FCS: Dynamical Systems Analysis

This problem sheet introduces techniques used to analyse dynamical systems. The aim is to determine the long-term behaviour of the system ie what states it is likely to be in and whether it can move between them. By the end you will be able to:

1. Find fixed points of a dynamical system
2. Investigate the stability of fixed points
3. Use cobweb plots to analyse dynamical systems
4. Use the phase-plane to analyse dynamical systems

Questions

1. *Exponential growth/decay **[3 marks]**:* Suppose we have the following equation:

$$\frac{dx}{dt} = kx \tag{1}$$

where x is population size and k governs the rate of growth, which we are to integrate using Euler integration ie using:

$$x(t+h) = x(t) + h\frac{dx}{dt} \tag{2}$$

To find and analyse fixed points of this system (and to make its dependence on h explicit) we must reformulate equations 1 and 2 so that $x(t+h) = f(x(t))$ for some function f.

(a) Use equations 1 and 2 to determine $f(x(t))$.

(b) Find the fixed point of the system (Tips: fixed points of a dynamical system where $x(t+h) = f(x)$ can be found by setting $x = f(x)$ and solving eg if the system was $x(t+h) = f(x(t)) = (x(t))^2$, we would set $x = x^2$ and solve to get $x = 1$ or $x = 0$).

(c) Denoting the fixed point as a_0, determine the stability of the system for $k > 0$ by evaluating $f'(a_0)$ ie calculate $\frac{df}{dx}$ and evaluate it at a_0. (Tips: Remember that $h > 0$. Also, the system is stable if $|f'(a_0)| < 1$, unstable if $|f'(a_0)| > 1$ and inconclusive if $|f'(a_0)| = 1$ eg the system $x(t+h) = f(x(t)) = (x(t))^2$ has fixed points $a_0 = 0, 1$. $f'(x) = \frac{df}{dx} = 2x$ so $|f'(a_0)| = |2a_0| = 0$ for $a_0 = 0$, which is therefore stable, and 2 for $a_0 = 1$ which is unstable).

(d) Investigate the stability for $k < 0$. If h is fixed, what range of values of k lead to a stable system and what to an unstable system (note that the range will be in terms of h). What is the relationship between h and k that makes $|f'(a_0)| = 1$ where a_0 is the fixed point of the system (and is the same as found in question 1c). Using your program from last week's seminar, see what happens to x if you use $k = -2$, $x(0) = 1$ and run it for a simulation length of $100s$ using different values for h. First use $h = 0.75s$, then use the value of h needed to make $|f'(a_0)| = 1$, which we shall call h_0. Finally, use $h = h_0 + 0.01$ then $h = h_0 - 0.01$. Note how much the long-term behaviour changes given such a small change in h. We therefore refer to $h = h_0$ as a *bifurcation point* of the system.

2. *GasNet, single neuron **[5 marks]**:* A GasNet neuron with positive self-recurrency and external input I updated iteratively is a discrete dynamical system governed by the following equation:

$$x(n+1) = f(x(n)) = \tanh\left(C\left(x(n) + I\right)\right)$$

where $x(n)$ is the neuron output at time-step n. C is a parameter which governs the slope of the transfer function tanh. Its default value is genetically-specified, but C is altered temporarily in the presence of a gaseous neuromodulator which may be emitted by other neurons.

(a) Suppose $I = 0$. By plotting $y = x$ and $y = f(x)$ between -2 and 2 (matlab has the function 'tanh') on the same graph, find (approximately) the fixed points of the system for the following values of C: $0.5, 1, 2, -0.9, -1$ and -1.1. Examine the stability of the fixed points by plotting $f'(x) = C(sech(Cx))^2$ between -2 and 2 (matlab has the function 'sech'). By examining the graphs and equations, deduce what range of C values lead to 3 fixed points rather than 1 and the stability of the resultant points.

Check your results using cobweb plots and plotting x over time, using the code fragments contained in GasNetEGs.m. Start from different values of x and examine the behaviour of the systems. Note the range of different behaviours for $C < 0$.

(b) Now set $C = 1.5$ and $I = 0.1$ and examine the fixed points and their stability using the techniques of question 2a. How do the number and behaviour of the fixed points change if I is increased from 0.1 to 0.15? Start both systems at $x(0) = -0.9$ and plot x against time to see the differences between the systems.

(c) In an evolved GasNet network, neuron A has default $C = 0.5$ and no external input ie $I = 0$ and governs a robot's wheel-motor. It is influenced by gas from neuron B, which it has an inhibitory electrical connection to. Neuron B emits gas continuously but at a low concentration, meaning that Neuron A's C-value remains unchanged until the gas has built up for 5 time-steps. At this point, the gas concentration crosses a threshold value and neuron A's C value at subsequent time-steps is changed to -2. This situation continues until neuron A's output $x \geq 0.95$. This inhibits neuron B strongly, stopping gas emission, so that A's C-value returns to 0.5, and the gas concentration returns to zero.

Change your program to include a gas concentration parameter and 2 'if' clauses. The 1st checks whether x is ≥ 0.95, setting the gas concentration to 0 if it is, and incrementing the gas concentration by 1 if not. The second checks whether gas concentration is ≥ 5. If it is, change neuron A's C-value as specified, if not, revert to the old value. Iterate through this system starting from an initial value of $x = 1$ for neuron A and zero gas concentration and observe the motor output. How would you describe the resultant behaviour?

3. *GasNet analysis, 2 neurons [4* ***marks****]:* Suppose we have a network of 2 GasNet neurons, A and B. Neuron A has default $C = 0.5$, positive self-recurrency and external input from a light sensor $I(n)$. Neuron B has $C = 1.2$, positive self-recurrency and receives inhibitory input from A. Denoting the outputs of A and B at the n'th time-step as $x(n)$ and $y(n)$ respectively, the governing equations are:

$$x(n+1) = f(x(n), y(n)) = \tanh(0.5\,(x(n) + I(n))) \quad (3)$$

$$y(n+1) = g(x(n), y(n)) = \tanh(1.2\,(y(n) - x(n))) \quad (4)$$

(a) Assuming $I(n) = 0$ determine the fixed point of neuron A, x_0, by plotting $x = x$ and $x = f(x)$. Use this value to plot $y = g(x_0, y)$ and thus determine the fixed points of the system (remember that at fixed points of the system, both neurons must be at a fixed point).

(b) Write a program that plots x and y over time and x against y (ie a phase-plane plot of x and y) using the code fragments in GasNetEGs.m. Use the program to investigate the behaviour of the system starting from different initial values of x and y (eg $x = -0.5$ and $y = 0$, $x = -0.5$ and $y = 0.5$ etc) and thus determine the stability of the fixed points.

(c) Suppose a light starts flashing so that every 10 time-steps the light sensor inputs 0.5 to neuron A. Alter the program so that $I(n) = 0.5$ every 10 time-steps (note 'mod(n,10)' returns a zero every time n is a multiple of 10). What happens if you start at $x = -0.5$ and $y = 0.5$ and run the system for 100 iterations?

(d) Suppose now that the light is switched off, but that every 5th time-step neuron B is affected by gas so that for that time-step its C-value switches from $C = 1.2$ to $C = -1.2$. Again investigate the behaviour of the system starting from $x = -0.5$ and $y = 0.5$.

(e) Finally, investigate the behaviour of the system starting from $x = -0.5$ and $y = 0.5$ if the light switched back on ie so that every 5th time-step the C-value of B switches from $C = 1.2$ to $C = -1.2$ and every 10th time-step $I(n) = 0.5$.

4. *Extra Credit [3* ***marks****]:* Explore what other systems you can get from GasNet neurons. Here 'other' means systems that have qualitatively different stable or quasi-stable states to the ones already investigated.