FCS Worksheet: Difference Equations

This seminar introduces numerical integration of differential equations using Euler's method and gives an intuition of 1st order dynamical systems and their analysis. By the end you should be able to:

1. Use the Euler method of numerical integration to solve differential equations
2. Numerically integrate a 1st order differential equation such as the equation governing a Continuous Time Recurrent Neural Network (CTRNN).
3. Numerically integrate a system of differential equations such as predator-prey models

I will now present some differential equations to be integrated numerically using Euler's method for initial value problems (IVPs), which proceeds as follows. If we have a variable $x(t)$ which varies over time as specified by $\frac{dx}{dt}$ then:

$$x(t+h) = x(t) + h\frac{dx}{dt} \tag{1}$$

where $\frac{dx}{dt}$ is evaluated at time t ie using $x(t)$ and t. As this is an IVP the *initial* value of x (usually $x(0)$ ie x at $t = 0$) must be specified, as must the size of the time-step (otherwise known as step-size) h.

1. We will start with an equation exhibiting exponential population growth

   $$\frac{dx}{dt} = kx \tag{2}$$

   where x is the population size and the parameter $k > 0$ governs the rate of growth. We want to code the following algorithm:

   ```
   OldX = initial value
   for TStep = 1:NumTimeSteps
       Calculate NewX at an advanced time-step using OldX, h and growth rate
       plot the new point
       Pause and view the graph
       OldX = NewX
   end
   ```

   Tips:

   (a) Write a sub-function which takes x, time-step h and growth rate and returns x at the advanced time-step using Euler integration specified in equation 1 and $\frac{dx}{dt}$ specified in equation 2.

   (b) If you 'hold' the plots as you go along, you can plot the new point using 'plot([TStep-1, TStep]*h,[OldX, x])'. Note that t*h is how much time has elapsed as t counts the number of time-steps of size h taken. As a corollary, the number of time steps needed is the length of the simulation divided by the time-step. If you want the axes to remain constant throughout use the 'axis' command (see help axis for details).

   (c) Pass the time-step size h, growth-rate, length of simulation and initial value from the command line by writing your function so that it takes arguments.

   Run the algorithm with a time-step $h = 0.05s$, a growth-rate of $k = 6$, a simulation length of $2s$ and an initial value at $t = 0$ of $x(0) = 0.1$. What happens to x? What if you start with $x(0) = 0$? Perturb x a little from 0 and see what happens. What does this tell you about the stationary point of the system? **[2 marks]**

2. Now implement exponential decay:

   $$\frac{dx}{dt} = -kx$$

   where again $k > 0$, starting at $x(0) = 0.8$ and $k = 1$ with a simulation length of 2 and a time-step of $h = 0.05$ (you can use the function you used for exponential growth if you pass in a negative k).

   Repeat for $k = 4$. What if you start with $x(0) = 0$? Perturb x a little from 0 and see what happens this time. What does this tell you about the stationary point of the system? **[1 mark]**

3. Next use the following equation which introduces a limit on the population size (as would be seen if there was a limit on food etc):

$$\frac{dx}{dt} = 4x(1 - x)$$

start with $x(0) = 0.1$, a simulation length of 2, a time-step of $h = 0.05$ and see what happens to x. Start at $x(0) = 0$ and $x(0) = 1$. Perturb x a little from these points and see what happens. Comment on the results. **[2 marks]**

4. The equation:

$$\frac{dx}{dt} = \frac{1}{\tau}(-x + I(t))$$

represents a single ctrnn style neuron with, time-dependent input I(t). Run this equation with $x(0) = 0$, $\tau = 1$, time-step $h = 0.05$, simulation length of 4 and:

$$I(t) = \left\{ \begin{array}{cc} 1 & t \leq 2 \\ 0 & else \end{array} \right\}$$

where t is the time that has elapsed since the system is started and NOT the time-step. What happens for $\tau = 5$? What about if τ is set to be equal to the length of the time-step h? Before running this, try to predict what will happen by putting $\tau = h$ into the numerical integration equations and solving. Finally, set $\tau = h/2$. What happens to the system? Can you explain your results by examining the numerical integration equations? **[3 marks]**

5. **Bonus marks:** The following question deals with 2 competing populations, one predator (represented by y) and one prey (represented by x). The population dynamics are represented by the following equations:

$$\frac{dx}{dt} = rx - axy \quad (3)$$

$$\frac{dy}{dt} = faxy - qy \quad (4)$$

The prey population size x is governed by 2 factors: the amount of prey born and the amount lost to predation. The number of prey born increases at a rate equal to the product of the population size (since the more prey there are, the more are born) and r, the prey growth rate. Similarly, the amount lost to predation is proportional to the prey population size, but it is also proportional to the number of predators and a, the attack efficiency of the predators. This factor is negative as the effect of predation will clearly be negative. These elements make up the 2 factors on the right-hand side of equation 3.

The predator population size y is also affected by 2 factors, one positive and one negative, which can be seen on the right-hand side of equation 4. It is affected by the a similar predation factor to the prey, though it is now positive and its effect is lessened by the factor f, the rate at which predators turn prey into offspring. The second factor affecting the predator population is a self-limiting factor which is the product of the number of predators and q, the starvation rate for predators.

Write a new function which takes an x and y value and returns values at an advanced time-step, using Euler integration to solve equations 3 and 4. Run the simulation for 30 seconds with a time-step of $h = 0.01$, starting with $x(0) = 100$ and $y(0) = 5$, $r = 0.6$, $a = 0.05$, $f = 0.1$ and $q = 0.4$. Plot x and y on the same graph over time and see what happens. On a separate graph, plot x against y to see the *phase-plane* behaviour. Finally, see what happens to the population if a time-step of 0.5 is used. **[2 marks]**