

Levels of Description: A Novel Approach to Dynamical Hierarchies.

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Abstract. We present a novel formal interpretation of “dynamical hierarchies” based on information theory, in which each level is a near state-determined system, and where levels are related to one another in a partial ordering. This re-formulation moves away from previous definitions which have considered unique hierarchies of structures or objects arranged in aggregates. Instead, we consider hierarchies of dynamical systems: these are more suited to describing living systems, which are not mere aggregates, but organisations. Transformations from lower to higher levels in a hierarchy are re-descriptions that lose information. There are two criteria for partial ordering. One is a state-dependence criterion enforcing predictability within a level. The second is a distinctness criterion which enforces the idea that the higher-level description must do more than just throw information away. We hope this will be a useful tool for empirical studies of both computational and physical dynamical hierarchies.

1 Introduction.

Several recent articles have proposed formal definitions of “dynamical hierarchies” [2, 11, 6, 7, 3]. The term “dynamical hierarchy” seeks to capture the seemingly intuitively “obvious” notion that living organisms are self-constructing and

self-maintaining (i.e. dynamical), and composite (i.e. hierarchical) organisations. However, it is acknowledged by the authors themselves, and confirmed in this paper, that no satisfactory formal definition exists. All previous definitions have failed to capture a crucial property of systems which make them so “obviously” dynamical hierarchies. We recast dynamical hierarchies within the dynamical systems framework. Dynamical systems approaches attempt to construct state-determined systems [1], which allow prediction of at least one property in which we are interested. The set of states at a given time is a description of the state of the system. But, given an infinite set of observable properties of objects, why should we choose to describe them in any particular way?

We propose a formal method that uses information theory to produce a partial ordering of near state-determined (i.e. state-dependent) dynamical systems, in which causality is only a property within a level, and in which functions that map lower levels to higher levels are re-descriptions where information is lost, and yet allows distinct predictions to be made. Our basic framework is applicable to any system in which entropy can be defined, and where the space of properties can be sampled (measurements made). In this way, it applies to both computational and physical systems. It is broader than the hyperstructure view of dynamical hierarchies because a system may admit many different partial orderings of higher levels, and because we consider hierarchies of systems rather than hierarchies within systems — thus the hierarchies need not be limited to objects within some given framework.

We discuss the implications for the notion of hyperstructures proposed by Baas [2], and the implications of this formulation for clarifying the recent dis-

agreements between Rasmussen [11] and Gross and McMullin [6].

2 The Hyperstructure Formalism for Defining Dynamical Hierarchies.

Baas [2] described a framework for defining dynamical hierarchies. In summary, a set of structures were arbitrarily defined as the lower level primitives, $S1$. They possessed properties obtained by an observation function, $Obs(S1)$. The structures then interacted (i.e. underwent some function, R , on their properties that changed the property values) resulting in second-order structures, $S2$. Another observation function was associated with these second-order structures, $Obs2(S2)$. Second-order structures thus consisted of first-order structures. A property, P , was defined as an emergent property of $S2$, if and only if P was an element of $Obs2(S2)$ and P was not an element of $Obs2(S1)$. So, P was emergent if and only if P was a “new” property, observable only on $S2$ objects, not on any $S1$ objects. The resulting $S2$ objects upon which P could be observed were called emergent structures. Hyperstructures were defined as consisting of families of second-order structures, i.e. at least third-order structures, $S3 = R(S2, Obs2(S2), Int(1,2))$, where $Int(1,2)$ were the interactions between lower-order structures.

The crucial problem with this formulation is that trivial “new” properties are allowed. We contribute a reformulation that captures relevant necessary conditions about properties that make them “meaningfully” and “interestingly” new. The hyperstructure description also allows patterns of interaction and causality between levels which make it non-operational within most empirical methods. We avoid such a complex causal framework, which we believe is unnecessarily

confusing.

3 Problems with Hyperstructures as an Operational Framework.

Rasmussen et al [11] used Baas' definition to analyse a model which they claimed had self-organised a hyperstructure. The first-order structures (S1) were monomers with observable properties such as "hydrophobic" and "hydrophilic" behaviour. These aggregated into polymers, second-order structures, with properties of phase transition, elasticity and amphipathic behaviour. These polymers formed micelles, and these were claimed to be third-order structures, i.e. hyperstructures, because they had new properties such as "inside", "outside" and permeability.

However, in a simplified model, Gross and McMullin [6] claimed that micelles were not, after all, third-order structures. They pointed out that the hyperstructure definition allowed for trivial higher-order structures; for example, in a pile of sand, arbitrary partitioning of pairs of adjacent grains into second-order structures is possible. Pairs of pairs of grains can then be designated third-order structures, and so forth. They wrote "obviously this is not what we mean when we talk about hierarchy." (ibid p360). Gross and McMullin argued for a further constraint in order to define a higher-order structure: that an Nth-order structure demands an interaction of (N-1)th-order structures either during formation or later (or both). Micelles could be formed in their simpler model by interaction of monomers alone, and hence according to their definition were only second-order structures. The problem with this solution is that interactions between grains of sand are also required for describing, say the distance between pairs

of grains of sand, as no two pairs of grains of sand may occupy the same space. The additional constraint forbidding null-interaction is necessary but certainly not sufficient to exclude trivial orders in a dynamical hierarchy.

Rasmussen et al [12] took issue with the sandpile criticism, saying that “New hyperstructures with no new properties would be trivial... Pairs of grains of sand fall into this category.” However, it is wrong to say that pairs of grains of sand possess no new properties. Pairs of grains of sand do possess new properties, i.e. orientation and distance relative to each other, which cannot be possessed by a single grain of sand. Pairs of pairs of grains of sand also have novel properties such as the variables needed to define their relative orientations in space. They can also define several planes, which a pair of grains of sand cannot. These “new” properties are not generally “interesting” to us.

Gross and Lenaert [7] attempted to fix the holes in the Baas formulation by giving as a necessary requirement for a novel property, that a new type of interaction occurs between structures at the lowest level. The previous definitions have been over-inclusive (e.g. allowing arbitrary trivial levels to be defined in a sandpile). Instead, Gross and Lenaert’s definition is over-exclusive; we do not believe that novel properties of biological organisms involve new types of interaction between fundamental particles. Also, like the other agent-based models, it considers higher-order structures as composed of aggregates of lower-order structures. This does not allow a glider in Conway’s Game of Life [4] to be defined as a higher-order structure, because a glider is not an aggregate of agents but rather a pattern.

The inadequacy of the hyperstructure definition of “dynamical hierarchies” is demonstrated by Alan Dorin and Jon McCormack [3] who present an “infinitely-leveled, self-assembling dynamical hierarchy” formed essentially of a grid containing triangles which stick together making larger and larger structures. These larger structures can be shown to possess (trivial) new properties and to have formed due to interactions between objects which did not possess all the properties of the resulting aggregate object. Although their system satisfies the formal definition of hierarchies elaborated by both Rasmussen et al, and Gross and McMullin, it is still entirely trivial and uninteresting¹.

In summary, neither Baas, Rasmussen et al, Gross and McMullin, Dorin and McCormack, nor Gross and Lenaert have, at the time of writing this paper, produced a formal definition of the necessary and sufficient requirements for non-trivial dynamical hierarchies. The problem with hyperstructures is twofold: first, no adequate formal definition of “(interesting) new properties” has been proposed; second, hyperstructures permit only hierarchies of *structure* as opposed to hierarchies of *organisation*[9]. Dorin and McCormack suggest that an information theory definition of a hierarchy may be valuable, which is exactly the approach we take in this paper.

4 Why Have Different Levels of Description?

The emphasis of our formal interpretation of dynamical hierarchies will be on producing *useful* descriptions of complex dynamical systems. These are descriptions which allow humans to manipulate and predict a system with ease, by

¹ One reason may be that the types of interaction shown by the primitives at higher levels does not change, i.e. there are no “scale bound allometries” in the physical laws of the system [5].

decomposing the system appropriately [14]. Our point here is that the value of higher-level descriptions is their function as a *cognitive tool* for humans, whose information-processing abilities are limited. Although it is not essential to our formal exposition, this perspective elucidates the crucial role of simplification in higher-level descriptions.

Why not just describe a system at the “lowest level”? Because it may be possible to choose a second level appropriately which allows it to be largely state-determined under restricted perturbations. For example, a closed thermodynamic system in which energy is conserved is an ordered dynamical system: a variable, energy, can be defined; equations can be written to predict its distribution and behavior in the absence of knowledge of the details of micro-structures at lower levels. In general, these predictions will not be perfect. The system will not be perfectly state-determined. Nevertheless thermodynamics is a useful descriptive formalism because it approximates a state-determined system under well defined conditions.

Our need for ordered state-determined systems in Rasmussen’s model is no exception. We wish to have “maximal grip” [10] , i.e. manipulability and predictability, upon properties such as permeability, elasticity, “inside” and “outside”. Our fascination with such properties starts at a young age when we start putting objects in our mouth. A desire to construct a *useful* state-determined dynamical system explains Rasmussen’s choice of properties by which to define his orders. The three levels which were previously described by Rasmussen using Baas’ formulation can now be reinterpreted in our framework. We can define at least three distinct near state-determined dynamical systems. The first level consists of properties of the monomers, e.g. their positions and velocities, and

a set of transformation functions programmed by Rasmussen according to what is known of hydrophobic and hydrophilic behaviour. The second level consists of reducing the number of variables and describing the elasticity of polymers for example. The third level involves a further dimensionality reduction to measurements of the permeability of micelles, along with a transformation function which describes how this permeability changes over time.

5 State-Determined Descriptive Levels

In order to make the problem of formalising our ideas more tractable, we begin by considering state-determined, discrete-time systems. These systems, such as cellular automata, are a highly restricted subset of dynamical systems in general. Nevertheless, they display a huge range of interesting behaviours.

Our approach at this point differs from Baas's in two major ways. First, instead of regarding the systems as being made up of objects, we see them simply as dynamical systems. That is to say, the systems have instantaneous states which consist of variables, and these states change over time, moving through a "state space". Second, we make explicit the role in which *levels of description* play in dynamical hierarchies. These differences allow us to place our formulation within the solid existing framework of dynamical systems theory.

5.1 A Simple System

We denote a (discrete-time) dynamical system by an uppercase S , with a subscript as in S_x when it is necessary to distinguish two or more systems.

The state of S_x at time t is written $s_x(t) \in \mathbb{S}_x$, where \mathbb{S}_x is the state space of S_x .

Let us begin by considering a discrete, deterministic dynamical system S_a . The system describes a predetermined trajectory through its state space \mathbb{S}_a according to a transition function $F_a : \mathbb{S}_a \rightarrow \mathbb{S}_a$. Formally,

$$\begin{aligned} s_a(0) &= C_a \\ s_a(t+1) &= F_a(s_a(t)) \end{aligned}$$

where C_a is the initial state of the system. In an object-centred formulation, $s_a(t)$ would be the instantaneous set of observable properties of the first-order objects forming S_a .

Now, still working within the object-centred framework, suppose that the system can also be described in terms of second-order objects whose properties are functions of the properties of more than one first-order object. These second-order objects define a second-order system S_b , whose state $s_b(t) \in \mathbb{S}_b$ is described in terms of the properties of the second-order objects. A *description* function $d_b : \mathbb{S}_a \rightarrow \mathbb{S}_b$ transforms a first-order state $s_a(t)$ of S_a into a second-order state $s_b(t)$ of S_b .

$$\begin{aligned} s_b(t) &= d_b(s_a(t)) \\ \therefore s_b(0) &= d_b(C_a) \\ s_b(t+1) &= d_b(F_a(s_a(t))) \end{aligned}$$

The movement of S_b in its second-order state space \mathbb{S}_b is entirely determined by the position of the first-order system S_a , and the first-order transition function F_a . However, if we have chosen the second-order description appropriately, S_b

itself will be *state-determined*. In other words, S_b will have a transition function $F_b : \mathbb{S}_b \rightarrow \mathbb{S}_b$ such that

$$s_b(t + 1) = F_b(s_b(t))$$

(See diagram 1.)

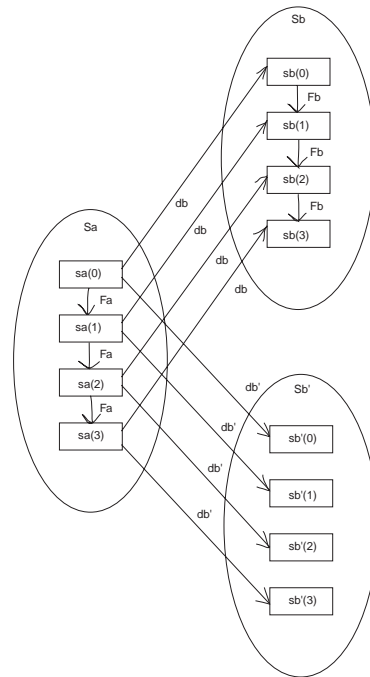


Fig. 1. The diagram shows two alternative higher-order descriptions of S_a . S_b has its own transition function F_b , but S'_b is not state-determined and does not have a transition function.

Of course, one could trivially construct state-determined descriptions of a system if the function d_b were permitted to be one-to-one (effectively, invertible). This does not seem intuitively to constitute a higher-order description. Therefore we propose a necessary condition for one description being higher-order than

the one it is based on: it must be *shorter* in some reasonable sense. For discrete systems, a reasonable sense of “shorter” is that the transformation from s_a to s_b involves a *loss of information*². This follows necessarily from disallowing the trivial case where d_b is one-to-one.

5.2 Conservative Example

The simplest example of a dynamical hierarchical description is in a *conservative* system, in which the first-order laws of the system result in the conservation through time of some high-order quantity $x(t) = X(s_a(t))$, such as energy or particle count.

$$X(s_a(t+1)) = X(s_a(t))$$

Then, the measurement of $x(t)$ represents a higher-order state-determined description of the system, because its value at time $t+1$ can be determined entirely from its value at time t , without knowing any of the system’s lower-order states. The state of the higher-order system S_x is simply the value of $x(t)$. Note that by using a dynamical systems approach, we do not have to worry about objects; object-based systems are a special instance of the more general case of a dynamical system.

6 Broadening Our Horizons

Our formulation so far distinguishes only between systems which are perfectly state-determined, and systems which are not. However, in most systems the higher-order (and sometimes lower-order) laws governing the system’s trajectory are only approximate, or involve some randomness. For instance, see [8] for an

² Or equivalently, a *reduction of dimensionality*.

example in which higher-order objects with stochastic laws emerge from a lower-order deterministic system.

We will turn to classical information theory to define a notion of *state-dependence*, a weaker version of state-determinism. We will see that this leads to a notion of *distinctness*. Taken together, these measures allow us to define a formal relation between two dynamical systems, which we call *hyperdescription*.

6.1 Basics: Entropy and Conditional Entropy

The central notion in classical information theory is that of *entropy*, which is a property of a probability distribution on a random variable. For a discrete distribution, the entropy $H[X]$ of the random variable X is defined as

$$H[X] = - \sum_x P(X = x) \log_2(P(X = x))$$

and the conditional entropy $H[X | Y = y]$ is defined as

$$H[X | Y] = - \sum_y \sum_x P(X = x | Y = y) \log_2(P(X = x | Y = y))$$

Entropy is often interpreted as the degree of uncertainty of a random variable's outcome. Using entropy to measure our systems introduces an additional complication to our analysis - in order to talk about the entropy of dynamical system states, we need to regard the states $s_x(t)$ as random variables (e.g. arising from an unknown initial state). This is analogous³ to considering how the laws of the system operate over the entire state space.

³ Most strongly so for deterministic systems with a uniform initial distribution.

6.2 State-Dependence

Instead of requiring that a higher-level system be *state-determined*, we want to be able to allow systems which are “largely” state-determined. Formally, we want $H[s_b(t+1) | s_b(t)]$ to be small. We certainly don’t want $H[s_b(t+1) | s_b(t)]$ to be as large as $H[s_b(t)]$, which would mean that the system’s current state tells us nothing about its next state.

Define the *state-dependence value*⁴ $r(S, t)$ of an entropic system as

$$r(S, t) = 1 - \frac{H[s(t+1) | s(t)]}{H[s(t+1)]}$$

(for non-entropic systems, where $H[s(t+1)] = 0$, r is undefined). Higher values correspond to “better” descriptions. It can be verified formally that $0 \leq r(S, t) \leq 1$.

6.3 Distinctness

Any higher-level description which does not lose much information about the causally relevant properties of the underlying system will necessarily be *state-dependent*. This is because whatever method is used to construct S_b from S_a , S_b must “know” at least as much about its own next state as it does about S_a ’s next state. It is only systems which somehow “know” *more* about their own behaviour than their underlying system’s behaviour which are interestingly “new” in the sense of being more than just noisy descriptions of the lower-level system.

Define the *distinctness value* $q(S_b, S_a, t)$ of S_b from S_a as

$$q(S_b, S_a, t) = 1 - \frac{H[s_b(t+1) | s_b(t)]}{H[s_a(t+1) | s_b(t)]}$$

⁴ This measure is similar in certain respects to the measure of *entropy density* for sequences [13].

(for systems where $H[s_a(t+1)|s_b(t)] = 0$, q is undefined). Distinctness expresses the “something extra” you get from choosing the right partial description of a system. (See Figure 2.) As before, higher values are better, and $0 \leq q(S_b, S_a, t) \leq 1$.

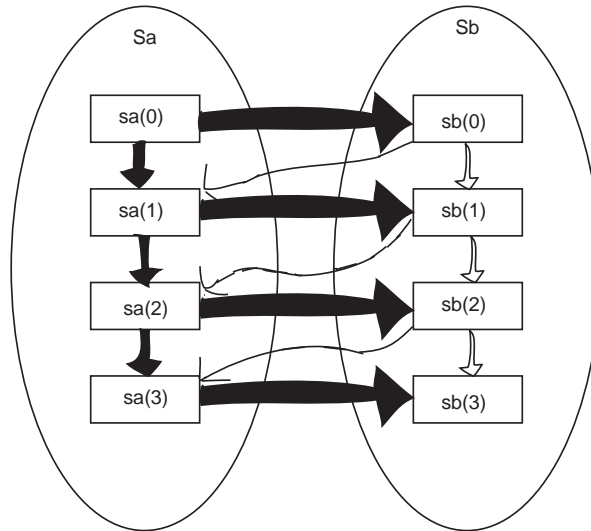


Fig. 2. System S_b and S_a proceed according to non-deterministic rules. The thicknesses of the arrows indicate how well states can be predicted from other states.

6.4 Definition of Hyperdescription

We say that a dynamical system S_b *hyperdescribes* at time t another system S_a (and thereby forms a *hyperdescription of S_a*) if and only if its *state-dependence* and *distinctness* values are greater than zero: -

<i>Specification</i>	There exists a non-random “description function” $d_b : \mathbb{S}_a \rightarrow \mathbb{S}_b$ such that $s_b(t) = d_b(s_a(t))$ and $s_b(t + 1) = d_b(s_a(t + 1))$
<i>State-Dependence</i>	$H[s_b(t + 1) s_b(t)] < H[s_b(t + 1)]$
<i>Distinctness</i>	$H[s_b(t + 1) s_b(t)] < H[s_a(t + 1) s_b(t)]$

N.B. The last two conditions are strict inequalities⁵.

Intuitively,

<i>Specification</i>	S_b ’s instantaneous state is simply a redescription of S_a ’s instantaneous state.
<i>State-Dependence</i>	The instantaneous state of S_b provides information about the next state of S_b .
<i>Distinctness</i>	The instantaneous state of S_b predicts the next state of S_b better than it does the next state of S_a .

When these conditions hold for all t (with the same d_b for all t), we simply say that S_b hyperdescribes S_a .

The Importance of the Distinctness Condition Whilst the meanings of the *specification* and *state-dependence* conditions are obvious, some more explanation is required to describe the role of the *distinctness* condition. Of the three formal conditions for hyperdescription, it is the hardest one to understand intuitively. However, the condition itself is crucial. It defines a sense in which a higher-level description of a system can *follow a new set of laws* compared to its underlying system. This is illustrated by the following examples.

⁵ As it happens, in both cases the $<$ sign can be equivalently replaced by a \neq sign.

Examples Suppose that S_1 is described by one variable $s_1(t) \in \{a, b, c\}$, and that its initial state is random (with uniform probability). S_1 is governed by the following simple law:

$$s_1(t+1) = \begin{cases} a & \text{if } s_1(t) = a \\ b & \text{if } s_1(t) = c \\ c & \text{if } s_1(t) = b \end{cases}$$

(See Figure 3.)

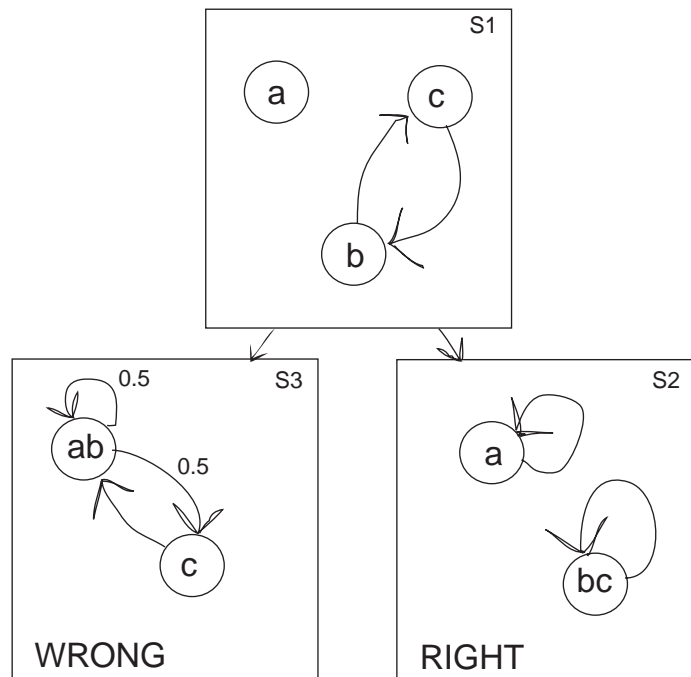


Fig. 3. Right and wrong decompositions of a simple dynamical system according to the distinctness criterion.

Hyperdescription Define another system S_2 whose state $s_2(t) \in \{a, bc\}$ is defined by

$$s_2(t) = \begin{cases} a & \text{if } s_1(t) = a \\ bc & \text{if } s_1(t) = b \text{ or } s_1(t) = c \end{cases}$$

Then it is easy to show that S_2 hyperdescribes S_1 . since

$$\begin{aligned} H[s_2(t+1) | s_2(t)] &= 0 & H[s_2(t+1)] &\approx 0.918 \\ H[s_1(t+1) | s_2(t)] &= \frac{2}{3} \end{aligned}$$

which fulfils both the hyperdescription inequalities.

Non-Hyperdescription With S_1 as given above, construct S_3 with state $s_3 \in \{ab, c\}$ defined as

$$s_3(t) = \begin{cases} ab & \text{if } s_1(t) = a \text{ or } s_1(t) = b \\ c & \text{if } s_1(t) = c \end{cases}$$

This time,

$$\begin{aligned} H[s_2(t+1) | s_2(t)] &= 0 & H[s_2(t+1)] &= \frac{2}{3} \\ H[s_1(t+1) | s_2(t)] &= \frac{2}{3} \end{aligned}$$

which fulfils *state-dependence* but not *distinctness*.

6.5 Formal Consequences of the Hyperdescription Definition

For discrete-time systems whose states are discrete random variables, it can be shown that: -

1. Hyperdescription entails partial information loss. This corresponds to the intuitive notion that a higher-order description must in some sense be *shorter* than a lower-order one, whilst still *capturing some dynamic information* about the lower-order system.
2. Hyperdescription defines a *partial order* on discrete-space dynamical systems. A partial order is a mathematical concept which allows for the comparison in a meaningful direction of some, but not all, pairs of elements of a set.⁶
3. There exist systems which cannot be hyperdescribed. Specifically, state-independent systems cannot be hyperdescribed.

The formal proofs of points 1. and 2., which are the significant points, can be found in the appendix. Information loss, which we postulated as a *condition* of higher-order description in the deterministic case, turns out to emerge as a *consequence* of the hyperdescription criteria.

7 The Implications of Hyperdescriptions For Dynamical Hierarchies

Multiple levels of hyperdescriptions are not the same as dynamical hierarchies. We believe a necessary (but not sufficient) condition for a dynamical hierarchy is that the different levels form hyperdescriptions or (in some formal sense) near-hyperdescriptions of one another. In this section we discuss some properties of hyperdescriptions which could plausibly be imputed to all dynamical hierarchies.

Hyperdescriptions are dynamical systems descriptions of other dynamical systems. Perhaps a dynamical hierarchy does not always have to consist of “objects”

⁶ Consequently, we could denote the hyperdescription relation by the $>$ symbol, with the symbols $<$ and \leq having obvious meanings. The \leq relation forms a *partial order* in the mathematical sense on dynamical systems. All the logic normally associated with these symbols can then be applied (except that for partial orders, $S_a \not\leq S_b$ does not imply $S_a \leq S_b$).

and “higher-order objects” made out of a collection of lower-order ones. Such constructs certainly can be used to construct higher-order descriptions of systems, but they are not the only way to do so. Indeed, even in an object-centred view of dynamical hierarchies, this observation is useful — although organisms are made of cells, the individual cells come and go over time. Although cells in turn are made of molecules, the individual molecules also come and go. CA gliders are made of CA cells, but a different set of CA cells at every cycle. All these higher-order objects are *patterns* in lower-order objects, i.e. they are features of a *description*, rather than a *collection* (in the sense of [9], an *organisation* rather than a *structure*).

It is not always possible to identify what levels of hyperdescription exist for a real system. Once we view dynamical hierarchies as being ordered levels of description, an important consequence is that we can’t always tell just by looking at a system whether a dynamical hierarchy exists or what its levels are. A significant part of the debate over the *ansatz*[6, 11, 12] has been whether micelles form second- or third-order structures in Rasmussen et al.’s artificial chemistry. In our formulation, this can only make sense if it is known that there are no intermediate levels of description between first-order, second-order and third-order descriptions. This is a statement about all possible *higher-order descriptions* of a system - the number of which grows hyperexponentially with the size of the system. It would be better in general simply to say that the respective levels are, or are not, higher-order than one another.

Hyperdescription defines a partial, rather than total, ordering on dynamical systems. There is absolutely no reason why levels of description in a dynamical hierarchy should be organised in a linear fashion. There might be multiple, alternative, equally good possible decompositions of a system, so that differentiating between them would have to be on subjective rather than formal grounds.

Gross & Lenaert [7] make the point that “for some purposes conclusions about the composition of [a natural system] may be different than those for other purposes”. We would claim that in general this extends to computational systems as well.

8 Going Beyond Hyperdescriptions

Our purpose in presenting the idea of hyperdescriptions is partly to propose a new formal tool for the science of dynamical hierarchies, and partly to provide a taste of what can be done with a combination of information theory and dynamical systems theory. We envisage that the hyperdescription criteria will be practically most useful in choosing between different descriptions of systems where: -

1. Preferably, the system is discrete, or can be discretised, in both time and space.
2. Candidate higher-level descriptions already exist.
3. Variables at a higher level are completely specified by variables at a lower level.
4. Conditional entropy values can be reasonably estimated, or at least ranked by magnitude.

With the possible exception of the last point, these restrictions are only shortcomings of the current hyperdescription formulation. They are not fundamental limitations of our view that dynamical hierarchies should be seen *formally* in terms of informational relations between systems, and *philosophically* in terms of making complex systems simpler for an observer to understand. Hopefully, such a view will yield better practical tools, and will broaden our understanding of dynamical hierarchies further, in the future.

9 Appendix: Formal Proofs

This section contains proofs of the theorems that “hyperdescription entails partial information loss” and “hyperdescription defines a partial order” for dynamical systems whose initial states are discrete random variables.

For these proofs, we use the notation $S_b \mathcal{R}_t S_a$ to indicate that S_b hyperdescribes S_a at time t . The proofs use the fact that entropy (including conditional entropy) is non-negative for all discrete random variables.

We will also use the result that for any discrete random variables X, Y, Z

$$H[X | Y] = 0 \quad \Rightarrow \quad H[X | Z] \leq H[Y | Z] \quad ([1.1])$$

which can be proven as follows⁷: -

$$\begin{aligned} H[X | Y] &= 0 \\ \Rightarrow H[X | Y, Z] &= 0 \\ \text{So } H[Y | Z] &= H[Y | Z] + H[X | Y, Z] \\ &= H[X, Y | Z] \\ &= H[X | Z] + H[Y | X, Z] \end{aligned}$$

Since $H[Y | X, Z] \geq 0$ always,

$$H[Y | Z] \geq H[X | Z]$$

Lastly, we will observe that the specification condition trivially entails $S_b \mathcal{R}_t S_a \Rightarrow H[s_b(t) | s_a(t)] = H[s_b(t+1) | s_a(t+1)] = 0$.

9.1 Hyperdescription Entails Partial Information Loss

We wish to show that if $S_b \mathcal{R}_t S_a$, then

⁷ Many thanks to Charles Goldie for this proof.

$$0 < H[s_a(t+1) | s_b(t+1)] < H[s_a(t+1)] \quad ([1.2])$$

For the sake of visual clarity, define $A' = s_a(t+1)$, $B = s_b(t)$, $B' = s_b(t+1)$.

For the first half of the inequality,

$$\begin{aligned} H[A' | B] &> H[B' | B] && (S_b \mathcal{R}_t S_a, \text{distinctness}) \\ \Rightarrow H[A' | B'] &\neq 0 && (\text{from [1.1]}) \\ \Rightarrow H[A' | B'] &> 0 && ([1.2.1]) \end{aligned}$$

For the second half of the inequality,

$$\begin{aligned} H[B' | A'] &= 0 && (S_b \mathcal{R}_t S_a, \text{specification}) \\ H[A' | B'] &= H[B' | A'] + H[A'] - H[B'] \\ &= H[A'] - H[B'] \\ \text{but } H[B'] &> H[B' | B] \quad (\geq 0) && (S_b \mathcal{R}_t S_a, \text{state-dependence}) \\ \Rightarrow H[A' | B'] &< H[A'] && ([1.2.2]) \end{aligned}$$

which, taken with [1.2.1], gives us [1.2].

9.2 Hyperdescription Defines a Partial Order

A relation \leq is a partial order if it is *transitive*, *antisymmetric* and *reflexive*.

We will prove that the hyperdescription relation \mathcal{R}_t is transitive, antisymmetric and *irreflexive* for discrete-space systems, and hence use it to construct a partial ordering on such systems.

Irreflexivity We wish to show $\neg(S \mathcal{R}_t S)$ for all S . This is trivial due to the above inequality [1.2.1] and the fact that $H[X | X] = 0$ for all X .

Antisymmetry Assume that $S_b \mathcal{R}_t S_a$ where $S_a \neq S_b$. Then we have to prove $\neg(S_a \mathcal{R}_t S_b)$. This follows trivially from the *specification* condition and the inequality [1.2.1].

Transitivity Assume that $S_c \mathcal{R}_t S_b$ and $S_b \mathcal{R}_t S_a$. Then we wish to prove that $S_c \mathcal{R}_t S_a$. To do this we have to prove three conditions hold: *specification*, *state-dependence* and *distinctness*.

1. **Specification of S_c by S_a**

This follows trivially from the definition and from composition of functions.

2. **State-dependence of S_c**

The *state-dependence* of S_c is already given by the assumption that $S_c \mathcal{R}_t S_b$.

3. **Distinctness of S_c from S_a**

Using the same convention as before, define $A' = s_a(t+1)$, $B' = s_b(t+1)$, $C = s_c(t)$, $C' = s_c(t+1)$.

$$\text{Observe } H[B' | A'] = 0 \quad (S_b \mathcal{R}_t S_a, \text{ specification})$$

$$\Rightarrow H[A' | C] \geq H[B' | C] \quad (\text{from [1.1]})$$

$$\text{But } H[B' | C] > H[C' | C] \quad (S_c \mathcal{R}_t S_b, \text{ distinctness})$$

$$\text{therefore } H[A' | C] > H[C' | C]$$

which is precisely the *distinctness* condition for S_c from S_a . This completes the transitivity proof that

$$(S_c \mathcal{R}_t S_b \quad \wedge \quad S_b \mathcal{R}_t S_a) \quad \Rightarrow \quad S_c \mathcal{R}_t S_a$$

Defining a Partial Order Finally, we define a partial order relation \leq on discrete-space systems as follows: -

$$S_a \leq S_b \quad \text{iff} \quad (S_b \mathcal{R}_t S_a \quad \vee \quad S_a = S_b)$$

where correspondingly $S_a > S_b$ is simply defined as $S_a \mathcal{R}_t S_b$. It is easy to verify that \leq forms a partial order.

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