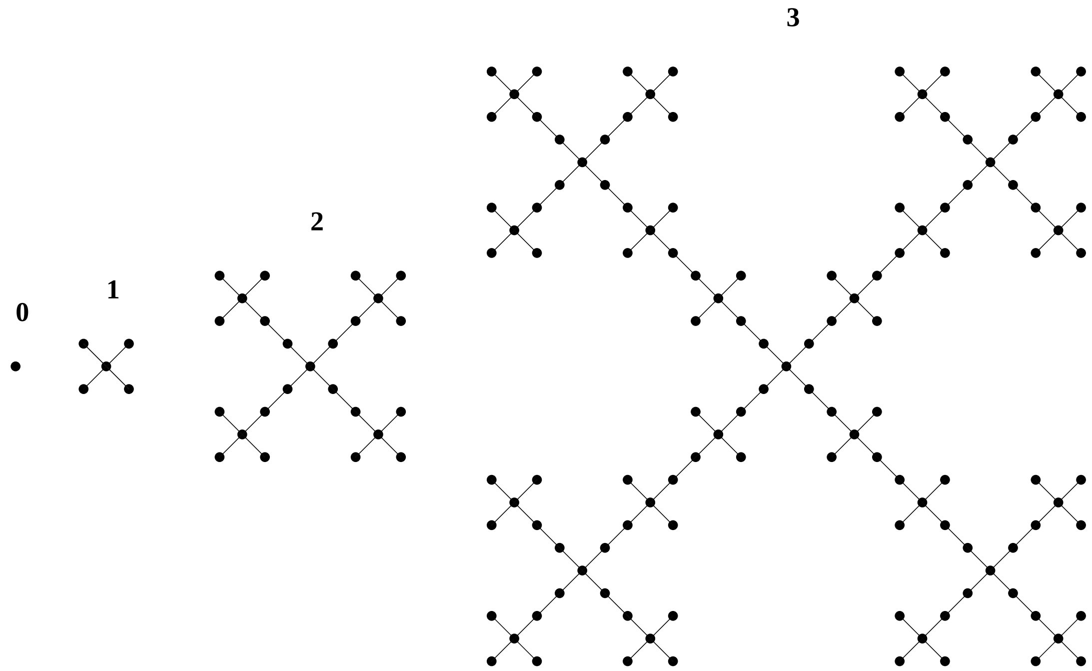

Figure 5: Self-contained construction

arranged. The construction steps can't be distinguished from those in figure 5.

Remark. 1. It becomes clear when looking at examples that the above construction algorithms are equivalent.

2. The construction is – of course – not unique. The result strongly depends on the choice of the nodes in $\mathcal{G}_{n+1}$ which carry the bonds of $\mathcal{G}_n$ in the first construction or $\mathcal{G}_1$ in the second one respectively. In our example all "marginal" nodes of the generator are equivalent because of the symmetry of the generator and therefore the construction is unique.

3. Seen from the viewpoint of the second construction it becomes clear that the local neighborhood of any node doesn't change in the course of the further construction. Therefore we can investigate any property of $\mathcal{G}$ in some $\mathcal{G}_N$ with sufficiently large N. Thus the infinite continuation of construction steps needn't worry us at all.

4. The first construction scheme provides us with the analogon of the similarity transforms of the self-similar set. These transforms correspond to the mapping of $\mathcal{G}$ on $\tilde{\mathcal{G}}$ where $\tilde{\mathcal{G}}$ is formed from $\mathcal{G}$ like some $\mathcal{G}_{n+1}$ from $\mathcal{G}_n$. Clearly $\mathcal{G}$ is invariant under this mapping.

As we can see from our example, all three construction algorithms, the self-contained ones as well as the one based on a self-similar set, are equivalent provided the self-similar set and the choice of the generator match. Seen in this light we can use all the construction principles simultaneously in our arguments.

4.2.3 Dimension of Self-Similar Graphs

Now we calculate the dimension of the graphs we get by the above construction using some self-similar set M. For the sake of simplicity we assume that $\mathcal{G}_1$ has a central node x_0 in the sense that all "marginal" nodes which carry the "outer" bonds have all the same distance r to this node. We further assume that $\frac{1}{c}$ (c the contraction parameter) is a natural number which is true in most of the well known examples of self-similar sets and finally that the self-similar set produces a connected adjoint graph. Then it is easy to see that starting from node x_0 we can exactly reach all nodes of construction step $k+1$ after $n_{k+1} = r + 2r\, n_k + n_k = (2r+1)\, n_k + r$ steps in the graph, with - of course - $n_0 = 0$. Thus $|\mathcal{U}_{n_k}(x_0)|$ is equal to the number of nodes in construction step k, i.e. $|\mathcal{U}_{n_k}(x_0)| = N_{\delta_k} = N_{c^k}$.[3] Explicitly we get for n_k

$$n_k = \sum_{j=0}^{k-1} (2r+1)^j\, r = r\, \frac{(2r+1)^k - 1}{2r} \quad \forall k \geq 1 \tag{45}$$

Now let us relate r to the contraction parameter c of the self-similar set. We assumed that the graph constructed from the self-similar set is connected. This implies that there are $\frac{1}{c}$ nodes on the "diagonal" of the generator, i.e. $2r+1 = \frac{1}{c}$. Now we have for the internal scaling dimension of $\mathcal{G}$

$$\lim_{k\to\infty} D_{n_k}(x_0) = \lim_{k\to\infty} \frac{\ln(N_{c^k})}{\ln\left(r\frac{(2r+1)^k-1}{2r}\right)} \tag{46}$$

$$= \lim_{k\to\infty} \frac{\ln(N_{c^k})}{\ln((2r+1)^k) + \ln\left(\frac{1-(2r+1)^{-k}}{2r}\right)} \tag{47}$$

$$= \lim_{k\to\infty} \frac{\ln(N_{c^k})}{-\ln(c^k) + \ln\left(\frac{1-(2r+1)^{-k}}{2r}\right)} = \dim_B(M) \tag{48}$$

in which $\dim_B(M)$ is the box counting dimension of M. Of course lemmas 3.8 and 3.9 provide us with the knowledge that this is the dimension of $\mathcal{G}$ starting from any node.

Thus we established equality of the box counting dimension of self-similar sets and the internal scaling dimension of the adjoint self-similar graphs under the assumptions stated above.

[3] N_{δ_k} is the number of cubes of edge length δ_k intersecting M, see the calculation of the box counting dimension in e.g. [6].

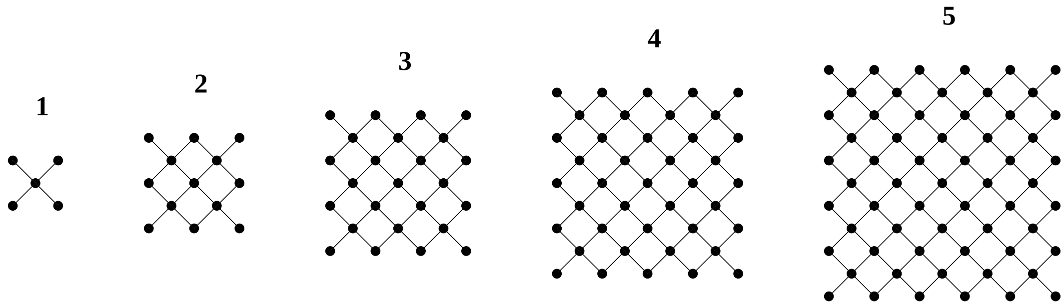

Figure 6: Some generators

Remark. The assumed existence of a central node x_0 is not essential for the equality of the dimensions of the fractal and the graph. The equality still holds in a more general context, e.g. for fractals like the Sirpinski Triangle. It is difficult though to give a general proof for arbitrary self-similar sets.

4.2.4 Approximation of a Two Dimensional Lattice

In this paragraph we want to show how it now becomes possible to do a dimensional approximation of a n-dimensional cubic lattice. Again, for the sake of simplicity, we discuss the idea only with a two-dimensional lattice but the generalization to n dimensions is obvious.

We introduce generators as shown in figure 6. With these we get graphs of dimensions

$$D_S^{(l)} = \frac{\ln(2l^2 + 2l + 1)}{\ln(2l + 1)} \tag{49}$$

in which l is the number which labels the generators in figure 6. Obviously we have

$$\lim_{l\to\infty} D_S^{(l)} = \lim_{l\to\infty} \frac{\ln(2l^2 + 2l + 1)}{\ln(2l + 1)} = \lim_{l\to\infty} \frac{2\ln(l) + \ln(2 + \frac{2}{l} + \frac{1}{l^2})}{\ln(l) + \ln(2 + \frac{1}{l})} = 2\ . \tag{50}$$

In this sense we have a dimensional approximation of a two-dimensional lattice as alleged above. This might have some relevance in connection with the dimensional regularization used in many renormalization approaches to quantum field theory.

Remark. The generators above correspond to fractal sets known as “sponges”, see e.g. [7]. We can construct such “sponges” for any dimension n, we just need to modify the generators appropriately.

4.2.5 How to Change the Dimension of a Graph

To enlarge the dimension of a graph it is necessary to add either bonds or nodes to the graph. In the former case we showed that adding only bonds between nodes

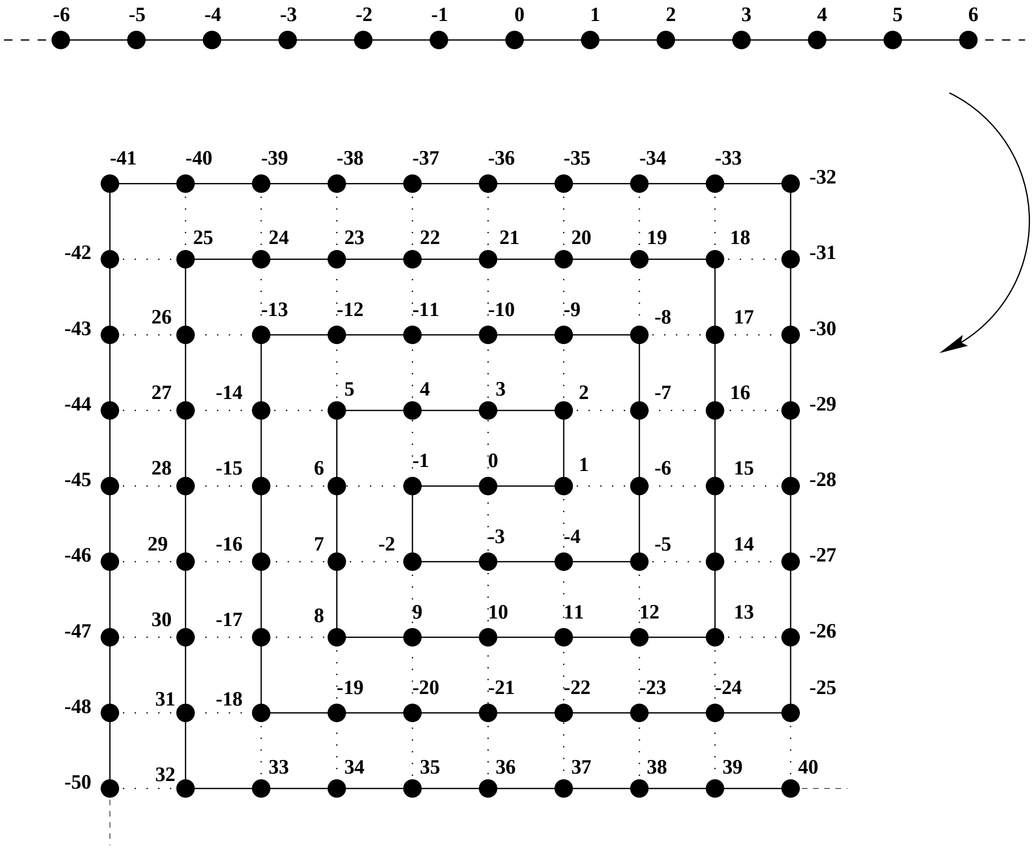

Figure 7: Deforming a one-dimensional graph into a two-dimensional one

with original distance less than some $k \in \mathbb{N}$ does not change the dimension. We want to illustrate this with an example. Let us try to get a two-dimensional lattice starting from an one-dimensional one. The procedure is shown in figure 7. The dotted bonds are those we added. As is easily seen, the former distance between the newly connected nodes grows unboundedly with n, the number of the nodes in the original graph.

If we choose to add nodes instead, it is equivalent to adding bonds to new nodes which formerly had infinite distance to the nodes of the original graph. This also illustrates the general result because adding finitely many nodes certainly doesn't change the dimension.

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