Formal Computational Skills

Dynamical Systems Analysis

Dynamical Systems Analysis

Why? Often have sets of differential equations describing a dynamical system.

Analysis determines how the system behaves over time, in particular investigating future behaviour of the system given any current state ie the long-term behaviour of the system. Can solve equations for particular starting point(s), but this is often not enough to enable us to understand the system

Therefore use complementary analysis focused on finding equilibrium states (or stationary/critical/fixed points) where system remains unchanged over time

Also try to classify these states/points as stable/unstable by investigating the behaviour of the system near them

Eg coin balanced on a table: how many equilibria?

3: Heads, Tails, Edge

Start on its edge, it is in equilibrium … is it stable?

… No! Small movement away (perturbation) means that coin will end up in one of 2 different equilibria: Heads/ Tails.

Both stable as perturbation does not result in a change in state

Given a noisy world, coin will end up in a stable state

Note however, may not be able to tell which state it will end up in … idea of chaotic systems

Similarly, think of a ball at rest in a dark landscape. It’s either on top of a hill or at the bottom of a valley. To find out which, push it (perturb it), and see if it comes back. (What about flat bits?)

By the end you will be able to:

- Find fixed points of a system
- Classify fixed points as stable/unstable etc
- Use graphical methods (direction and cobweb plots) to analyse the behaviour of systems
- Use phase-plane analysis to analyse the behaviour of systems

1st half: 1 (space) dimensional systems

2nd half: 2 (and higher) dimensional systems

Will NOT talk about:

- Proofs of many of the stability theorems – focus will be on how to use them to analyse your systems
- The many special cases – these lectures are a primer

Fixed points of 1D systems

Want to analyse multi-dimensional nonlinear dynamical systems

start with simple 1D systems eg

$$dx/dt = f(x, t)$$

At a fixed point, x doesn't change as time increases, ie:

$$dx/dt = 0$$

So, to find fixed points, set: $f(x, t) = 0$ and solve

Note: same procedure for finding maxima and minima

Eg: $dx/dt = 6x(1 - x)$ What are the fixed points?

Set: $dx/dt = 0$ ie: $6x(1-x) = 0$

Maths fact: if $AB = 0$ either $A = 0$ or $B = 0$

so either $6x = 0 \Rightarrow x = 0$ or $1 - x = 0 \Rightarrow x = 1$

Therefore 2 fixed points: stability??

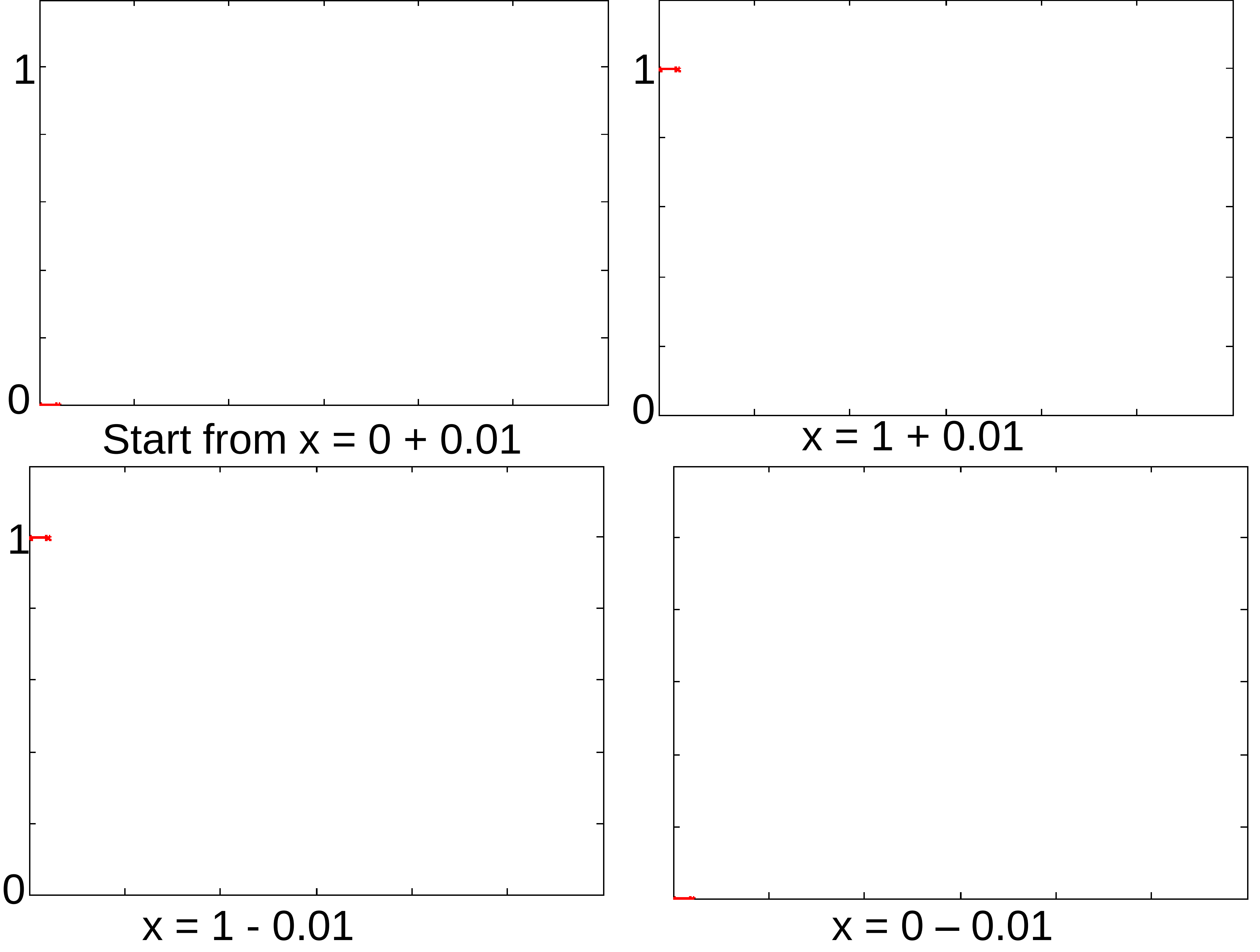
Perturb the points and see what happens …

But to perturb them must have a model of the system.

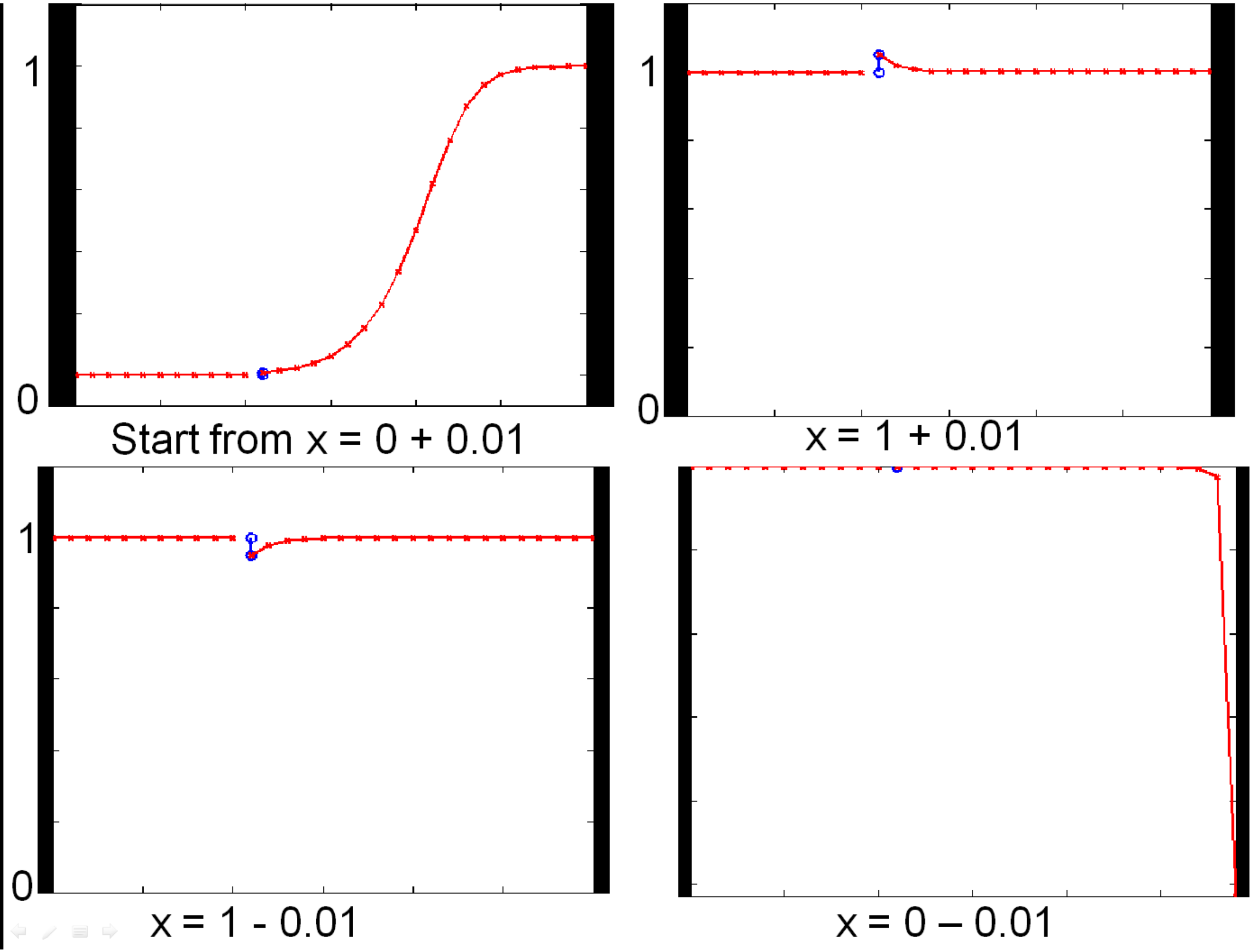
Use difference equation: $x(t+h) = x(t) + h\, dx/dt$ from various different initial x's

Eg start from: $x = 0 + 0.01$, $x = 0 - 0.01$,

$x = 1 + 0.01$, $x = 1 - 0.01$



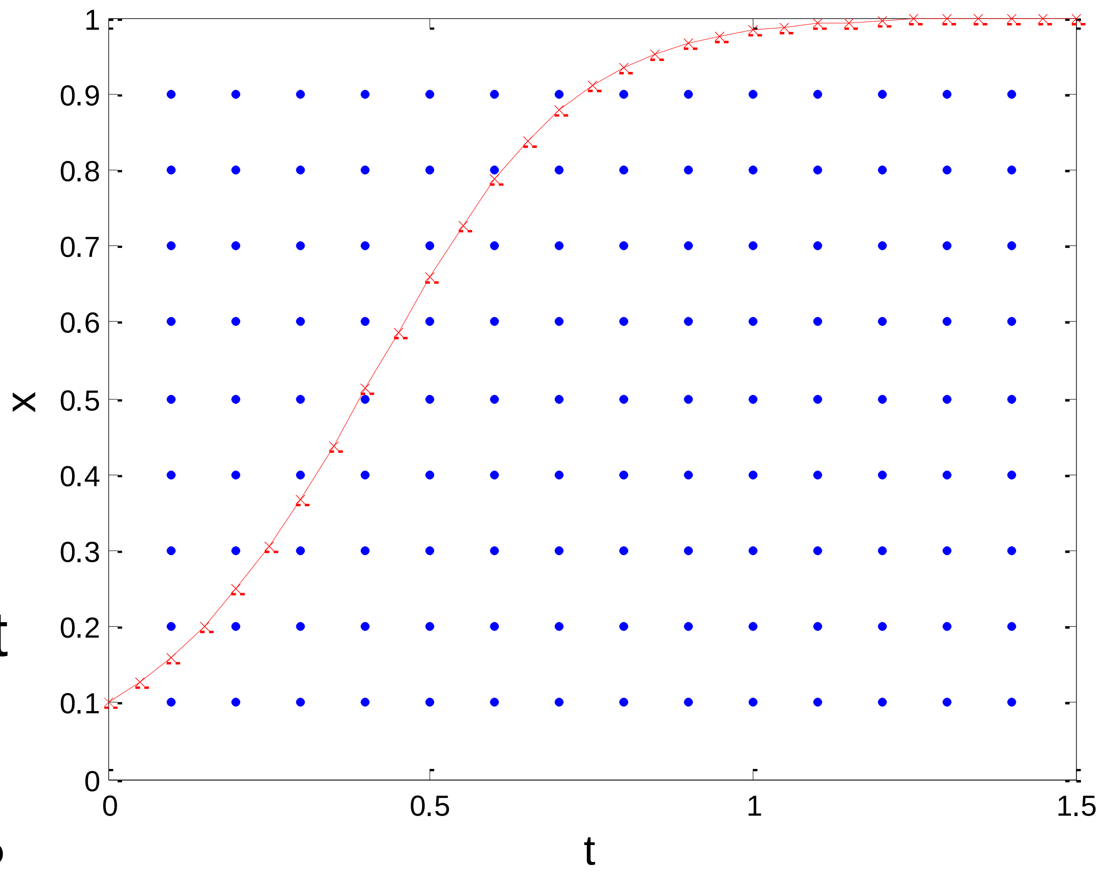
1
0
Start from x = 0 + 0.01
1
0
x = 1 + 0.01
1
0
x = 1 - 0.01
x = 0 – 0.01



So by perturbing looks like x=0 unstable and x=1 stable x=0 unstable and x=1 stable. What if we want more info?

What happens if we don't start from a fixed point? Iterate the system to get behaviour

What happens if we start at any starting positions eg x(0) =0.8 or any of the points shown by blue dots?

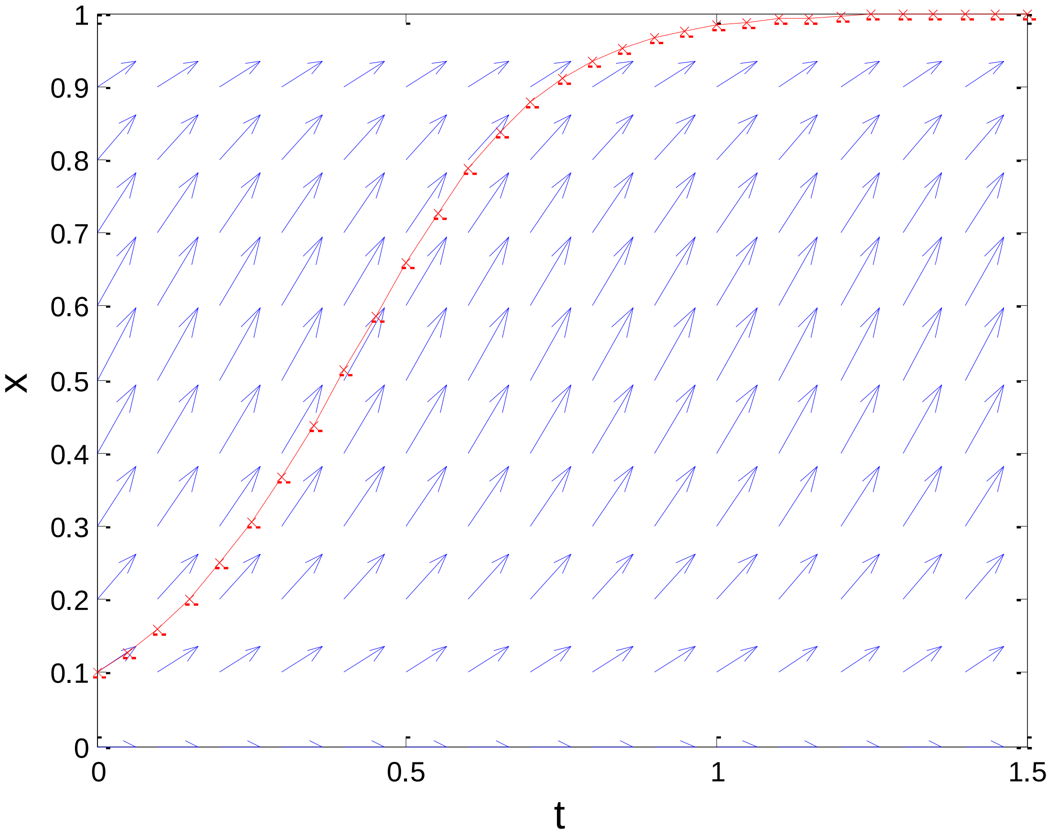


Use direction fields. Idea is to plot x against t and, at a grid of points on the plane, draw an arrow indicating the direction of subsequent points x(t+h), thus showing the movement of x for different starting conditions.

Direction fields

What is this direction? remember: x(t+h) = x(t) + h dx/dt

Drawing an arrow from x(t) to x(t+h), shows the direction of next point so arrow starts at (t, x(t)) and goes to (t+h, x(t)+hdx/dt)



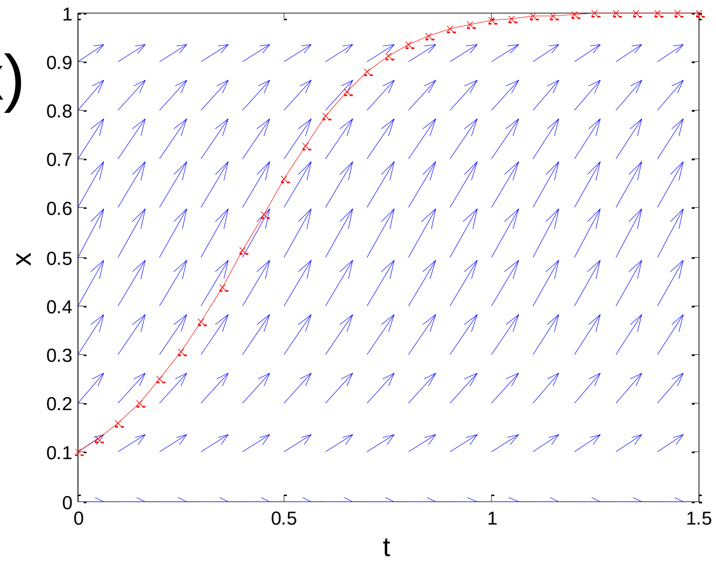
Matlab code to generate direction : use ‘quiver’

```
[T,X]=meshgrid([0:0.1:1.4],
        [0:0.1:0.9]);
NewT= ones(size(T));
NewX=6*X.*(1-X);
quiver(T,X,NewT,NewX)
```

Don’t need h as it is scaled automatically

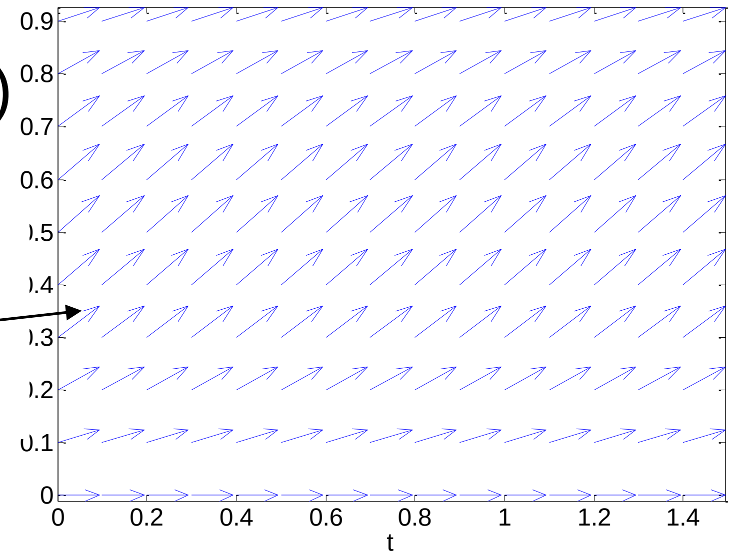
Can use them to eg see the influence of parameters on systems

6x(1-x)

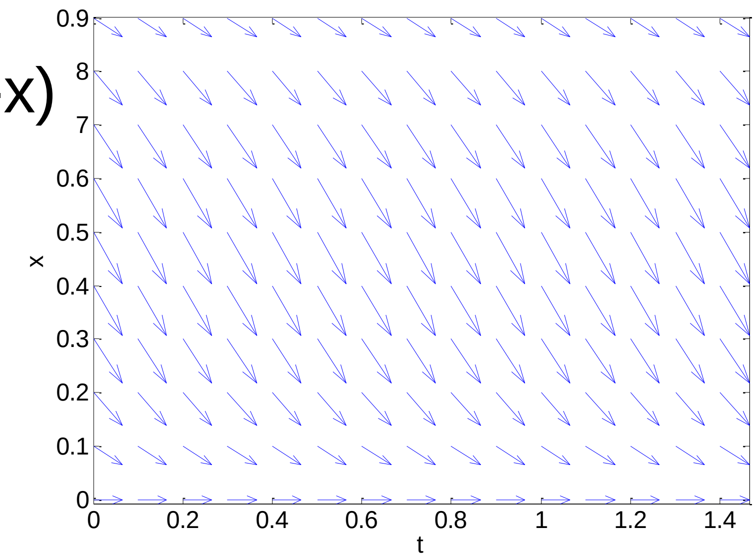


3x(1-x)

Rise is less steep

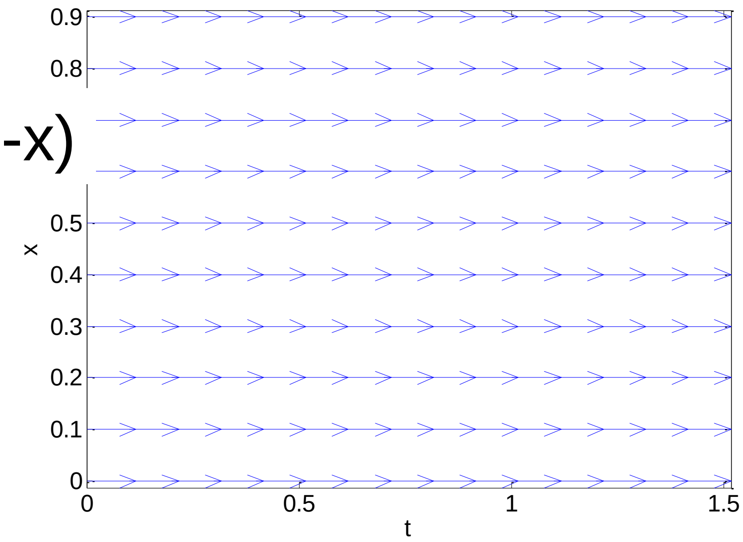


-6x(1-x)



Now 0 is stable and 1 unstable

0x(1-x)



All points stable: degenerate solution

Recap:

Have looked at analysis of 1D systems

Analysis focussed on finding fixed points and analysing stability

Showed that fixed points could be found by solving:

$$dx/dt = 0$$

Next we will examine discrete dynamical systems then systems with more dimensions!

Introduce cobweb plots as an analytical aid

Discrete dynamical systems

Now widen discussion to discrete dynamical systems where:

$$x(n+1) = f(x(n))$$

for a sequence of time steps, $t_0, t_1, t_2, \ldots, t_n, t_{n+1}, \ldots$

ie $x(n)$ is shorthand for "the value of x at the n'th timestep t_n"

Fixed points are those where $x(n+1) = x(n)$ so find them by solving: $a = f(a)$

[NB this is same as if dx/dt=0 since

Define $\quad f(x(n)) = x(n) + h\,dx/dt$,

then if: $\quad dx/dt = 0 \,, \quad x(n+1) = f(x(n)) = x(n) + 0 = x(n)$]

However, the dependency on the step-size h is now explicit

Important as stability can be determined by value of h used

Cobweb Plots

Often instructive to solve a = f(a) graphically ie plot y=a and y=f(a) on the same axes.

Eg $dx/dt=6x(1-x)$ with time-step $h=0.1$:

$x(n+1)=x(n) + h dx/dt$

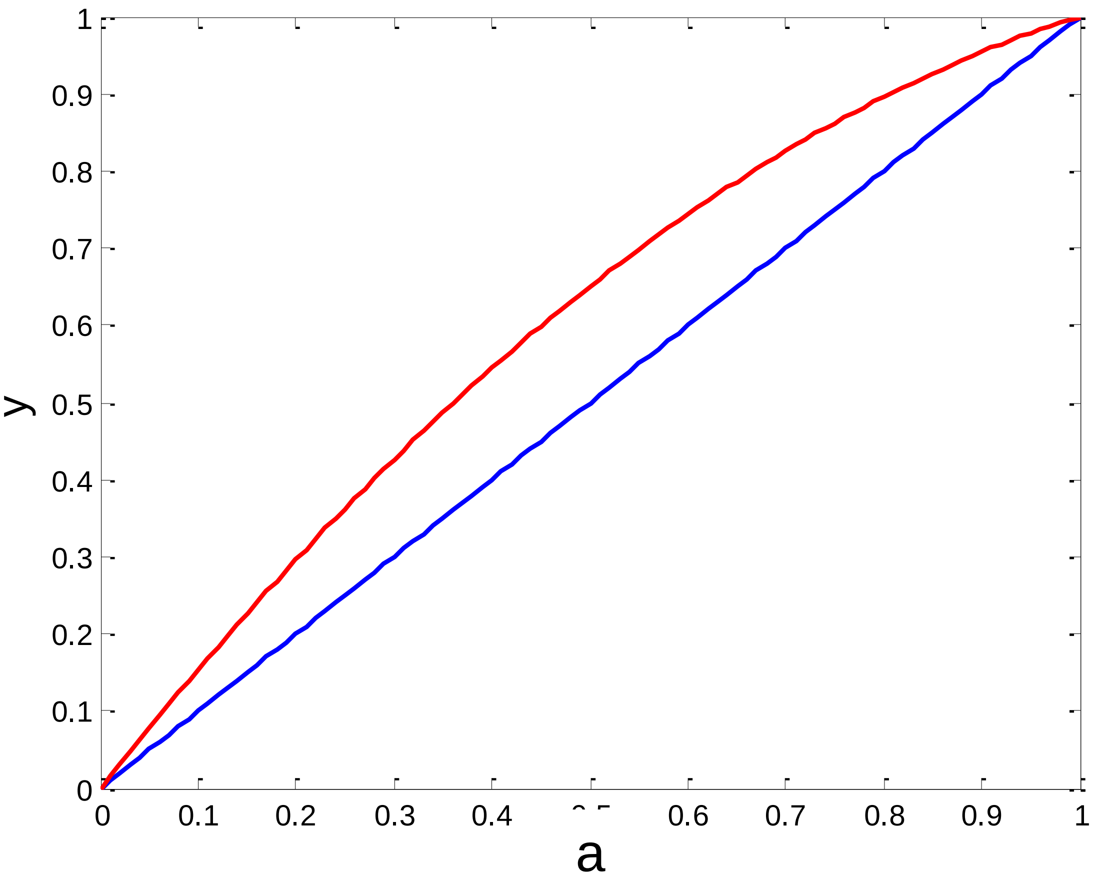
$x(n+1)=x(n) + h(6x(1-x))$

So: $f(x) = x + 0.6x(1-x)$

so plot:

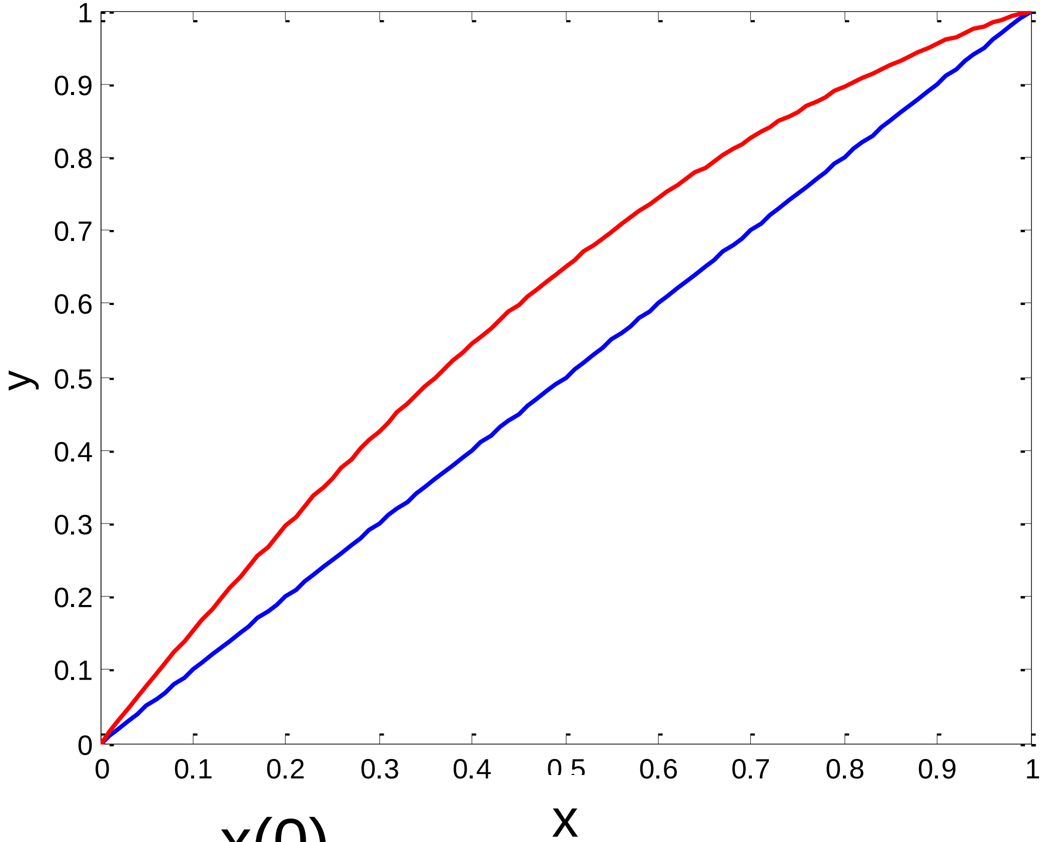
$y=f(a)=a + 0.6 a(1-a)$ (red)

and $y = a$ (blue)



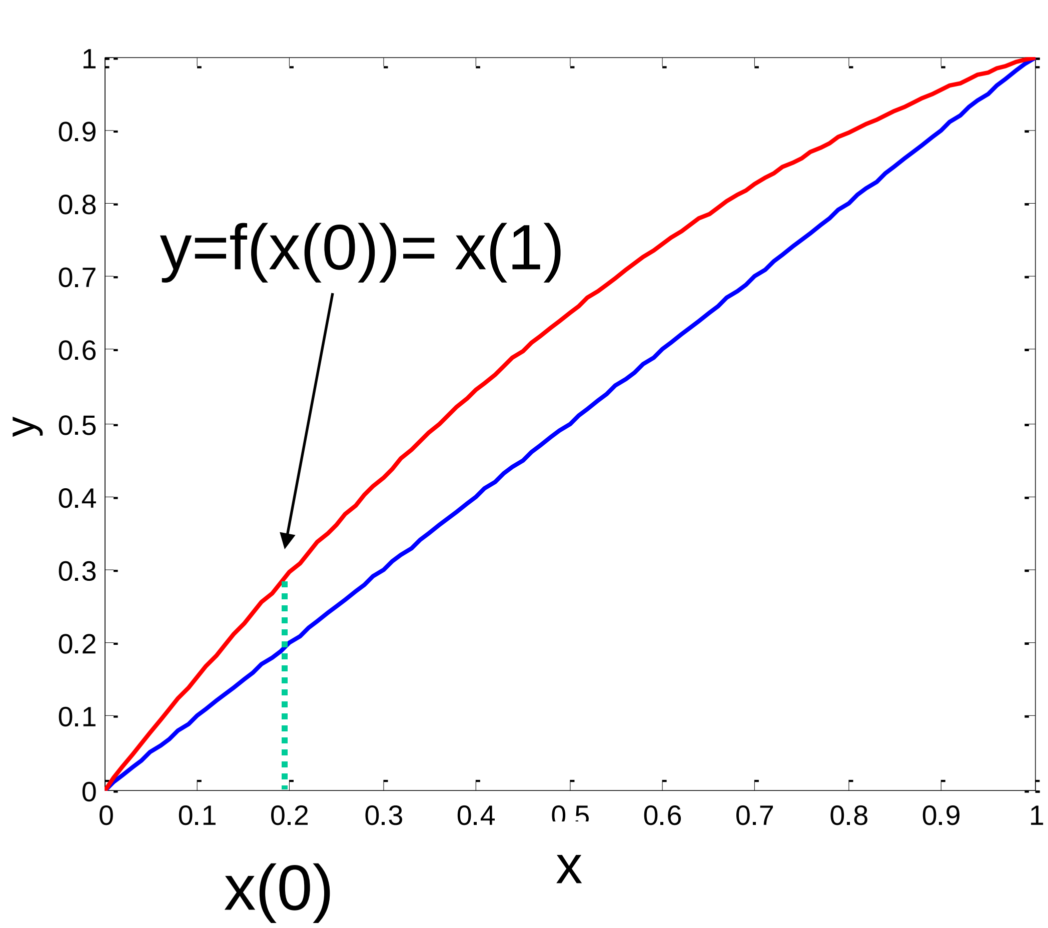
Where they cross, f(a) = a ie the fixed points ie a= 0 and a = 1.

However, these plots can also tell us about the stability of the fixed points via the following procedure:



x(0)

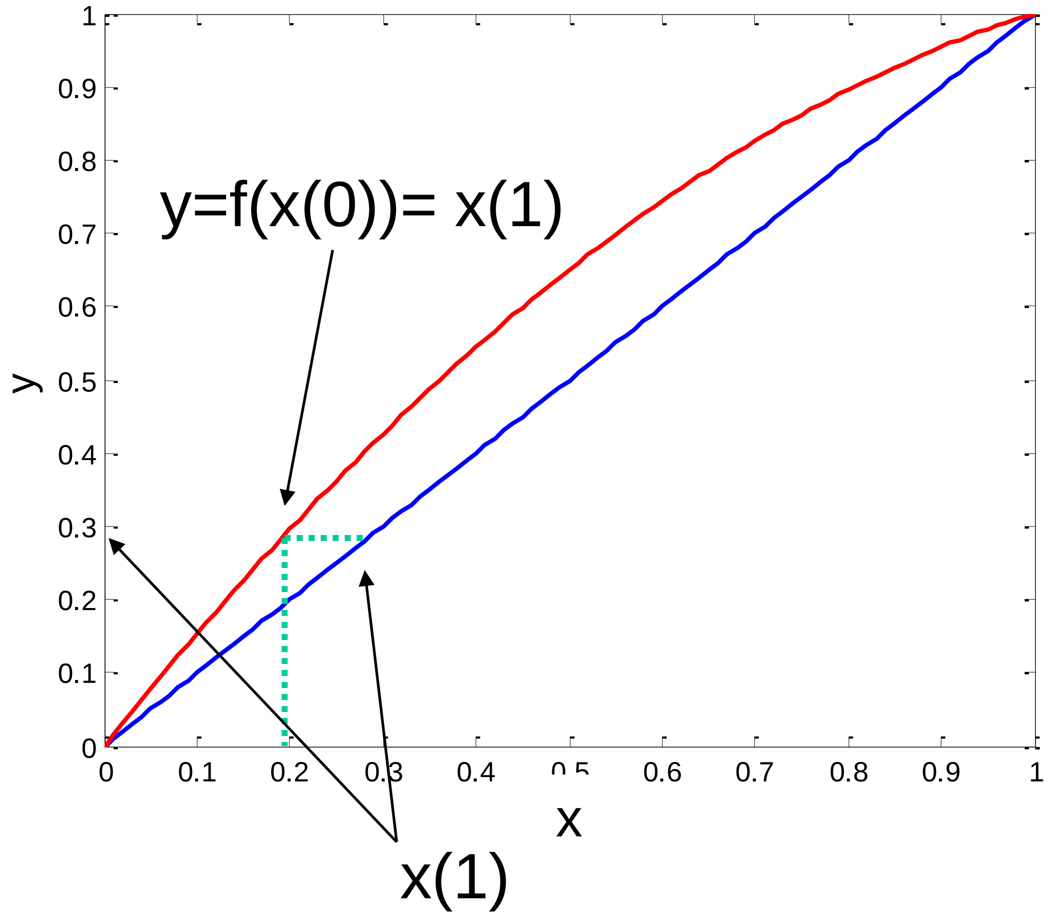
Start at a point x(0)=0.2



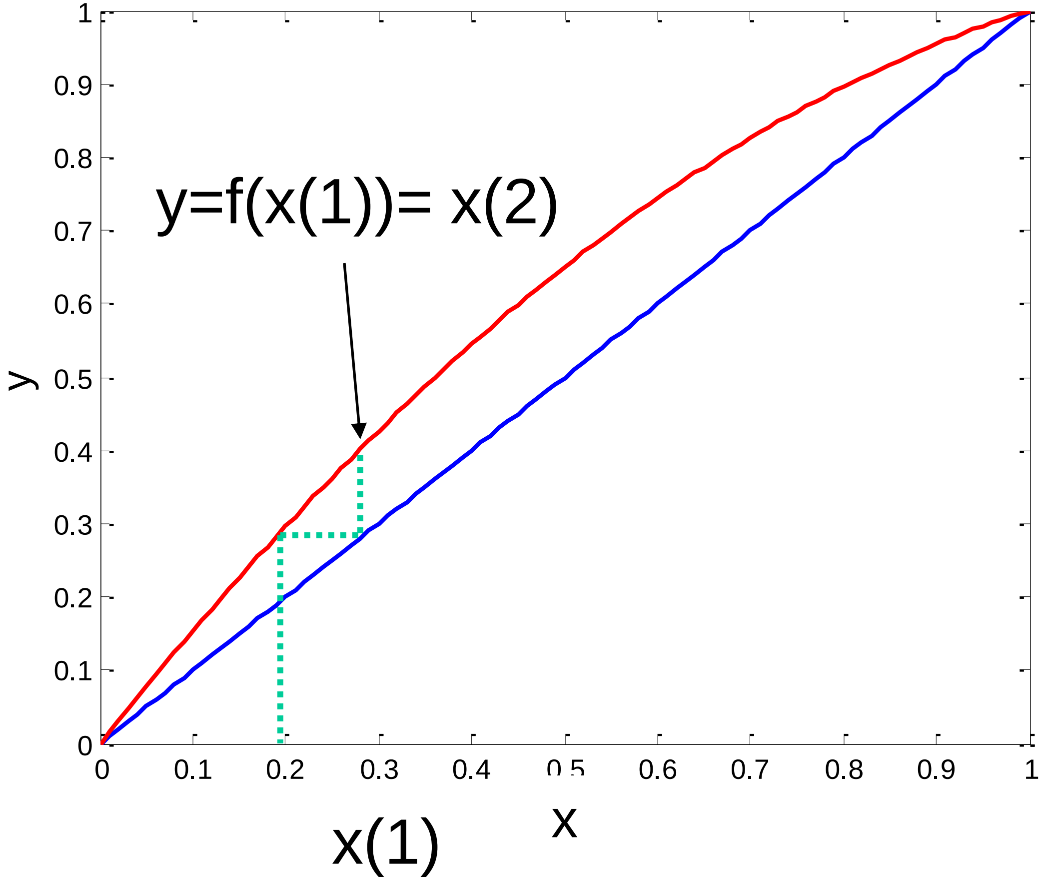
Start at a point x(0)=0.2

Go vertically up to the curve f(x)

This is f(x(0))= x(1)



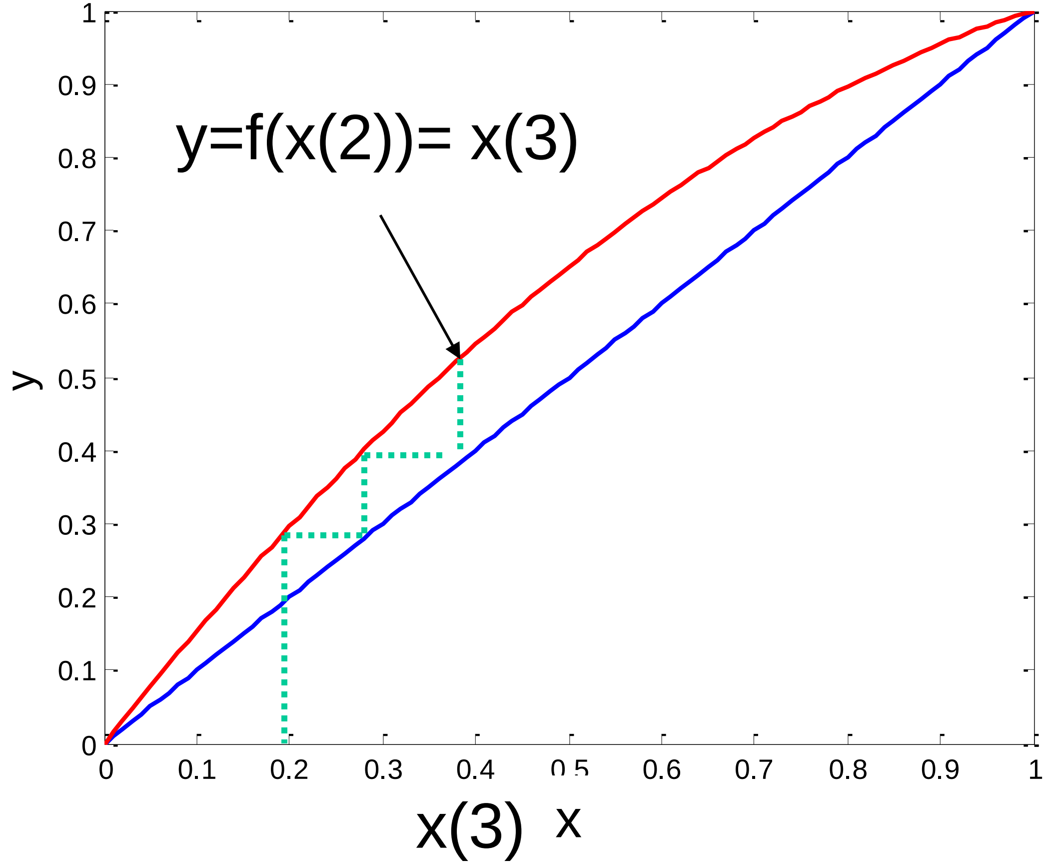
Go horizontally from this point to the line y=x as this is x(1) on the x-axis.



Go horizontally from this point to the line y=x as this is x(1) on the x-axis.

Get x(2) from going vertically from this point to f(x)

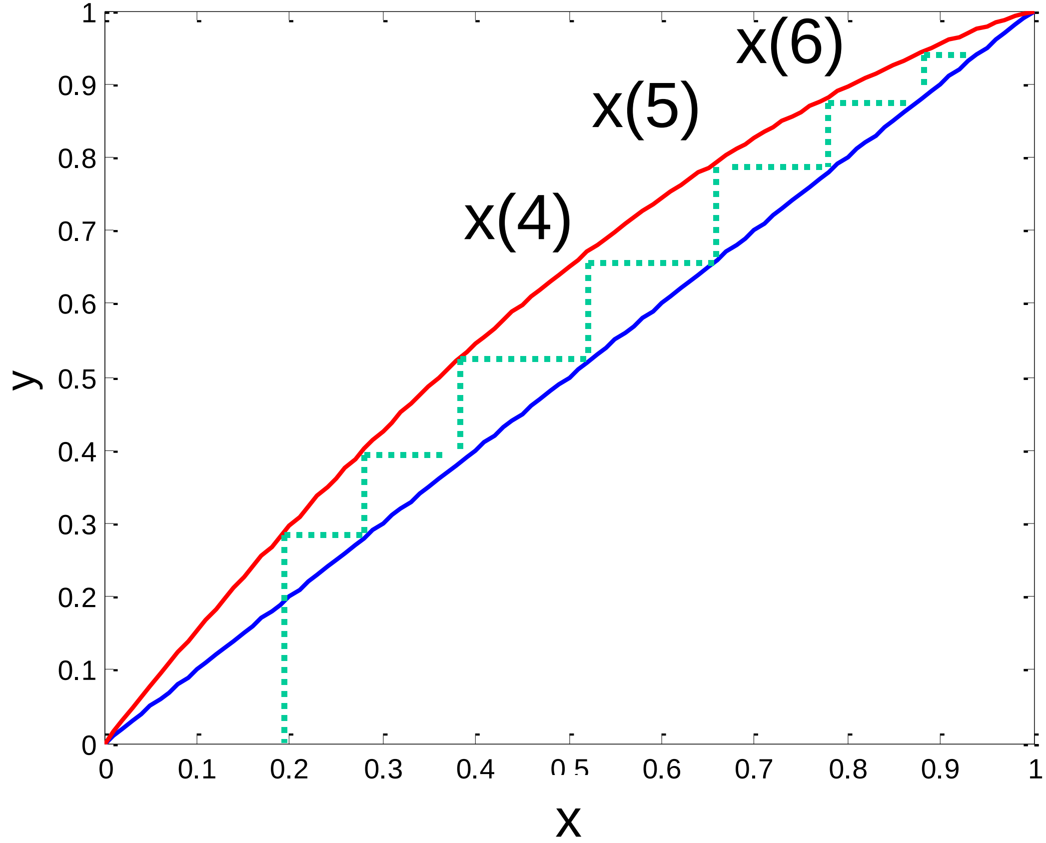
This is f(x(1))= x(2)



Go horizontally from this point to the line y=x to get x(2) on the x-axis

Then get x(3) from going vertically from this point to f(x) since:

f(x(2)) = x(3)



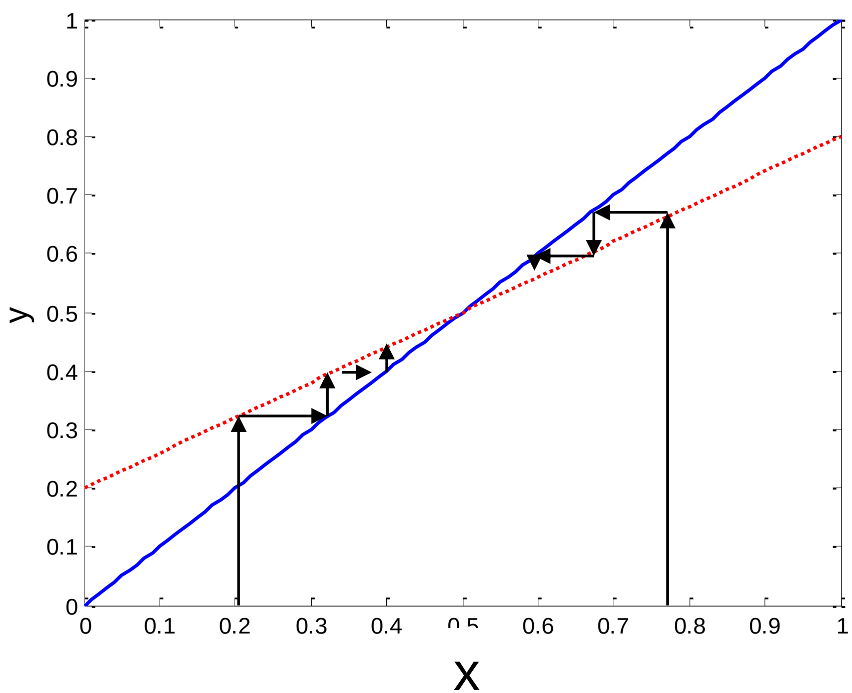
Then get x(4), x(5) etc etc

Can therefore classify stability of fixed points by seeing which starting points lead to which fixed point

Try eg’s on sheet. Start from values indicated

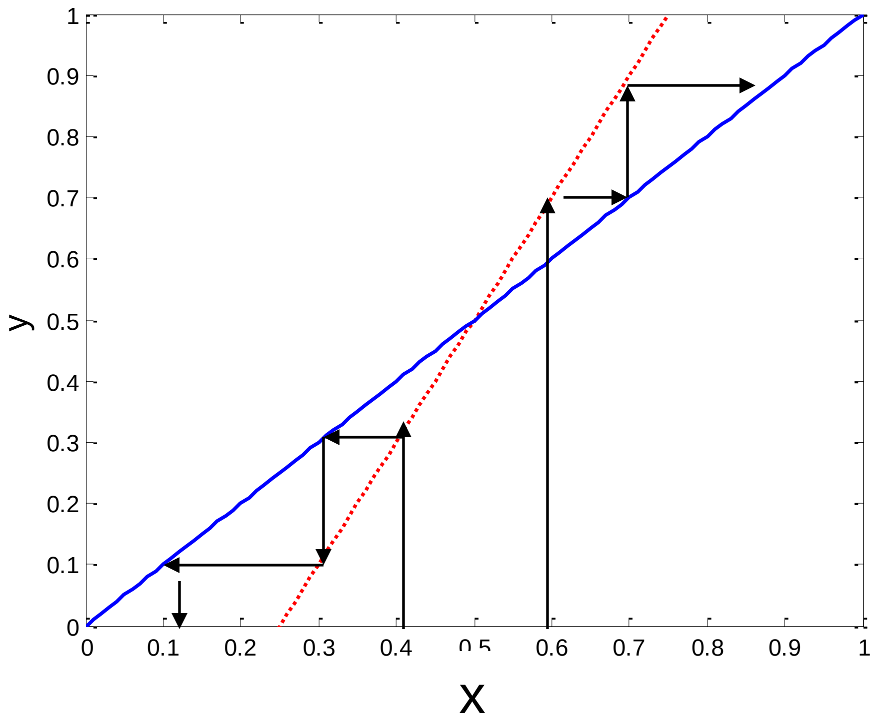
Solid (blue) lines are y=x, dashed (red) lines y=f(x)

y = 0.6x + 0.2



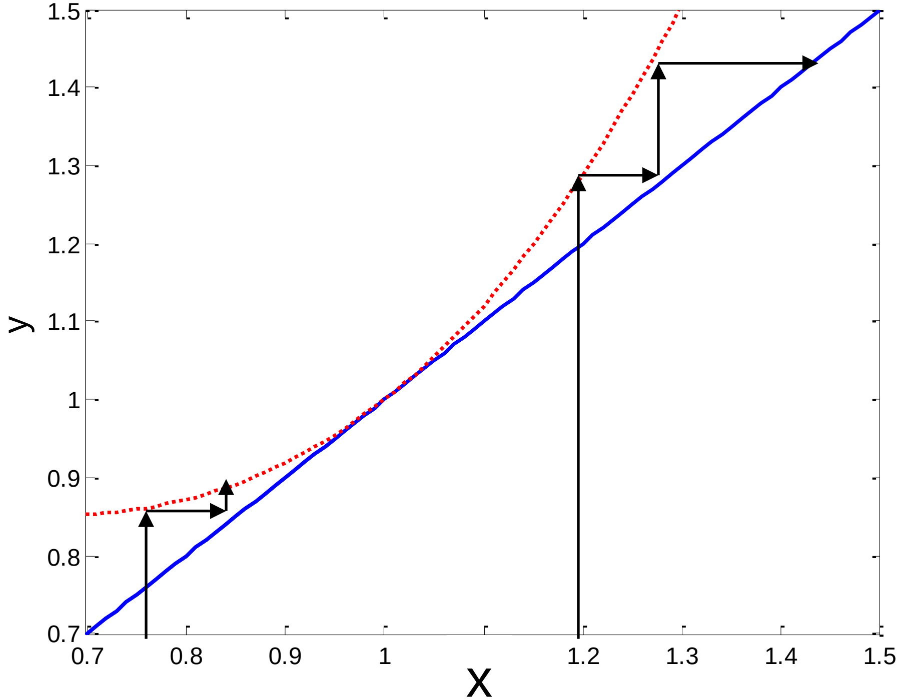
Stable

y = 2x - 0.5



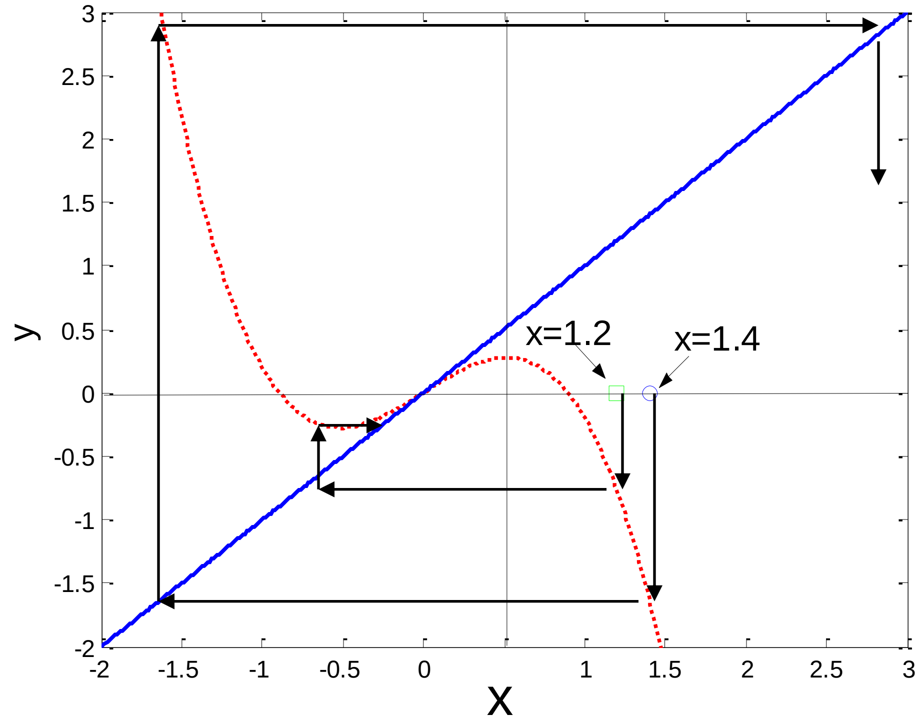
Unstable

$y = x^3 - x^2 + 1$



Semi-Stable from below

$y = 0.8x - x^3$

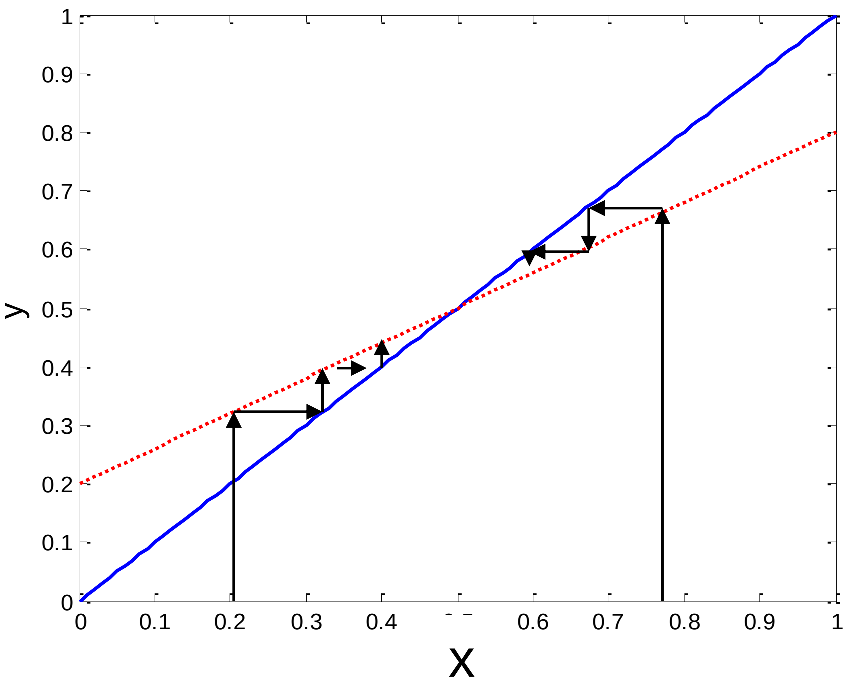


x=1.2 stable, x=1.4 unstable.

Last eg illustrates the concept of a basin of attraction: the range of values of a that will lead to a stable point if started from (or if passed through)

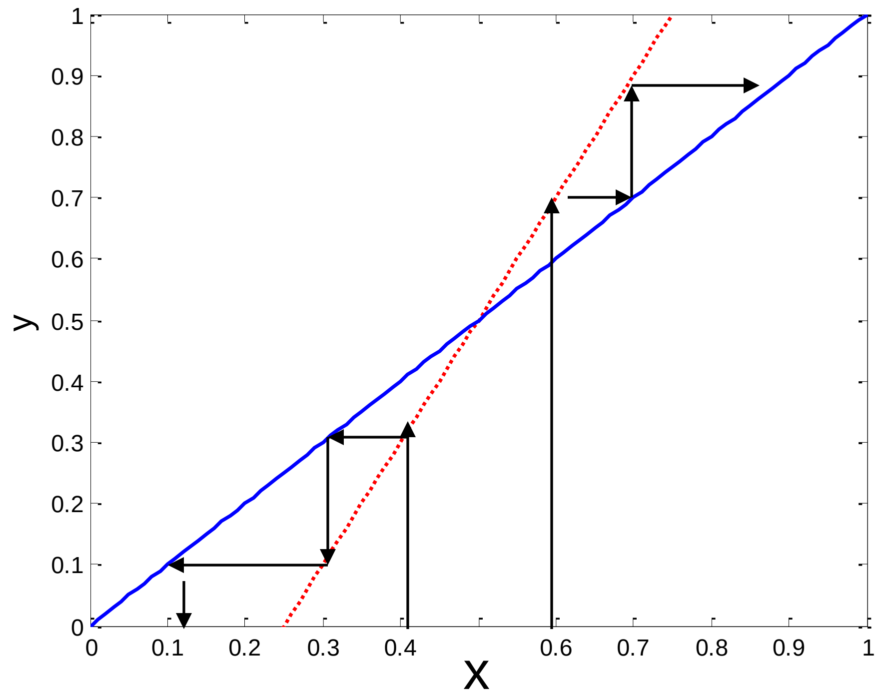
Notice that if the gradient of y=f(x) evaluated at a_0 (the fixed point) is > gradient of y=x (which equals 1) the point is unstable

$y = 0.6x + 0.2$



Stable

$y = 2x - 0.5$



Unstable

Leads us to following theorem ...

Classifying Fixed Points

If a_0 is a fixed point

if $|f'(a_0)| > 1$, a_0 is unstable
if $|f'(a_0)| < 1$, a_0 is stable
if $|f'(a_0)| = 1$, inconclusive.

(where $f'(a_0)$ means the derivative of f, df/dx (or df/da etc) evaluated at a_0 and $|f'(a_0)|$ means the absolute value of it)

if $|f'(a_0)| = 1$ need to use higher derivatives or other methods

Can be semi-stable from above or below or periodic …

eg $x(n+1) = -x(n) + 4$:

So $f(x) =$ $-x + 4$

$df/dx = f'(x) =$ -1

so $|f'(a)| = 1$,

but notice what happens if we start with $x(0) = -2$:

$x(0)=-2$,
$x(1) = -x + 4 =$ $-(-2) + 4 = 6$
$x(2) =$ $-6 + 4 = -2$
$x(3) =$ $-(-2) + 4 = 6$ etc

periodic with period 2

Check the Egs:

1) $f(x) = 0.6x + 0.2$

2) $f(x) = 2x - 0.5$

3) $f(x) = x^3 – x^2 + 1$

4) $f(x) = 0.8x - x^3$

First determine $df/dx = f'(x)$

Then calculate $f'(x)$ at the fixed points

Determine f'(x):

1) $f(x) = 0.6x + 0.2$

$f'(x) = 0.6$

2) $f(x) = 2x - 0.5$

$f'(x) = 2$

3) $f(x) = x^3 – x^2 + 1$

$f'(x) =3x^2 – 2x$

4) $f(x) = 0.8x – x^3$

$f'(x) =0.8 – 3x^2$

Calculate f'(x) at the fixed point:

1) $f(x) = 0.6x + 0.2$, $f'(x) = 0.6 < 1$ so stable

2) $f(x) = 2x - 0.5$, $f'(x) = 2 > 1$ so unstable

3) $f(x) = x^3 - x^2 + 1$, $f'(x) = 3x^2 - 2x$

$a_0=1$ so:
$f'(a_0) = 3(1^2) - 2(1) = 1$
so inconclusive (semi-stable)

4) $f(x) = 0.8x - x^3$, $f'(x) = 0.8 - 3x^2$

$a_0=0$ so:
$f'(a_0) = 0.8 - 3(0^2) = 0.8 < 1$
so stable

Summary

So far:

- how to find fixed points of a dynamical system
- concept of stability and its dependence on parameters
- Direction fields for determining behaviour
- cobweb plots for stability
- how to check the stability of a fixed point

In seminars:

- introduce a few more examples
- show how the wrong choice of a time-step leads to instability
- work through an example of this analysis used for GasNets

Now

- 2D (and higher) systems...

2 and higher dimensional systems

In higher-dimensional systems movement of trajectories can exhibit a wider range of dynamical behaviour

Fixed points still exist, but can be more interesting depending on how trajectories approach or repel from the equilibrium point eg system could spiral in to a stable point

Also, other types of stability exist eg saddle-nodes, and importantly cyclic/periodic behaviour: limit cycles

More interesting, but more difficult to analyse… We will cover:

- How to find fixed points
- Classifying fixed points for linear systems
- Phase-plane (phase-space) analysis of behaviour of system

2D systems

Analyse 2D (and multi-D) systems in a similar way to 1d systems.

Won't go into proofs (see eg introduction to ordinary differential equations, Saperstone and refs at end) but will give general procedure

Suppose we have the following system

$$\frac{dx}{dt} = \dot{x} = f(x, y)$$

$$\frac{dy}{dt} = \dot{y} = g(x, y)$$

1. Find fixed points
2. Examine stability of fixed points
3. Examine the phase plane and isoclines/trajectories

Find Fixed Points

Find fixed points as before ie solve dx/dt = 0 and dy/dt = 0 ie solve: f(x,y) = 0 and g(x,y) = 0 to get fixed points (x_0, y_0)

Eg predator-prey from last seminar:

$$\dot{x} = 0.6x - 0.05xy$$

$$\dot{y} = 0.005xy - 0.4y$$

Set 0 = 0.6x – 0.05xy
so: 0 = x(0.6 - 0.05y)
so: x = 0 or y = 12

and: 0 = 0.005xy – 0.4y
so: 0 = y(0.005x – 0.4)
so: y = 0 or x = 80

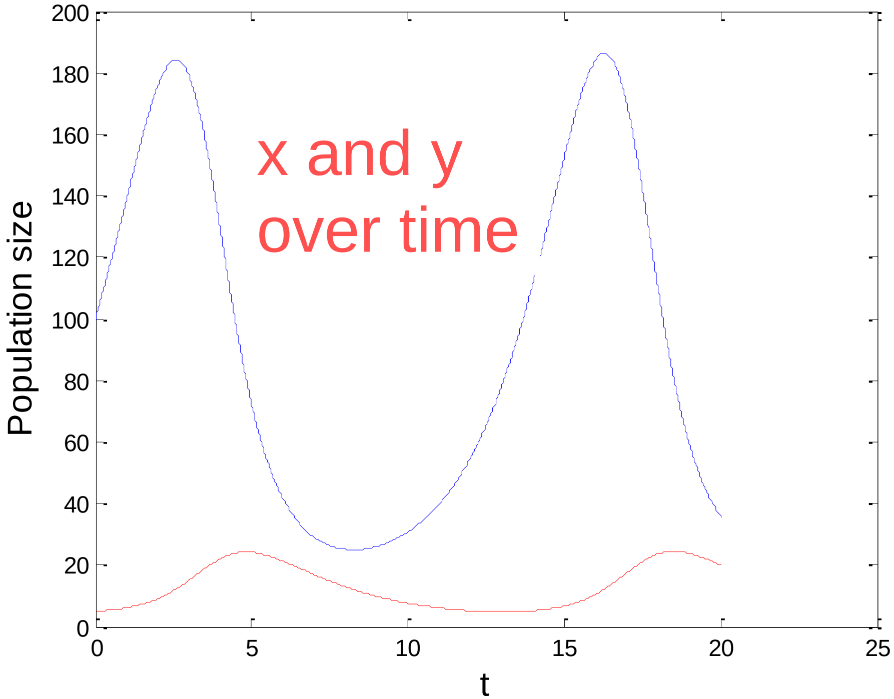
If x=0, dy/dt = -0.4y ie need y=0 for dy/dt=0 so fixed point at (0,0)

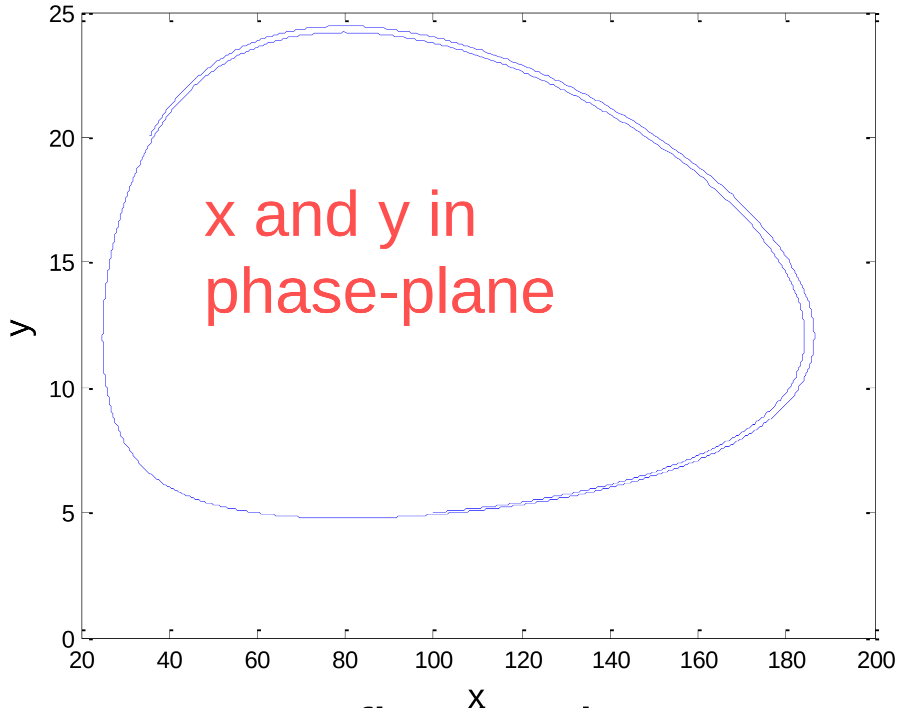
Similarly, if y=0, dx/dt = 0 for x=0 so again, (0,0) is fixed point

Other fixed point at (80, 12)

Examine behaviour at/near fixed points but view in phase-space (or phase-plane) where x plotted against y rather than against time

eg predator-prey from last week: phase plane gives extra info





Cyclic behaviour is fixed but system not at a fixed point: complications of higher dimensions

Need phase plane AND fixed points to analyse behaviour: [DEMO]

Classify Fixed Points

Suppose $\underline{x}_0 = (x_0, y_0)^T$ is a fixed point. Define the Jacobian:

$$J(\underline{x_0}) = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix}$$

where $f_x(x, y) = \dfrac{\partial f}{\partial x}$ and $f_y(x, y) = \dfrac{\partial f}{\partial y}$

Find eigenvalues and eigenvectors of J evaluated at the fixed point:

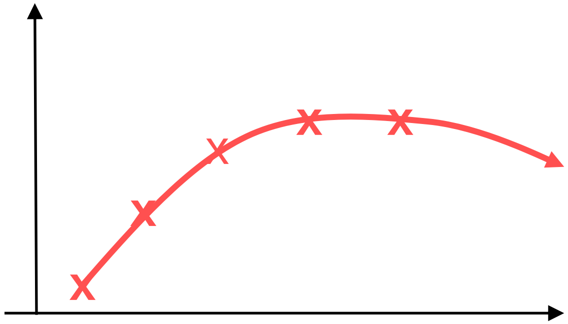
1. If eigenvalues have negative real parts, x_0 asymptotically stable
2. If at least one has positive real part, x_0 unstable
3. If eigenvalues are pure imaginary, stable or unstable

Complex numbers are made up of a real part (normal number) and an imaginary part eg 4 + 3i where $i = \sqrt{-1}$

Remember from matrix lecture: Intuition is direction of $\underline{x}$ is unchanged by being transformed by A so it reflects the principal direction (or axis) of the transformation

ie Repeatedly transform $\underline{v}$ by A.

Start at $\underline{v}$ then A$\underline{v}$ then AA$\underline{v}$= $A^2\underline{v}$ etc …

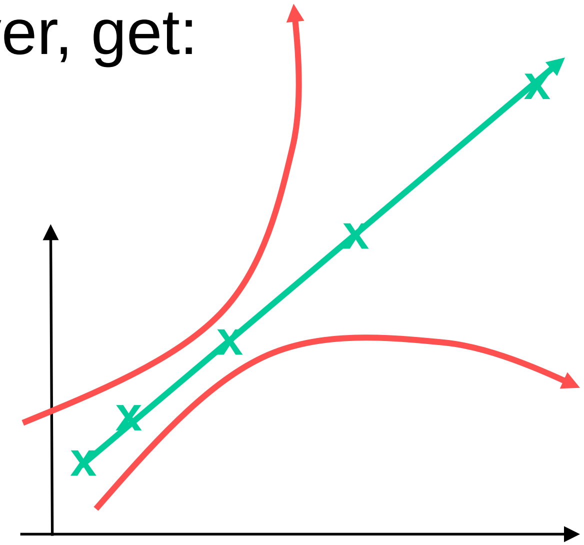


Most starting points result in curved trajectories

Starting from an eigenvector, $\underline{x}$ however, get:

$A\underline{x} = \lambda\underline{x}$, $A^2\underline{x} = \lambda^2\underline{x}$, $A^3\underline{x} = \lambda^3\underline{x}$,
$A^4\underline{x} = \lambda^4\underline{x}$, …

So trajectory is a straight line

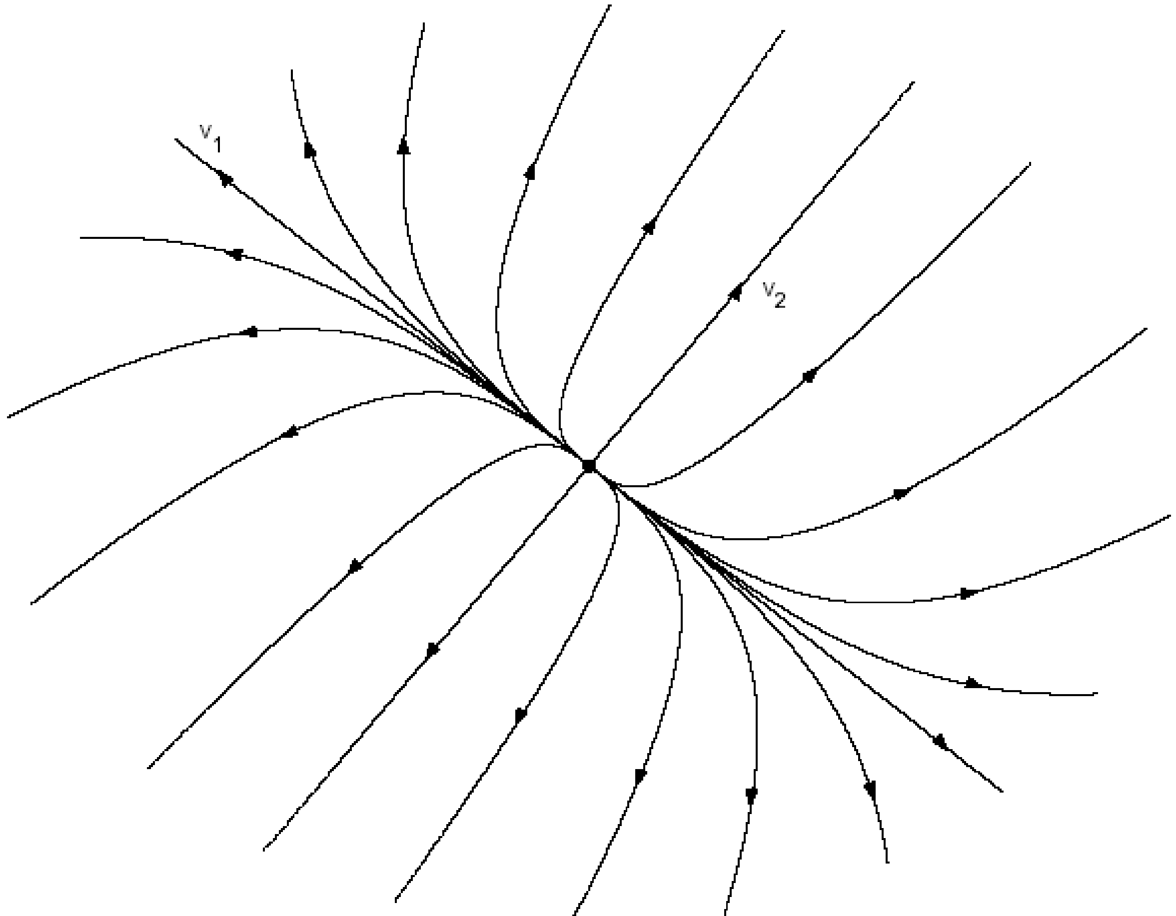


Note if $|\lambda| > 1$, $\underline{x}$ expands. If not, will contract

Fixed Points of Linear Systems

Various behaviours depending on the eigenvalues (e_i) and eigenvectors ($\underline{v}_i$) of J

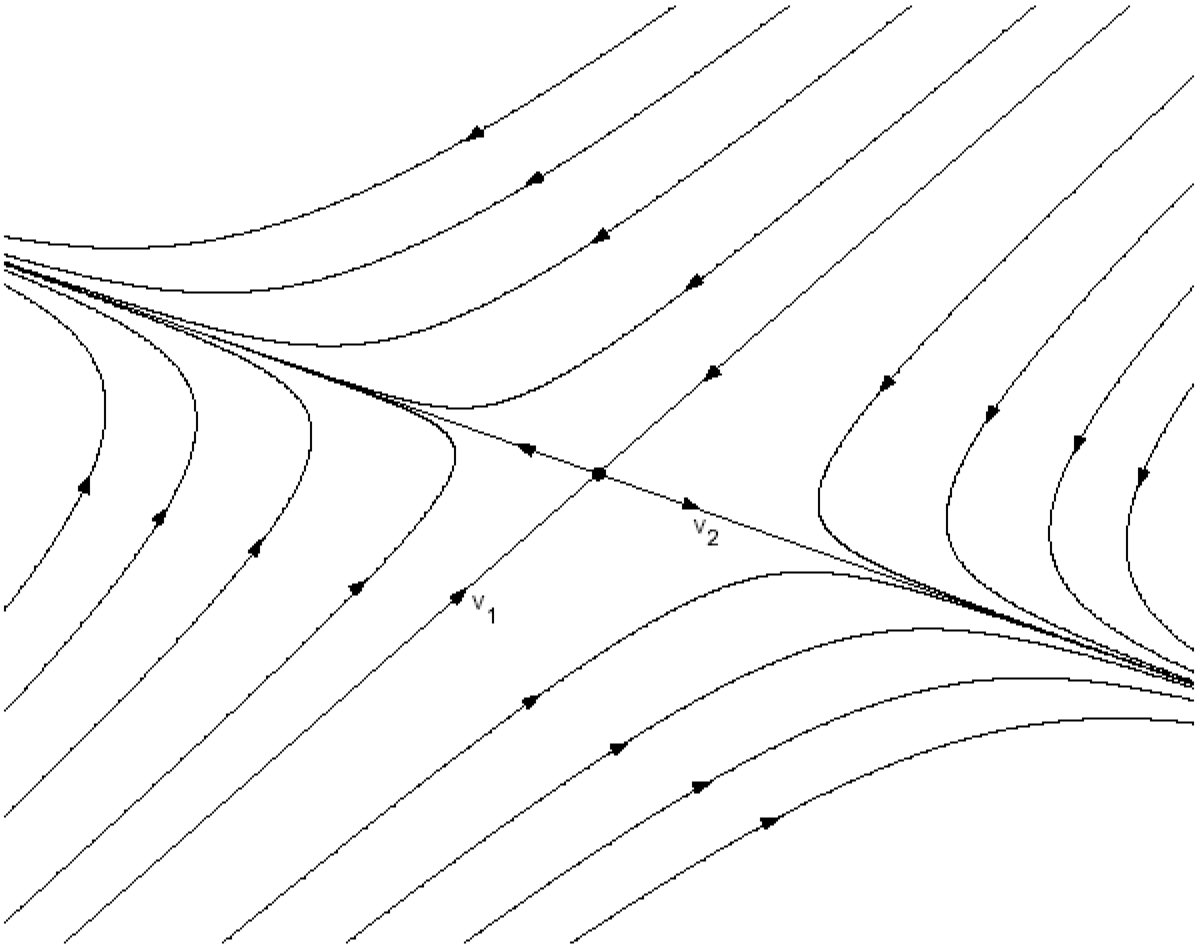
In general, points attracted along negative eigenvalues and repelled by positive. Axes of attraction etc are eigenvectors eg



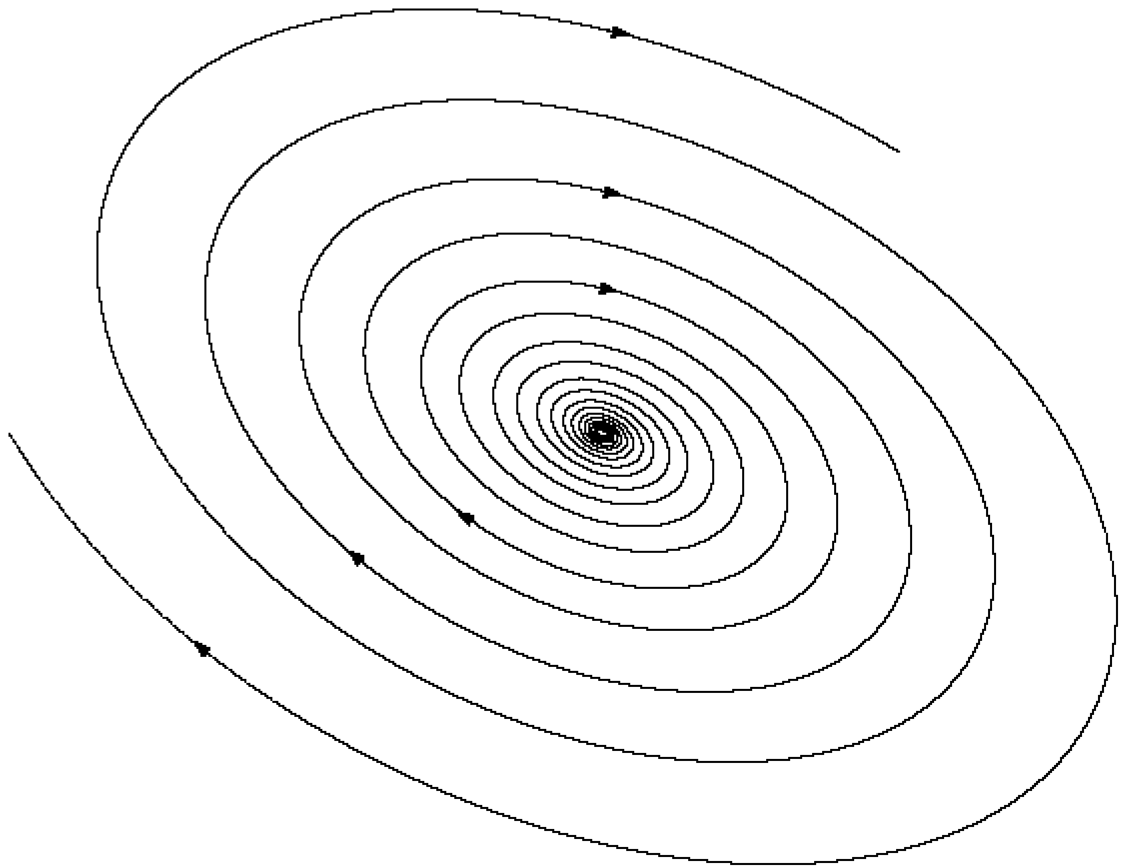
Unstable node,

$e_1 > e_2 > 0$

(Stable node is same but arrows pointing the other way)

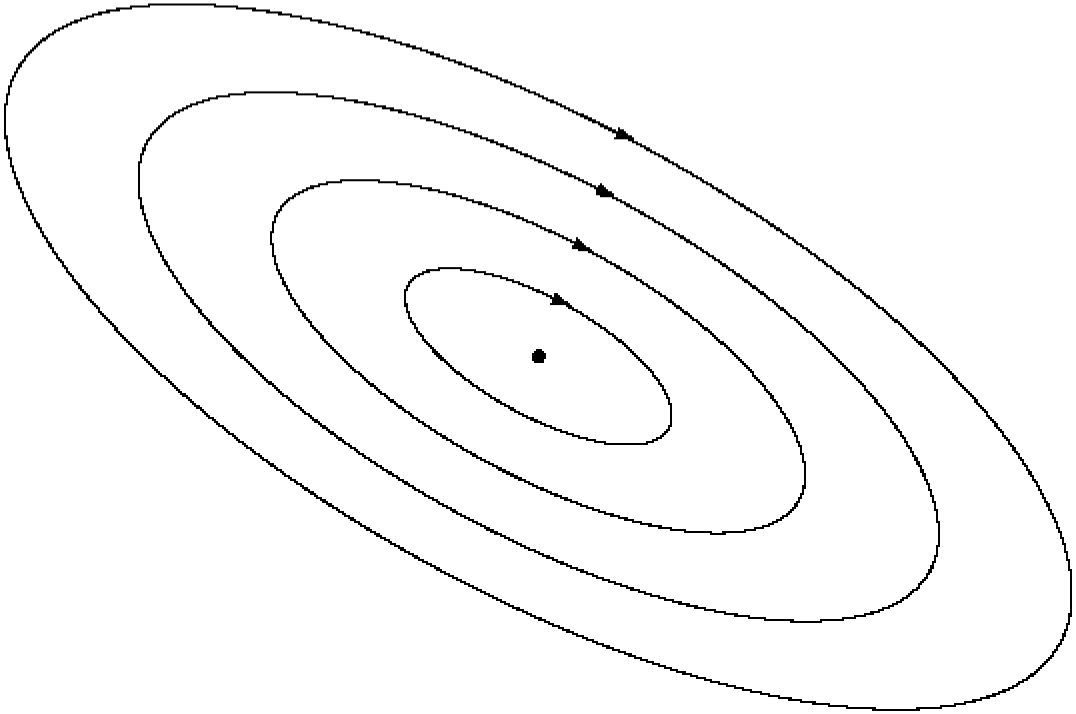


Saddle point, $e_1 < 0 < e_2$



Unstable focus, complex e_i's, real part > 0

For a stable focus, real part < 0



Linear centre, complex e_i's, real part = 0

For non-linear equations, behaviour near the fixed points will be ‘almost like’ the behaviour of a linear system depending how ‘almost linear’ it is

Behaviour gets less linear-like the further away trajectories get from the fixed point

Back to eg:

$$\dot{x} = 0.6x - 0.05xy$$
$$\dot{y} = 0.005xy - 0.4y$$

$f_x = 0.6 - 0.05y$, $f_y = 0.05x$,

$g_x = 0.005y$, $g_y = 0.005x - 0.4$

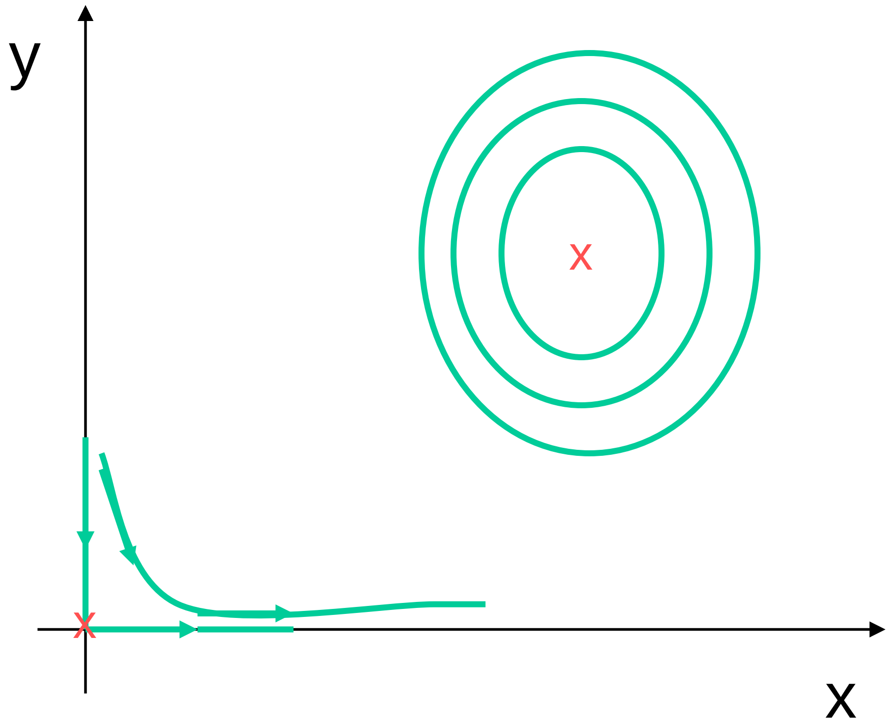
$$J(\underline{x_0}) = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix}$$

so: $$J(0,0) = \begin{pmatrix} 0.6 & 0 \\ 0 & -0.4 \end{pmatrix} \qquad J(80,12) = \begin{pmatrix} 0 & 0.6 \\ 4 & 0 \end{pmatrix}$$

For J(0,0) eigenvalues 0.6 and –0.4 and eigenvectors (1,0) and (0,1). Unstable (a saddle point) with main axes coordinate axes

For J(80,12) eigenvalues are pure imaginary: need more info but will be like a centre…. **to the phase-plane!**

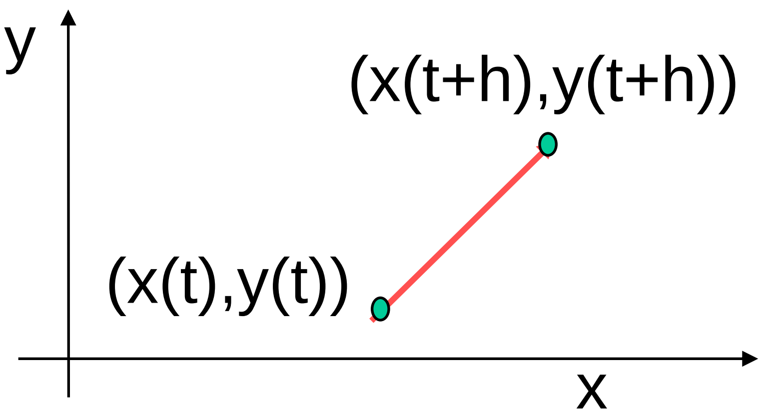
have a saddle node at (0,0) and a centre at (80, 12)



To get a more accurate picture, we can look at all the direction vectors in the phase plane

Phase plane analysis

Similar to direction fields except we use a plot of x against y



Want to examine behaviour of the dynamical system from different starting points

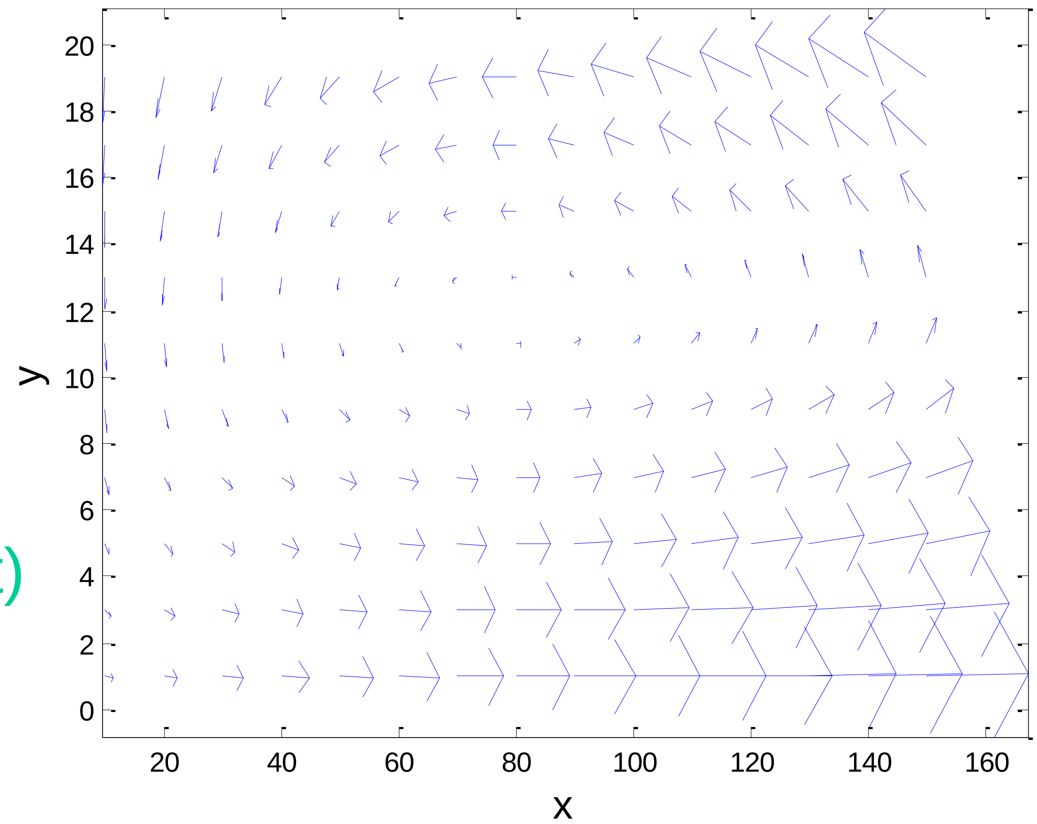
1. Select a set of starting points (x(t),y(t))

2. By Euler evaluate:

(x(t+h),y(t+h))=

(x(t)+hdx/dt, y(t) +hdy/dt)

plot an arrow depicting the direction of movement



Isoclines

Often helpful to plot isoclines on the phase plane

Isoclines are curves where dx/dt or dy/dt are constant

Found by setting $dx/dt = c$ and $dy/dt = c$ and solving

The most useful are nullclines, where dx/dt or $dy/dt = 0$ since

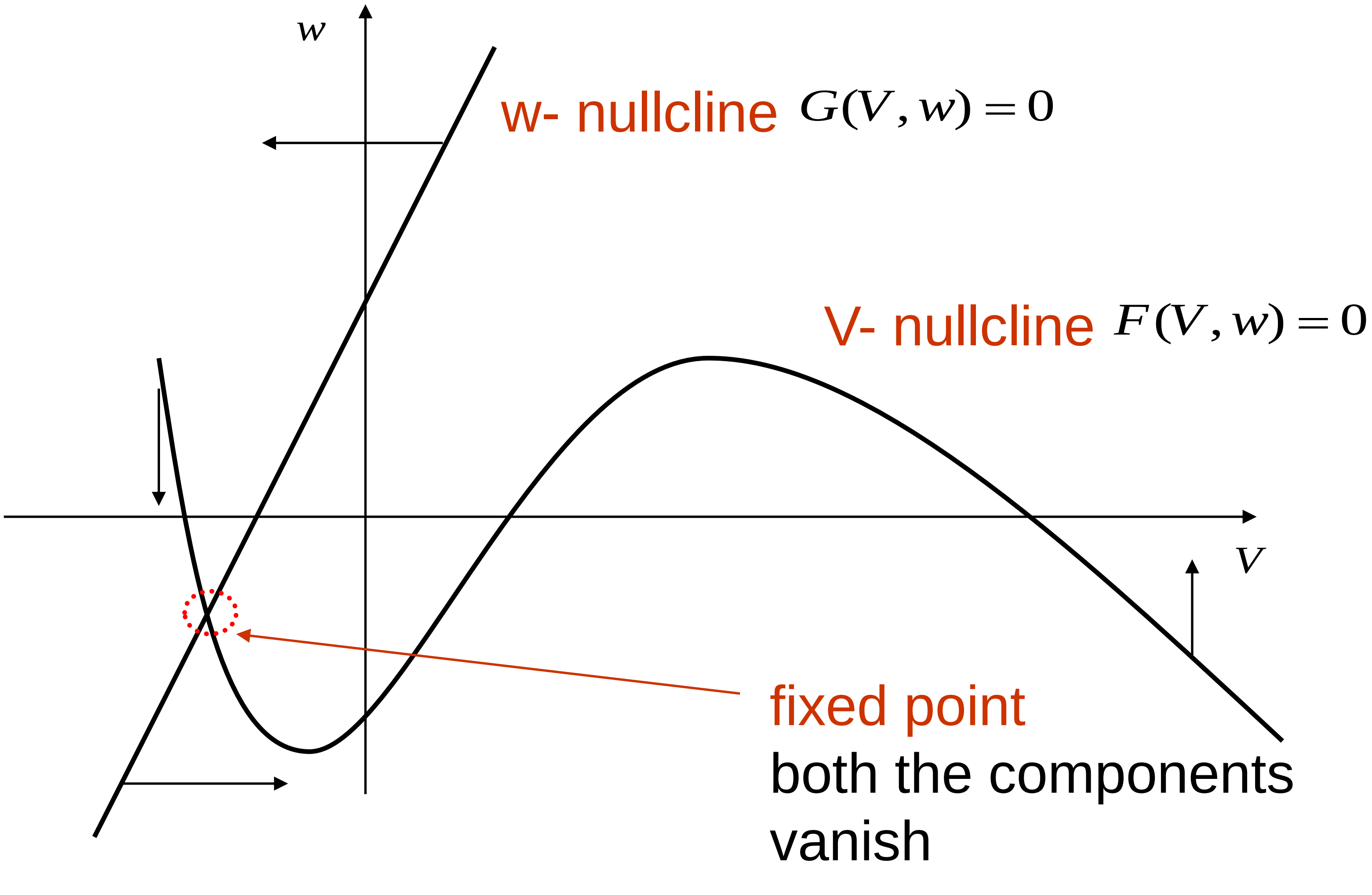
1. Points where the nullclines cross are fixed points

2. Trajectories cross the nullclines at right angles so we know in which direction they are moving

Why? New point $(x(t+h), y(t+h)) = (x(t)+h\,dx/dt, \; y(t)+h\,dy/dt)$

If: $dx/dt = 0$ $x(t+h) = x(t)$ so movement is vertical

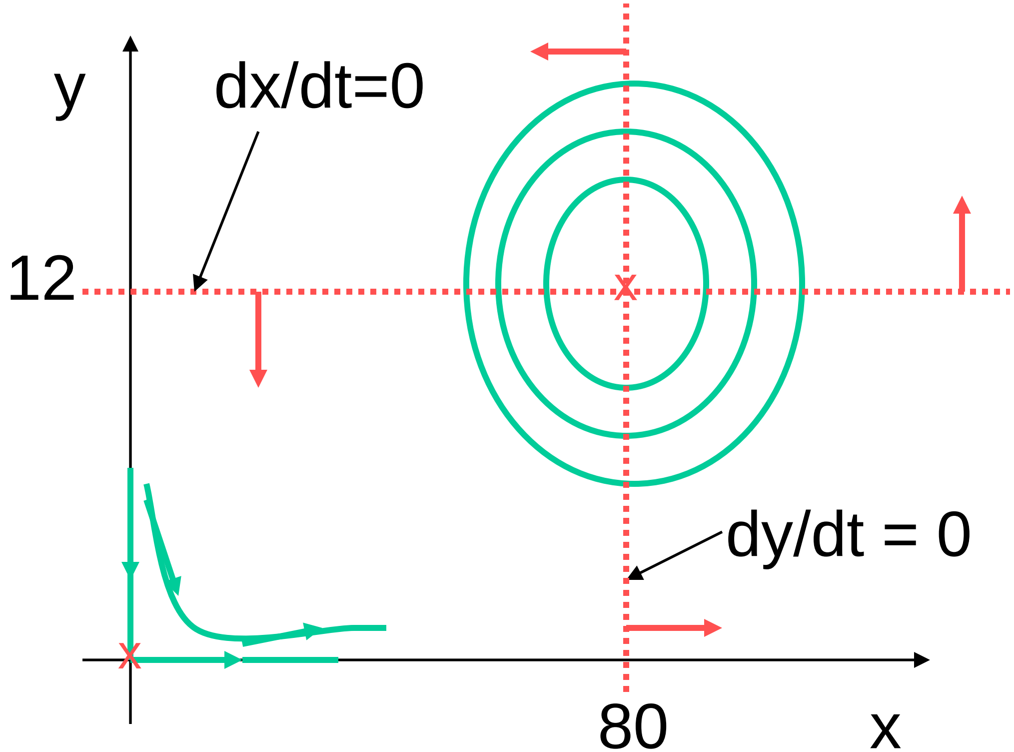
If: $dy/dt = 0$ $y(t+h) = y(t)$ so movement is horizontal can be more useful in a more complicated example eg Fitzhugh-Nagumo



eg

$$\dot{x} = 0.6x - 0.05xy$$

$$\dot{y} = 0.005xy - 0.4y$$



if dx/dt=0 => y= 12

$\Rightarrow$ dy/dt =0.06x – 4.8

$\Rightarrow$ dy/dt< 0 if x< 80,

$\Rightarrow$ dy/dt > 0 else

so tells us direction of rotation of the centre

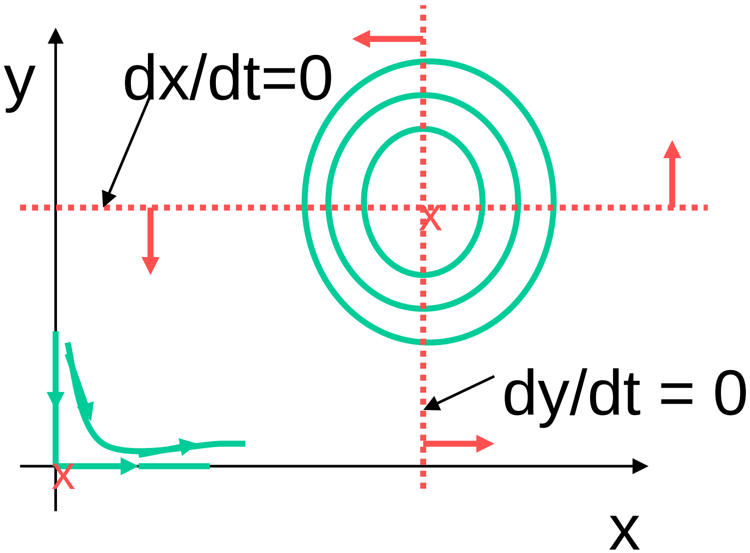
Similarly, dy/dt=0 => x=80 and dx/dt = 0.6(80) – 0.05(80)y ie dx/dt = 48 – 4y

$\Rightarrow$ dx/dt< 0 if y<12,

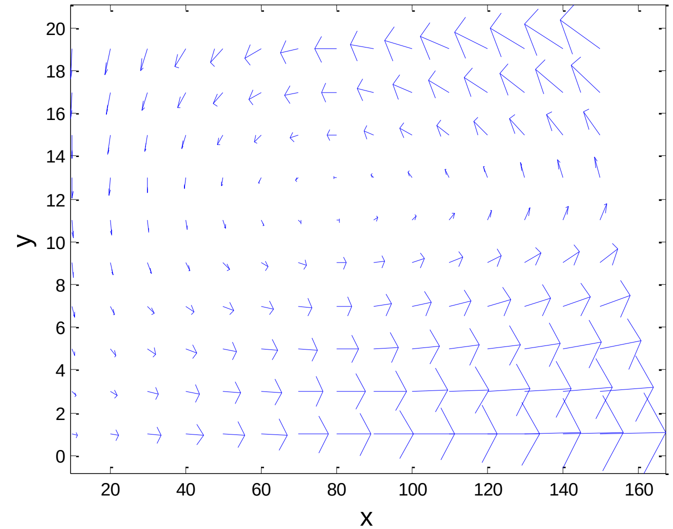
$\Rightarrow$ dx/dt > 0 else

To see the behaviour of the system, use all the info gathered

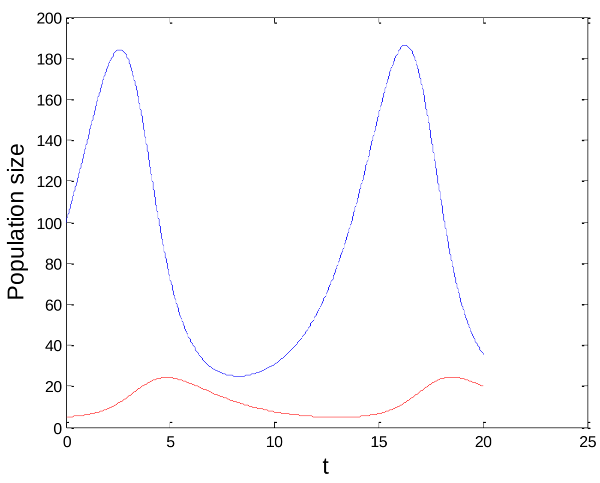
stability analysis and isoclines

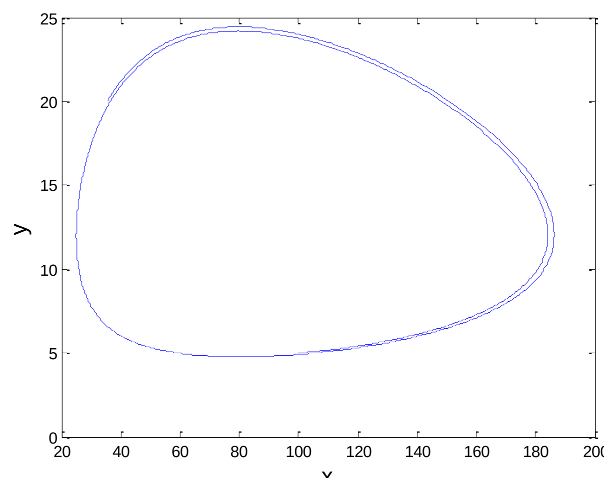


phase-plane



trajectories in phase space and over time





and build up a picture of what will happen

Chaos and stability

Higher dimensional systems are VERY often chaotic

"No one [chaos scientist he interviewed] could quite agree on [a definition of] the word itself" (Gleik 1988, quoted in http://mathworld.wolfram.com/Chaos.html, 11/08)

My notion of chaos is unpredictability: things starting arbitrarily close together can get arbitrarily far apart (but do not necessarily do so)

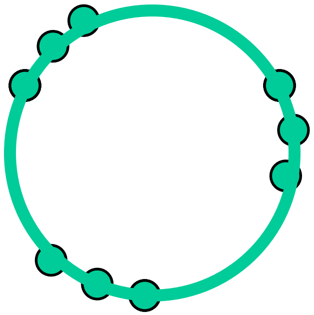
For real systems, means that finest-grain of detail of initial conditions is important ie the butterfly effect

For analysis, means it is a pain if not impossible unless constrained somehow, which is often the case

Have the concept of Lyapunov Stability, things getting arbitrarily close to a fixed point, but maybe not being there all the time, and looser notions

Limit Cycles

Can also approach a set of points: consider a periodic attractor with period 3;



What about period π?

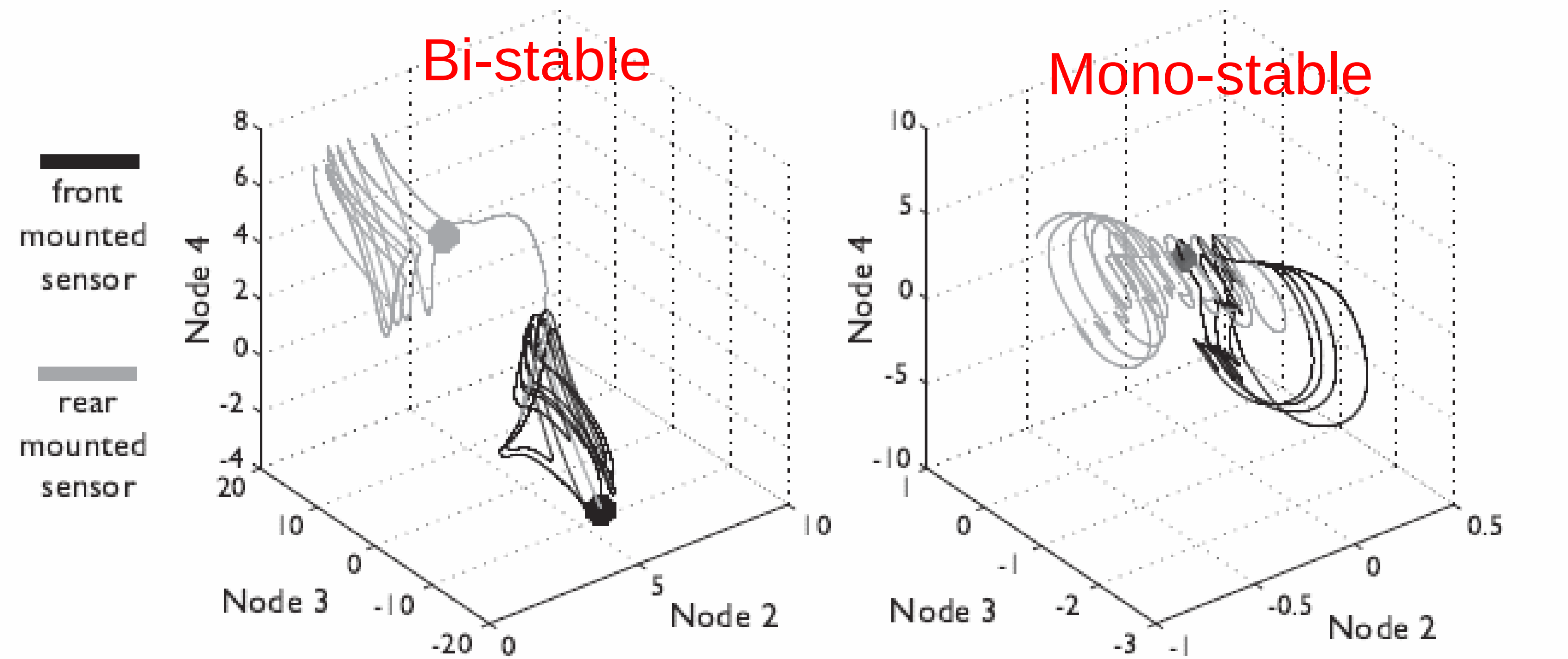
Limit cycles or orbits also become possible, where the state of the system ‘cycles’ through a set of states.

Limit Cycle: “An attracting set to which orbits or trajectories converge and upon which trajectories are periodic.” (http://mathworld.wolfram.com/LimitCycle.html, 11/08)

There are many different kinds of limit cycles eg periodic or chaotic, stable, unstable, or sometimes even half-stable (Strogatz, 1994).

Limit cycles are important because eg they represent systems that oscillate even in the absence of external input.

One Man's Transient is Another Man's Limit Cycle?



From Buckley et al. (2008). Monostable controllers for adaptive behaviour. In Proc SAB 10

- 'Transient': any point not in/at(/cycling round?) an equilibrium
- Buckley et al showed you can get bi-stability from a mono-stable system
- Shows the importance of transients
- Also an issue of 'strict' definitions? Both have 2 attractors … discuss

Multi-dimensional 1st order systems

If system is more than 2 dimensional, can use the same techniques

ie find Jacobian (will be an nxn matrix if we have n equations)

However can only view 2 or 3 variables at once

… and for non-linear gets pretty complicated pretty quickly but we can but try

Summary

This lecture:

- how to find fixed points of a dynamical system
- How to analyse stability of fixed points
- the use of the phase plane to determine the behaviour

In seminars:

- work through analysis of a GasNet neuron
- analyse dynamics of 2 linked GasNet neurons (NB similar to analysis of CTRNN neurons)

Refs

Many refs for dynamical systems. Some will be more suited to your level and you may find others after looking at these :

General refs:

1. Saperstone, (1998) Introduction to ordinary differential equations
2. Goldstein, H. (1980). Classical Mechanics.
3. Strogatz, S. (1994) *Nonlinear Dynamics and Chaos, with Applications to Physics, Biology, Chemistry, and Engineering.*
4. http://mathworld.wolfram.com/
5. Gleick, J. (1988) *Chaos: Making a New Science.*

More on analysis and applications

1. Rosen, R. (1970). Dynamical System Theory in Biology, volume I. Stability theory and its applications.
2. Rubinow, S. I. (1975). Introduction to Mathematical Biology.

Good for discrete systems

1. Sandefur, J. T. (1990). Discrete Dynamical Systems: Theory and Applications.