Information flow in a kinetic Ising model peaks in the disordered phase: Supplemental material

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1: CALCULATION OF PAIRWISE MUTUAL INFORMATION

It is clear from homogeneity that S_i has the same distribution for any site i and, similarly, that S_i, S_j have the same *joint* distribution for any pair of neighbouring sites i, j. Thus we have $I_{pw} = \mathrm{I}(S_i : S_j) = 2\mathrm{H}(S_i) - \mathrm{H}(S_i, S_j)$ for any fixed choice of lattice neighbours i, j (we note, though, that in sample the lattice-averaged form will yield a more efficient estimator). Firstly, $\mathrm{H}(S_i) = -\sum_\sigma p_\sigma \log p_\sigma$ where $p_\sigma \equiv \mathbf{P}(S_i = \sigma)$ and the sum is over spins $\sigma = \pm 1$. Firstly, we show that p_σ is as given in [1], eq. 6. In the calculations that follow, we make frequent use of the identity

$$\delta(\sigma, \sigma') = \tfrac{1}{2}(1 + \sigma\sigma') \tag{1}$$

for spins $\sigma, \sigma' = \pm 1$. We have

$$\begin{aligned}
\mathbf{P}(S_i = \sigma) &= \sum_{\boldsymbol{s}} \mathbf{P}(\boldsymbol{S} = \boldsymbol{s})\, \mathbf{P}(\, S_i = \sigma \mid \boldsymbol{S} = \boldsymbol{s}) && \text{conditioning on } \boldsymbol{S} \\
&= \sum_{\boldsymbol{s}} \Pi(\boldsymbol{s}) \delta(s_i, \sigma) \\
&= \sum_{\boldsymbol{s}} \Pi(\boldsymbol{s}) \cdot \tfrac{1}{2}(1 + \sigma s_i) && \text{by (1)} \\
&= \tfrac{1}{2}(1 + \sigma \langle S_i \rangle) \to \tfrac{1}{2}(1 + \sigma \mathcal{M}) && \text{as } N \to \infty
\end{aligned}$$

as required. Next we show that $p_{\sigma\sigma'}$ is also as in [1], eq. 6. We have $\mathrm{H}(S_i, S_j) = -\sum_{\sigma,\sigma'} p_{\sigma\sigma'} \log p_{\sigma\sigma'}$ where $p_{\sigma\sigma'} \equiv \mathbf{P}(S_i = \sigma, S_j = \sigma')$, and

$$\begin{aligned}
\mathbf{P}(S_i = \sigma, S_j = \sigma') &= \sum_{\boldsymbol{s}} \mathbf{P}(\boldsymbol{S} = \boldsymbol{s})\, \mathbf{P}(\, S_i = \sigma, S_j = \sigma' \mid \boldsymbol{S} = \boldsymbol{s}) \\
&= \sum_{\boldsymbol{s}} \Pi(\boldsymbol{s}) \delta(s_i, \sigma) \delta(s_j, \sigma') \\
&= \tfrac{1}{4} \sum_{\boldsymbol{s}} \Pi(\boldsymbol{s})(1 + \sigma s_i + \sigma' s_j + \sigma\sigma' s_i s_j) \\
&= \tfrac{1}{4}(1 + \sigma \langle S_i \rangle + \sigma' \langle S_j \rangle + \sigma\sigma' \langle S_i S_j \rangle) \to \tfrac{1}{4}[1 + (\sigma + \sigma')\mathcal{M} - \tfrac{1}{2}\sigma\sigma'\mathcal{U}] && \text{as } N \to \infty
\end{aligned}$$

as required. In the last step, we use $\langle S_i S_j \rangle \to -\frac{1}{2}\mathcal{U}$ as $N \to \infty$, which follows from $\mathcal{U} = \frac{1}{N}\langle \mathcal{H}(\boldsymbol{S}) \rangle$. I_{pw} is thus as in [1], eq. 5. Note that for $T < T_c$ the sign of $\mathcal{M}$ does not affect the result; i.e. I_{pw} is invariant to the direction in which symmetry breaks (this applies to the other information measures too).

2: CALCULATION OF PAIRWISE TRANSFER ENTROPY

We start by proving the following lemma: for arbitrary lattice neighbours i, j,

$$\langle S_i P_i(\boldsymbol{S}) \rangle \equiv 0 \tag{2}$$

$$\langle S_i S_j P_i(\boldsymbol{S}) \rangle \equiv 0 \tag{3}$$

$$\langle S_j P_i(\boldsymbol{S}) \rangle \equiv 0 \qquad \text{for } T \geq T_c \text{ only}\,. \tag{4}$$

Let $\Lambda_i^+ \equiv \{\boldsymbol{s}|s_i = +1\}$ and $\Lambda_i^- \equiv \{\boldsymbol{s}|s_i = -1\}$. Then

$$
\begin{aligned}
\langle S_i P_i(\boldsymbol{S})\rangle &= \sum_{\boldsymbol{s}} \Pi(\boldsymbol{s}) s_i P_i(\boldsymbol{s}) \\
&= \frac{1}{Z}\sum_{\boldsymbol{s}\in\Lambda_i^+} e^{-\beta\mathcal{H}(\boldsymbol{s})} P_i(\boldsymbol{s}) - \frac{1}{Z}\sum_{\boldsymbol{s}\in\Lambda_i^-} e^{-\beta\mathcal{H}(\boldsymbol{s})} P_i(\boldsymbol{s}) \\
&\qquad \text{Now we note that as } \boldsymbol{s} \text{ runs through } \Lambda_i^+,\ \boldsymbol{s}^i \text{ runs through } \Lambda_i^- \\
&= \frac{1}{Z}\sum_{\boldsymbol{s}\in\Lambda_i^+} e^{-\beta\mathcal{H}(\boldsymbol{s})} P_i(\boldsymbol{s}) - \frac{1}{Z}\sum_{\boldsymbol{s}\in\Lambda_i^+} e^{-\beta\mathcal{H}(\boldsymbol{s}^i)} P_i(\boldsymbol{s}^i) \\
&= \frac{1}{Z}\sum_{\boldsymbol{s}\in\Lambda_i^+} \left\{ e^{-\beta\mathcal{H}(\boldsymbol{s})} \frac{1}{1+e^{\beta\Delta H_i(\boldsymbol{s})}} - e^{-\beta[\mathcal{H}(\boldsymbol{s})+\Delta H_i(\boldsymbol{s})]} \frac{1}{1+e^{-\beta\Delta H_i(\boldsymbol{s})}} \right\} \qquad \text{using } \Delta H_i(\boldsymbol{s}^i) = -\Delta H_i(\boldsymbol{s}) \\
&= \frac{1}{Z}\sum_{\boldsymbol{s}\in\Lambda_i^+} e^{-\beta\mathcal{H}(\boldsymbol{s})} \left\{ \frac{1}{1+e^{\beta\Delta H_i(\boldsymbol{s})}} - \frac{e^{-\beta\Delta H_i(\boldsymbol{s})}}{1+e^{-\beta\Delta H_i(\boldsymbol{s})}} \right\} \\
&= 0
\end{aligned}
$$

proving (2). A similar argument works for (3). If $T \geq T_c$ then since symmetry is unbroken, for each equilibrium state there is a corresponding equilibrium state with all spins reversed. For spin-reversed states, $\Delta H_i(\boldsymbol{s})$, and hence $P_i(\boldsymbol{s})$, is unchanged, so that $s_j P_i(\boldsymbol{s})$ has the opposite sign; (4) thus follows.

By homogeneity, the joint distribution of $S_i(t), S_i(t-1), S_j(t-1)$ is the same for any fixed pair of neighbouring sites i, j and we have $\mathcal{T}_{pw} = \mathrm{H}(\,S_i(t) \mid S_i(t-1)) - \mathrm{H}(\,S_i(t) \mid S_i(t-1), S_j(t-1))$. Firstly, $\mathrm{H}(\,S_i(t) \mid S_i(t-1)) = -\sum_\sigma p_\sigma \sum_{\sigma'} p_{\sigma'|\sigma} \log p_{\sigma'|\sigma}$, where we define $p_{\sigma'|\sigma} \equiv \mathbf{P}(\,S_i(t) = \sigma' \mid S_i(t-1) = \sigma)$. In the calculations that follow, we make use of the following explicit expression for the Markov transition probabilities in the Glauber kinetic model:

$$
P(\boldsymbol{s}'|\boldsymbol{s}) = \begin{cases} 1 - \frac{1}{N}\sum_k P_k(\boldsymbol{s}) & \boldsymbol{s}' = \boldsymbol{s} \\ \frac{1}{N} P_j(\boldsymbol{s}) & \boldsymbol{s}' = \boldsymbol{s}^j \\ 0 & \text{otherwise} \end{cases} . \tag{5}
$$

We have

$$
\begin{aligned}
&\mathbf{P}(S_i(t) = \sigma', S_i(t-1) = \sigma) \\
&= \sum_{\boldsymbol{s},\boldsymbol{s}'} \mathbf{P}(\,S_i(t) = \sigma', S_i(t-1) = \sigma \mid \boldsymbol{S}(t) = \boldsymbol{s}', \boldsymbol{S}(t-1) = \boldsymbol{s})\, \mathbf{P}(\boldsymbol{S}(t) = \boldsymbol{s}', \boldsymbol{S}(t-1) = \boldsymbol{s}) \\
&= \sum_{\boldsymbol{s}} \Pi(\boldsymbol{s})\delta(s_i,\sigma) \sum_{\boldsymbol{s}'} \delta(s_i',\sigma') P(\boldsymbol{s}'|\boldsymbol{s}) \\
&= \sum_{\boldsymbol{s}} \Pi(\boldsymbol{s})\delta(s_i,\sigma) \left\{ \delta(s_i,\sigma') \left[1 - \frac{1}{N}\sum_j P_j(\boldsymbol{s}) \right] + \sum_j \delta(s_i^j,\sigma') \frac{1}{N} P_j(\boldsymbol{s}) \right\} \qquad \text{by (5)} \\
&= \sum_{\boldsymbol{s}} \Pi(\boldsymbol{s})\delta(s_i,\sigma) \left\{ \delta(s_i,\sigma') - \frac{1}{N}\sum_j \left[\delta(s_i,\sigma') - \delta(s_i^j,\sigma') \right] P_j(\boldsymbol{s}) \right\} \\
&\qquad \text{Note that the term in square brackets vanishes unless } j = i \\
&= \sum_{\boldsymbol{s}} \Pi(\boldsymbol{s})\delta(s_i,\sigma) \left\{ \delta(s_i,\sigma') - \frac{1}{N} \left[\delta(s_i,\sigma') - \delta(s_i^i,\sigma') \right] P_i(\boldsymbol{s}) \right\} \\
&= \sum_{\boldsymbol{s}} \Pi(\boldsymbol{s})\delta(s_i,\sigma) \left\{ \delta(s_i,\sigma') - \frac{1}{N}\sigma' s_i P_i(\boldsymbol{s}) \right\} \qquad \text{since } s_i^i = -s_i \\
&= \sum_{\boldsymbol{s}} \Pi(\boldsymbol{s})\delta(s_i,\sigma)\delta(s_i,\sigma') - \frac{1}{N}\sigma\sigma' \sum_{\boldsymbol{s}} \Pi(\boldsymbol{s})\delta(s_i,\sigma) P_i(\boldsymbol{s})
\end{aligned}
$$

$$= \delta(\sigma,\sigma') \sum_{\boldsymbol{s}} \Pi(\boldsymbol{s}) \delta(s_i,\sigma) - \frac{1}{N} \sigma\sigma' \sum_{\boldsymbol{s}} \Pi(\boldsymbol{s}) \tfrac{1}{2}(1+\sigma s_i) P_i(\boldsymbol{s})$$
$$= \delta(\sigma,\sigma') p_\sigma - \frac{1}{N} \sigma\sigma' \tfrac{1}{2} \left(\langle P_i(\boldsymbol{S}) \rangle + \sigma \langle S_i P_i(\boldsymbol{S}) \rangle \right)$$
$$= \delta(\sigma,\sigma') p_\sigma - \frac{1}{N} \sigma\sigma' q \qquad \text{since by (2) } \langle S_i P_i(\boldsymbol{S}) \rangle \text{ vanishes},$$

with q as in [1], eq. 11. So

$$p_{\sigma'|\sigma} = \begin{cases} 1 - \dfrac{1}{N} \dfrac{q}{p_\sigma} & \sigma' = \sigma \\ \dfrac{1}{N} \dfrac{q}{p_\sigma} & \sigma' = -\sigma \end{cases} . \tag{6}$$

Next, $\mathrm{H}(\, S_i(t) \mid S_i(t-1), S_j(t-1)) = -\sum_{\sigma,\sigma'} p_{\sigma\sigma'} \sum_{\sigma''} p_{\sigma''|\sigma\sigma'} \log p_{\sigma''|\sigma\sigma'}$, where we define $p_{\sigma''|\sigma\sigma'} \equiv \mathbf{P}(\, S_i(t) = \sigma'' \mid S_i(t-1) = \sigma, S_j(t-1) = \sigma')$, and we may calculate along the same lines as above (we omit details) that

$$p_{\sigma''|\sigma\sigma'} = \begin{cases} 1 - \dfrac{1}{N} \dfrac{q_{\sigma'}}{p_{\sigma\sigma'}} & \sigma'' = \sigma \\ \dfrac{1}{N} \dfrac{q_{\sigma'}}{p_{\sigma\sigma'}} & \sigma'' = -\sigma \end{cases} \tag{7}$$

with $q_{\sigma'}$ again as in [1], eq. 11. Now, working to $\boldsymbol{O}\left(\frac{1}{N}\right)$,

$$\begin{aligned}
\mathcal{T}_{pw} &= -\sum_\sigma p_\sigma \sum_{\sigma'} p_{\sigma'|\sigma} \log p_{\sigma'|\sigma} + \sum_{\sigma,\sigma'} p_{\sigma\sigma'} \sum_{\sigma''} p_{\sigma''|\sigma\sigma'} \log p_{\sigma''|\sigma\sigma'} \\
&= -\sum_\sigma p_\sigma \left(p_{\sigma|\sigma} \log p_{\sigma|\sigma} + p_{-\sigma|\sigma} \log p_{-\sigma|\sigma} \right) \\
&\quad + \sum_{\sigma,\sigma'} p_{\sigma\sigma'} \left(p_{\sigma|\sigma\sigma'} \log p_{\sigma|\sigma\sigma'} + p_{-\sigma|\sigma\sigma'} \log p_{-\sigma|\sigma\sigma'} \right) \\
&= -\sum_\sigma p_\sigma \left[\left(1 - \frac{1}{N} \frac{q}{p_\sigma} \right) \log \left(1 - \frac{1}{N} \frac{q}{p_\sigma} \right) + \frac{1}{N} \frac{q}{p_\sigma} \log \left(\frac{1}{N} \frac{q}{p_\sigma} \right) \right] \\
&\quad + \sum_{\sigma,\sigma'} p_{\sigma\sigma'} \left[\left(1 - \frac{1}{N} \frac{q_{\sigma'}}{p_{\sigma\sigma'}} \right) \log \left(1 - \frac{1}{N} \frac{q_{\sigma'}}{p_{\sigma\sigma'}} \right) + \frac{1}{N} \frac{q_{\sigma'}}{p_{\sigma\sigma'}} \log \left(\frac{1}{N} \frac{q_{\sigma'}}{p_{\sigma\sigma'}} \right) \right] \\
&= -\frac{1}{N} \sum_\sigma q \left(\log \frac{q}{p_\sigma} - \log N - 1 \right) + \frac{1}{N} \sum_{\sigma,\sigma'} q_{\sigma'} \left(\log \frac{q_{\sigma'}}{p_{\sigma\sigma'}} - \log N - 1 \right) + \boldsymbol{O}\left(\frac{1}{N^2} \right) \\
&= -\frac{1}{N} \sum_\sigma q \log \frac{q}{p_\sigma} + \frac{1}{N} \sum_{\sigma,\sigma'} q_{\sigma'} \log \frac{q_{\sigma'}}{p_{\sigma\sigma'}} + \boldsymbol{O}\left(\frac{1}{N^2} \right)
\end{aligned}$$

as $N \to \infty$, where in the penultimate step we use $\log(1 + x/N) = x/N + \boldsymbol{O}\left(1/N^2\right)$ as $N \to \infty$ and in the last step we use the identity $\sum_{\sigma'} q_{\sigma'} \equiv q$, which follows directly from [1], eq. 11, so that the $(\log N + 1)$ terms cancel. Thus in the thermodynamic limit, we obtain [1], eq. 10.

3: CALCULATION OF GLOBAL TRANSFER ENTROPY

Once again by homogeneity we have $\mathcal{T}_{gl} = \mathrm{H}(\, S_i(t) \mid S_i(t-1)) - \mathrm{H}(\, S_i(t) \mid \boldsymbol{S}(t-1))$ for any fixed site i. The first term has been calculated above and for the second term $\mathrm{H}(\, S_i(t) \mid \boldsymbol{S}(t-1)) = -\sum_{\boldsymbol{s}} \Pi(\boldsymbol{s}) \sum_{\sigma'} p_i(\sigma'|\boldsymbol{s}) \log p_i(\sigma'|\boldsymbol{s})$

where $p_i(\sigma'|\boldsymbol{s}) \equiv \mathbf{P}(\, S_i(t) = \sigma' \mid \boldsymbol{S}(t-1) = \boldsymbol{s})$. We have

$$
\begin{aligned}
\mathbf{P}(\, S_i(t) = \sigma' \mid \boldsymbol{S}(t-1) = \boldsymbol{s}) &= \sum_{\boldsymbol{s}'} \mathbf{P}(\, S_i(t) = \sigma' \mid \boldsymbol{S}(t-1) = \boldsymbol{s}, \boldsymbol{S}(t) = \boldsymbol{s}')\, \mathbf{P}(\, \boldsymbol{S}(t) = \boldsymbol{s}' \mid \boldsymbol{S}(t-1) = \boldsymbol{s}) \\
&= \sum_{\boldsymbol{s}'} \delta(s'_i, \sigma') P(\boldsymbol{s}'|\boldsymbol{s}) \qquad \text{again, } \boldsymbol{s}' = \boldsymbol{s} \text{ or } \boldsymbol{s}' = \boldsymbol{s}^j \text{ for some } j \\
&= \delta(s_i, \sigma') \left[1 - \frac{1}{N} \sum_j P_j(\boldsymbol{s}) \right] + \sum_j \delta(s_i^j, \sigma') \frac{1}{N} P_j(\boldsymbol{s}) \\
&= \delta(s_i, \sigma') - \frac{1}{N} \sum_j \left[\delta(s_i, \sigma') - \delta(s_i^j, \sigma') \right] P_j(\boldsymbol{s}) \\
&= \delta(s_i, \sigma') - \frac{1}{N} \left[\delta(s_i, \sigma') - \delta(s_i^i, \sigma') \right] P_i(\boldsymbol{s}) \\
&= \delta(s_i, \sigma') - \frac{1}{N} \sigma' s_i P_i(\boldsymbol{s})\,,
\end{aligned}
$$

so

$$
p_i(\sigma'|\boldsymbol{s}) = \begin{cases} 1 - \dfrac{1}{N} P_i(\boldsymbol{s}) & \sigma' = s_i \\ \dfrac{1}{N} P_i(\boldsymbol{s}) & \sigma' = -s_i \end{cases} \,. \tag{8}
$$

By an argument analogous to that for the pairwise case,

$$
\begin{aligned}
\mathcal{T}_{gl} &= -\frac{1}{N} \sum_\sigma q \left(\log \frac{q}{p_\sigma} - \log N - 1 \right) + \frac{1}{N} \sum_{\boldsymbol{s}} \Pi(\boldsymbol{s}) P_i(\boldsymbol{s}) \left[\log P_i(\boldsymbol{s}) - \log N - 1 \right] + \boldsymbol{O}\left(\frac{1}{N^2} \right) \\
&= -\frac{1}{N} \sum_\sigma q \log \frac{q}{p_\sigma} + \frac{1}{N} \langle P_i(\boldsymbol{S}) \log P_i(\boldsymbol{S}) \rangle + \boldsymbol{O}\left(\frac{1}{N^2} \right)
\end{aligned}
$$

as $N \to \infty$, where in the last step we use $\sum_{\boldsymbol{s}} \Pi(\boldsymbol{s}) P_i(\boldsymbol{s}) = \langle P_i(\boldsymbol{s}) \rangle = 2q$, so that again the $(\log N + 1)$ terms cancel. Thus in the thermodynamic limit we obtain [1], eq. 13.

4: GRADIENT OF MUTUAL INFORMATION MEASURES AT CRITICALITY

In the thermodynamic limit $\mathcal{M} \equiv 0$ for $\beta \le \beta_c$, so that $-\sum_\sigma p_\sigma \log p_\sigma$ is constant with respect to β and $p_{\sigma\sigma'} = \frac{1}{4}(1 - \frac{1}{2}\sigma\sigma'\mathcal{U})$. Thus from [1], eqs. 5, 8 we may calculate that up to a constant

$$
I_{pw} = \tfrac{1}{2}(1 + \tfrac{1}{2}\mathcal{U}) \log(1 + \tfrac{1}{2}\mathcal{U}) + \tfrac{1}{2}(1 - \tfrac{1}{2}\mathcal{U}) \log(1 - \tfrac{1}{2}\mathcal{U}) \tag{9}
$$

$$
\frac{1}{N} I_{gl} = -\beta(\mathcal{U} - \mathcal{F}) \tag{10}
$$

For convenience we change to the variable $x \equiv 2\beta$, and denote partial differentiation with respect to x by a prime. From $\mathcal{U} = \dfrac{\partial}{\partial \beta}(\beta \mathcal{F})$ we find

$$
I'_{pw} = \tfrac{1}{4} \log \left(\frac{1 + \frac{1}{2}\mathcal{U}}{1 - \frac{1}{2}\mathcal{U}} \right) \cdot \mathcal{U}' \tag{11}
$$

$$
\frac{1}{N} I'_{gl} = -\tfrac{1}{2} x \mathcal{U}' \tag{12}
$$

We want to evaluate these quantities as $x \to x_c$ from below, where $x_c \equiv 2\beta_c = \log\left(\sqrt{2} + 1\right)$. We thus set $x = x_c - \varepsilon$ and let $\varepsilon \to 0$ from above. Setting $\kappa \equiv 2 \dfrac{\sinh x}{\cosh^2 x}$ we have ([1], TABLE I)

$$
\mathcal{U} = -\coth x \left[1 + \frac{2}{\pi} \left(\kappa \sinh x - 1 \right) K(\kappa) \right] \tag{13}
$$

where

$$K(\kappa) \equiv \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\kappa^2 \sin^2\theta}} \tag{14}$$

is the *complete elliptic integral of the first kind* [2]. Working to $\boldsymbol{O}(\varepsilon)$, we may calculate

$$\sinh x = 1 - \sqrt{2}\varepsilon + \boldsymbol{O}\left(\varepsilon^2\right) \tag{15}$$
$$\cosh x = \sqrt{2} - \varepsilon + \boldsymbol{O}\left(\varepsilon^2\right) \tag{16}$$
$$\tanh x = \tfrac{1}{\sqrt{2}} - \tfrac{1}{2}\varepsilon + \boldsymbol{O}\left(\varepsilon^2\right) \tag{17}$$
$$\coth x = \sqrt{2} + \varepsilon + \boldsymbol{O}\left(\varepsilon^2\right) \tag{18}$$

and to $\boldsymbol{O}\left(\varepsilon^2\right)$

$$\kappa = 1 - \varepsilon^2 + \boldsymbol{O}\left(\varepsilon^3\right) \tag{19}$$

First we evaluate $\mathcal{U}$ as $x \to x_c$ from below. From (13) we have

$$\mathcal{U} = -(\sqrt{2}+\varepsilon)\left[1 - \frac{2\sqrt{2}}{\pi} \cdot \varepsilon K\left(1-\varepsilon^2\right)\right] + \boldsymbol{O}\left(\varepsilon^2\right) \tag{20}$$

Now $K\left(1-\varepsilon^2\right) \to \infty$ logarithmically as $\varepsilon \to 0$ [3], so that $\varepsilon K\left(1-\varepsilon^2\right) \to 0$ and $\mathcal{U} \to -\sqrt{2}$ as $x \to x_c$ from below. Thus from (11) and (12) we see that both I'_{pw} and $\frac{1}{N} I'_{gl} \to -\frac{1}{2} x_c \mathcal{U}'$ as $x \to x_c$ from below. From (13) a straightforward calculation yields

$$\mathcal{U}' = -\frac{1}{\sinh x \cosh x}\mathcal{U} - \frac{8}{\pi}\frac{1}{\cosh^2 x}K(\kappa) + \frac{4}{\pi}\frac{(\kappa \sinh x - 1)^2}{\sinh x}K'(\kappa) \tag{21}$$

Now

$$K'(\kappa) = \frac{1}{\kappa(1-\kappa^2)}E(\kappa) - \frac{1}{\kappa}K(\kappa) \tag{22}$$

[2] where

$$E(\kappa) \equiv \int_0^{\pi/2} \sqrt{1-\kappa^2\sin^2\theta}\, d\theta \tag{23}$$

is the *complete elliptic integral of the second kind* [2]. Some algebra yields

$$\mathcal{U}' = -\frac{1}{\sinh x \cosh x}\mathcal{U} + \frac{4}{\pi}\frac{(\kappa \sinh x - 1)^2}{\kappa(1-\kappa^2)\sinh x}E(\kappa) - \frac{2}{\pi}\coth^2 x\, K(\kappa) \tag{24}$$

Using $E(1) = 1$ [2], we find

$$\mathcal{U}' \to 1 + \frac{4}{\pi} - \frac{4}{\pi}K(\kappa) \tag{25}$$

as $\varepsilon \to 0$. But $K(\kappa) \to \infty$ logarithmically as $\kappa \to 1$, so $\mathcal{U}' \to -\infty$ which implies $\dfrac{\partial I_{pw}}{\partial \beta}, \dfrac{1}{N}\dfrac{\partial I_{gl}}{\partial \beta} \to +\infty$ as $\beta \to \beta_c$ from below and finally, since $\dfrac{\partial}{\partial \beta} = -T^2 \dfrac{\partial}{\partial T}$, we have $\dfrac{\partial I_{pw}}{\partial T}, \dfrac{1}{N}\dfrac{\partial I_{gl}}{\partial T} \to -\infty$ logarithmically as $T \to T_c$ from above.

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